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Precautionary saving and changes in risk correlation

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Abstract

This note analyzes the effect of a change in the covariance between labor income risk and interest rate risk on the threshold level for prudence ensuring positive precautionary saving, recently derived by Baiardi, Magnani and Menegatti (2014). We show that this effect is different in different cases. An increase in the covariance between the two risks decreases (increases) the threshold level when the variance of labor income is smaller (larger) than the variance of the return on saving. An interpretation of these results in terms of elasticity of total variance with respect to saving is provided.

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1 Introduction

Precautionary saving theory studies the optimal choices of consumption and saving levels when future income is risky. The traditional literature examines two different kinds of risk: either labor income risk (Leland, 1968; Sandmo, 1970 and Drèze & Modigliani, 1972) or interest rate risk (Sandmo, 1970; Rothschild & Stiglitz, 1971).[1]

Two recent works analyze precautionary saving when both labor income risk and interest rate risk are introduced. Li (2012) studies this issue assuming positive quadrant dependence between the two risks. Baiardi, Magnani & Menegatti (2014, henceforth BMM) examine the same problem in the case of small risks without introducing any *a priori* assumption on their joint distribution. Both papers find that the presence of positive precautionary saving relies on the magnitude of the index of partial relative prudence, which needs to be higher than a threshold level. This threshold is equal to 2 under Li's assumption. But it is equal to a variable level K, which depends on the variance of each of the two risks and on their covariance, in the BMM framework. BMM consider some specific cases where different values for K are obtained.

The aim of this note is to re-examine the threshold derived by BMM, studying the effects of changes in the joint distribution of the two random variables which vary their covariance keeping their variances unchanged. We thus provide a characterization of the threshold K as a function of the covariance between the two risks, under the assumption that neither of their variances vary. Note that, under this assumption, every increase (decrease) in covariance causes a proportional increase (decrease) in correlation.[2]

There are two main reasons for analyzing changes in risk covariance and correlation.

First, both works cited above (Li, 2012 and BMM) provide results which depend on how the risks covariates. Li (2012) derives his results under the assumption of positive quadrant dependence which requires positive covariance.[3] Similarly, some of the results by BMM are derived under different specific assumptions on the sign of risk covariance. Given these results and since, in general, risk covariance may be *a priori* either negative or positive and either large or small, different levels for it affect precautionary saving choice.

Furthermore, the literature shows that the precautionary saving motive is related to the desire to increase the level of wealth in the period where an agent bears income

[1] For labor income risk see also Kimball (1990) and Menegatti (2001). The case where other risks, such as health or environmental risks, are also introduced is studied by Courbage & Rey (2007), Menegatti (2009a, 2009b) and Denuit, Eeckhoudt & Menegatti (2011)

[2] Analytically, risk correlation is a linearly homogeneous function of risk covariance.

[3] Positive covariance is necessary (but not sufficient) for positive quadrant dependence (see Lehmann, 1966 and Li, 2011).

risk.[4] Starting from this interpretation, one might expect that, in the presence of a larger correlation, which implies a larger uncertainty on income, the level of prudence required for precautionary saving would become lower. But the analysis below provides a formal study of this issue and shows that the conjecture, although plausible, is too simplistic. The relationship between correlation and precautionary saving choice is actually more complex.

The note has the following structure. Section 2 presents the model. Section 3 derives our main results. Section 4 provides an interpretation and Section 5 concludes.

2 The model

We examine a two-period model where a consumer has a Von Neumann–Morgenstern utility function $u(x)$ in period 0 and $v(x)$ in period 1, where x is consumption. Functions $u(x)$ and $v(x)$ are three times continuously differentiable. We denote by u_i, u_{ij}, u_{ijk} (respectively v_i, v_{ij}, v_{ijk}) first, second and third partial derivatives of u (respectively v).[5] We assume, as usual, that the agent is non-satiated ($u_1 > 0$ and $v_1 > 0$) and risk averse ($u_{11} < 0$ and $v_{11} < 0$). For simplicity we also introduce the hypothesis that the agent is prudent ($u_{111} > 0$ and $v_{111} > 0$).[6]

The consumer decision problem in the certainty case is:

$$\max_s u\left(y_0 - s\right) + v\left(y_1 + s\right) \tag{1}$$

where y_0, y_1, s and R denote respectively: first–period labor income, second-period labor income, saving and the return on saving. The optimal level of saving $s^\star$ is defined by the first-order condition:

$$u_1\left(y_0 - s^\star\right) = Rv_1\left(y_1 + Rs^\star\right). \tag{2}$$

Consider now the case where both labor income and the return on saving are uncertain. The uncertain future level of labor income is denoted by $\tilde{y}$, which is a random variable such that $\mathbb{E}\left[\tilde{y}\right] = y_1$. The uncertain level of the return on saving is $\tilde{R}$ which is a random variable such that $\mathbb{E}\ [\tilde{R}] = R$. The consumer decision problem becomes:

$$\max_s u\left(y_0 - s\right) + \mathbb{E}[v(\tilde{y} + \tilde{R}s)]. \tag{3}$$

[4]See Eeckhoudt and Schlesinger (2006) and Menegatti (2007).

[5]The intertemporal discount rate is embedded in function $v(x)$.

[6]Note that prudence is not sufficient for the condition derived by BMM to hold. Moreover, as stated by BMM, unlike in the previous literature, it is not necessary either. However imprudence is compatible with the condition derived by BMM only in some specific cases. We thus focus on the most relevant circumstance where the agent is prudent.

In the presence of two sources of uncertainty (i.e. income and interest rate risks), the optimal level of saving $s^{\star\star}$ is defined by the first-order condition:

$$u_1(y_0 - s^{\star\star}) = \mathbb{E}[\tilde{R}v_1(\tilde{y} + \tilde{R}s^{\star\star})]. \tag{4}$$

When $s^{\star\star}$ in Equation (4) is larger than $s^{\star}$ in Equation (2), the presence of uncertainty increases saving generating a positive 'precautionary saving'. If, on the other hand, $s^{\star\star} < s^{\star}$ uncertainty reduces saving and we have 'precautionary dissaving'.

3 Precautionary saving conditions when covariance changes

Given the framework described in Section 2, BMM provides a necessary and sufficient condition for positive precautionary saving in the case of small risks which involve the index of partial relative prudence (PRP). This index, introduced for the first time by Choi et al.(2001) and Eichner & Wagener (2004a,b), can be defined as

$$PRP = -s^{\star\star}\tilde{R}\frac{v_{111}(\tilde{y} + s^{\star\star}\tilde{R})}{v_{11}(\tilde{y} + s^{\star\star}\tilde{R})}. \tag{5}$$

Given this definition BMM show that:

Lemma 1 (**Baiardi, Magnani & Menegatti, 2014**). *In the presence of small risks,*

$$PRP \geq K \tag{6}$$

where

$$K = 2\frac{(s^{\star\star})^2 var[\tilde{R}] + s^{\star\star}cov[\tilde{y}, \tilde{R}]}{var[\tilde{y}] + (s^{\star\star})^2 var[\tilde{R}] + 2s^{\star\star}cov[\tilde{y}, \tilde{R}]}. \tag{7}$$

is a necessary and sufficient condition to have a positive precautionary saving ($s^{\star\star} \geq s^{\star}$).

Lemma 1 characterizes a threshold equal to K for PRP which depends on different variables describing the joint distribution of labor income risk and interest rate risk. BMM characterizes the threshold K in the two specific cases where the covariance between the two risks is positive and where it is negative (for positive s) and sufficiently low (below a given level).

Given these results and since labor income risk and interest rate risk can be either positively or negatively correlated, in this note, we generalize BMM analysis by studying the effect on K of changes in the levels of covariance. In order do to this we analyze a variation in the joint distribution of the two risks where the variances of both risks are unchanged and the covariance term varies.[7] This allows us to characterize the threshold K as a function of $cov[\tilde{y}, \tilde{R}]$. We perform our analysis in the benchmark case where the consumer is a net saver, i.e. when $s^{\star\star} \geq 0$, denoting this function by $K = K(cov[\tilde{y}, \tilde{R}])$.

In order to do this we first note that the Cauchy - Schwarz Inequality ensures that $\left|cov[\tilde{y}, \tilde{R}]\right| \leq \sqrt{var[\tilde{R}]var[\tilde{y}]}$. This implies that the domain of function $K(cov[\tilde{y}, \tilde{R}])$ is $\left[-\sqrt{var[\tilde{R}]var[\tilde{y}]}, \sqrt{var[\tilde{R}]var[\tilde{y}]}\right]$.

In the analysis of function $K(cov[\tilde{y}, \tilde{R}])$ we then distinguish three cases, depending on the comparison between $var[\tilde{y}]$ and $(s^{\star\star})^2 var[\tilde{R}]$, which are respectively the variance of labor income and the variance of the total return on saving $s^{\star\star}\tilde{R}$.[8] In the first case labor income is more variable than the return on saving ($var[\tilde{y}] > (s^{\star\star})^2 var[\tilde{R}]$) while the opposite happens in the second case ($var[\tilde{y}] < (s^{\star\star})^2 var[\tilde{R}]$). Lastly, the third case is when the two variances are equal ($var[\tilde{y}] = (s^{\star\star})^2 var[\tilde{R}]$)

Consider initially the circumstance where labor income is more variable than total return on saving ($var[\tilde{y}] > (s^{\star\star})^2 var[\tilde{R}]$). In this case the following results hold:

Proposition 1. *If $var[\tilde{y}] > (s^{\star\star})^2 var[\tilde{R}]$ holds then function $K(cov[\tilde{y}, \tilde{R}])$ is increasing in $cov[\tilde{y}, \tilde{R}]$ with an upper bound equal to 1.*

Proof. Consider Equation (7) and take the first derivative with respect to $cov[\tilde{y}, \tilde{r}]$ to obtain:[9]

$$\frac{\partial K(cov[\tilde{y}, \tilde{R}])}{\partial cov[\tilde{y}, \tilde{R}]} = 2s^{\star\star} \frac{var[\tilde{y}] - (s^{\star\star})^2 var[\tilde{R}]}{\left\{var[\tilde{y}] + (s^{\star\star})^2 var[\tilde{R}] + 2s^{\star\star} cov[\tilde{y}, \tilde{R}]\right\}^2} \tag{8}$$

Since $s^{\star\star} \geq 0$ holds, $K(cov[\tilde{y}, \tilde{R}])$ is increasing in $cov[\tilde{y}, \tilde{R}]$, if $var[\tilde{y}] > (s^{\star\star})^2 var[\tilde{R}]$.

By adding and subtracting $var[\tilde{y}]$ in the numerator of the fraction in Equation (7) we get:

$$K(cov[\tilde{y}, \tilde{R}]) = \frac{(s^{\star\star})^2 var[\tilde{R}] - var[\tilde{y}]}{var[\tilde{y}] + (s^{\star\star})^2 var[\tilde{R}] + 2s^{\star\star} cov[\tilde{y}, \tilde{R}]} + 1. \tag{9}$$

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[7]This idea could be interestingly linked to the concept of copula, recently introduced in statistics and used in several financial applications. For an introduction to this concept see Nelsen (2006) and Joe (2015).

[8]Note that $var[\tilde{R}]$ is the variance of the interest rate while $(s^{\star\star})^2 var[\tilde{R}]$ is the variance of total return on saving.

[9]Note that by the assumption that variances do not change, only the last term in the denominator of the fraction varies

Given this, if $var[\tilde{y}] > (s^{\star\star})^2 var[\tilde{R}]$, K is always smaller than 1. □

Corollary 1. *If* $var[\tilde{y}] > (s^{\star\star})^2 var[\tilde{R}]$*:*

- *when* $cov[\tilde{y}, \tilde{R}] < -s^{\star\star} var[\tilde{R}]$, $K(cov[\tilde{y}, \tilde{R}]) < 0$ *holds*
- *when* $cov[\tilde{y}, \tilde{R}] = -s^{\star\star} var[\tilde{R}]$, $K(cov[\tilde{y}, \tilde{R}]) = 0$ *holds*
- *when* $cov[\tilde{y}, \tilde{R}] > -s^{\star\star} var[\tilde{R}]$, $K(cov[\tilde{y}, \tilde{R}]) > 0$ *holds*

Proof. Follows from (7) since $var[\tilde{y}] + (s^{\star\star})^2 var[\tilde{R}] + 2s^{\star\star} cov[\tilde{y}, \tilde{R}] = var[\tilde{y} + \tilde{R}s^{\star\star}] > 0$. □

Figure 1 provides a qualitative description of the results above. Consider now the

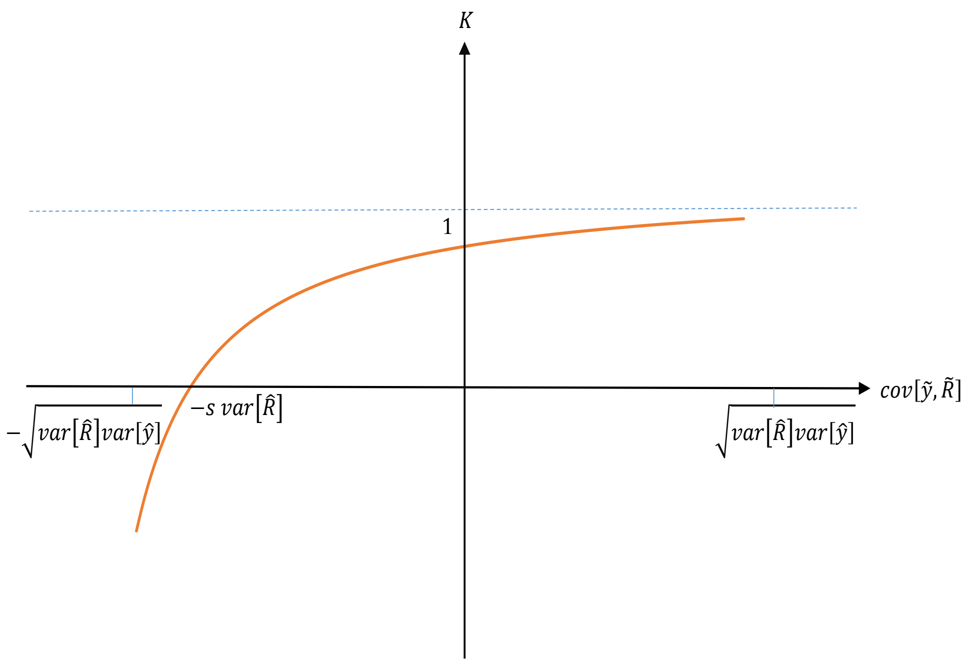

Figure 1: The function $K(cov[\tilde{y}, \tilde{R}])$ when $var[\tilde{y}] > (s^{\star\star})^2 var[\tilde{R}]$ holds

circumstance where the return on saving is more variable than labor income ($var[\tilde{y}] < (s^{\star\star})^2 var[\tilde{R}]$). In this case the following results hold:

Proposition 2. *If* $var[\tilde{y}] < (s^{\star\star})^2 var[\tilde{R}]$ *holds then* $K(cov[\tilde{y}, \tilde{R}])$ *is decreasing in* $cov[\tilde{y}, \tilde{R}]$ *with a lower bound equal to 1.*

Proof. Analogous to that of Proposition 1. □

Figure 2 provides a qualitative description of the above results. Lastly we consider

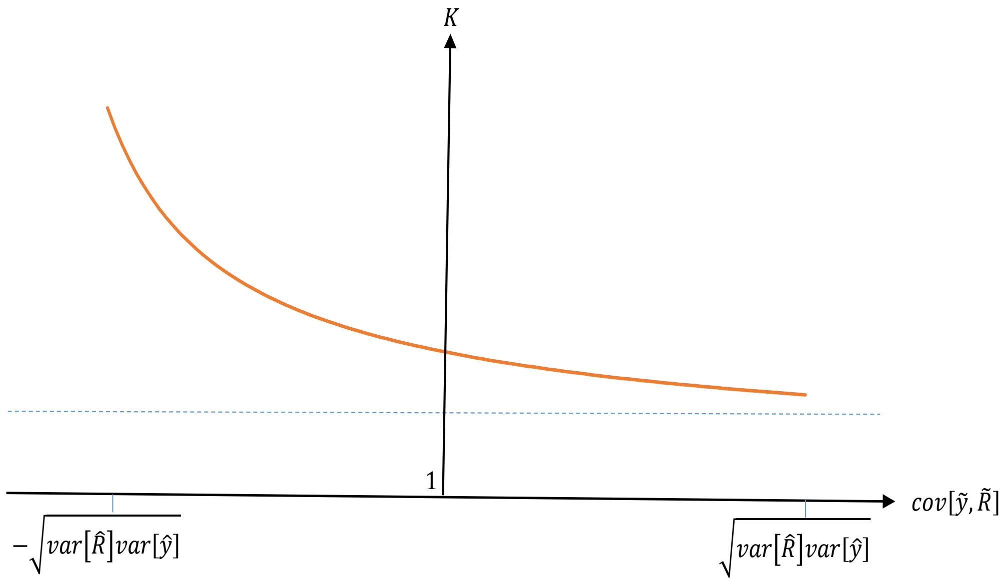

Figure 2: The function $K(cov[\tilde{y},\tilde{R}])$ when $var[\tilde{y}] < (s^{\star\star})^2 var[\tilde{R}]$ holds

the circumstance where the variance of labor income and the variance of total return on saving are equal ($var[\tilde{y}] = (s^{\star\star})^2 var[\tilde{R}]$). In this case we have:

Proposition 3. *If* $var[\tilde{y}] = (s^{\star\star})^2 var[\tilde{R}]$ *holds then the function* $K(cov[\tilde{y},\tilde{R}])$ *assumes the value 1 over the whole dominion.*

Proof. Follows straightforwardly from (9). □

Figure 3 provides a qualitative description of the above result. The comparison between the variance of labor income ($var[\tilde{y}]$) and the variance of the total return on saving ($s^{\star\star 2} var[\tilde{R}]$) is pivotal in determining the change in the threshold K caused by a different level of covariance. In particular, when the variance of labor income is larger than that of the total return on saving, the threshold increases with $cov[\tilde{y},\tilde{R}]$ toward the upper bound level 1. Hence the minimum level of partial relative prudence required for positive precautionary saving (represented by threshold K in Lemma 1) increases when the

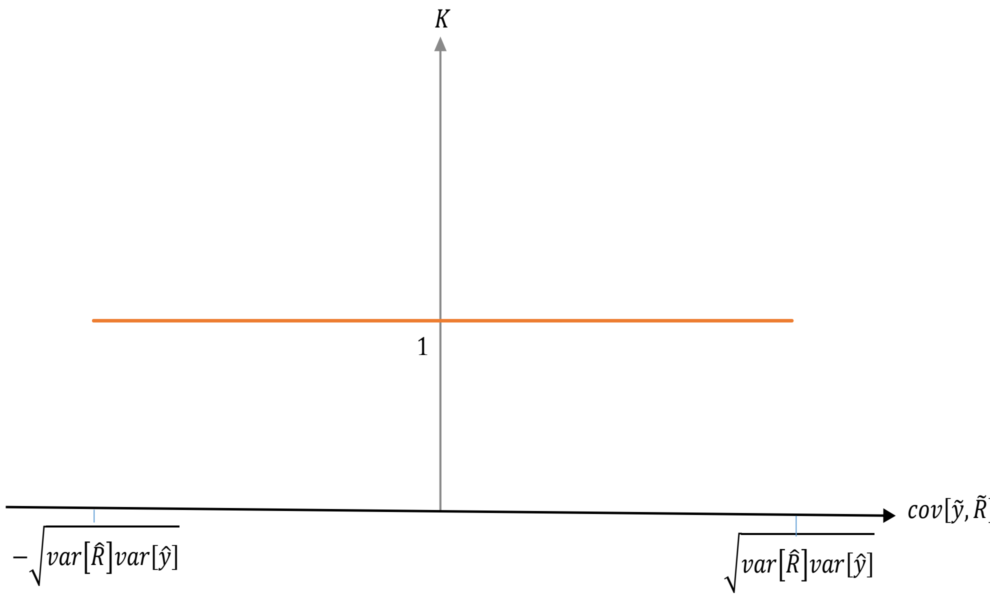

Figure 3: The function $K(cov[\tilde{y}, \tilde{R}])$ when $var[\tilde{y}] = (s^{\star\star})^2 var[\tilde{R}]$ holds

correlation between the two risks increases. But when the variance of the total return on saving exceeds that of labor income, the opposite happens. The threshold decreases with $cov[\tilde{y}, \tilde{R}]$ toward the lower bound level 1 and the minimum level of partial relative prudence required for positive precautionary saving decreases when the correlation between the two risks decreases. Lastly, the threshold is constant at value 1 when the two variances are exactly equal.

4 Discussion of results

In order to interpret the results derived in Section 3, we can simply re-write Equation (7). Note first that $var[\tilde{y}] + (s^{\star\star})^2 var[\tilde{R}] + 2s^{\star\star} cov[\tilde{y}, \tilde{R}] = var[\tilde{y} + \tilde{R}s^{\star\star}]$, which means that the denominator of the fraction in Equation (7) is the variance of second-period income. Simple algebra therefore yields:

$$K = \frac{dvar[\tilde{y} + \tilde{R}s^{\star\star}]}{ds^{\star\star}} \times \frac{s^{\star\star}}{var[\tilde{y} + \tilde{R}s^{\star\star}]} \tag{10}$$

which means that the threshold K is the elasticity of the variance of second-period income with respect to saving. This implies that $\frac{\partial K(cov[\tilde{y},\tilde{R}])}{\partial cov[\tilde{y},\tilde{R}]}$ in Equation (8) is the derivative of this elasticity with respect to the covariance between the two risks. When this derivative is positive, the elasticity is increasing in the covariance term; when it is negative the elasticity is decreasing in the covariance term. This finding supplies us with some useful insights which help us interpret the results in the previous Section.

In order to do this, note also that when the agent increases saving by one unit we have two different effects. First, second-period expected income increases. Second, the variance of second-period income increases. Since the agent is risk averse and prudent, she likes the first effect and dislikes the second. As recalled in Section 1, prudence is related to the wish to increase wealth in the period where there is risk, so this implies that the agent desires the increment in saving only if her level of prudence is sufficiently high, as stated in Equation (7).

Consider now what happens when the covariance between the two risks increases. Assume first that $var[\tilde{y}] > (s^{\star\star})^2 var[\tilde{R}]$. Under this assumption the increase in the covariance term implies that the elasticity of the variance of second-period income with respect to saving increases too. This means in turn that the second effect described above becomes stronger. As a consequence, only an agent with a higher level of prudence is willing to accept the increment of one unit of saving, implying that K must increase.

Assume now that $var[\tilde{y}] < (s^{\star\star})^2 var[\tilde{R}]$. In this case, an increase in the covariance reduces the elasticity of the variance of second-period income with respect to saving. Hence the second effect of the increment of saving becomes weaker, implying that this increment is also accepted by an agent with a lower level of prudence, and hence that K is lower.

Lastly, in case $var[\tilde{y}] = (s^{\star\star})^2 var[\tilde{R}]$, a change in the covariance does not affect the elasticity of the variance of second-period income with respect to saving. The second effect of the increment of saving is thus unchanged, which implies that K is constant when covariance changes.

The interpretation in terms of elasticity also allows us to provide an insight on the different values of the function K in the three cases studied above. Consider first the case where $var[\tilde{y}] > (s^{\star\star})^2 var[\tilde{R}]$. Here, the variance of second-period income mainly depends on labor income variance, and a variation in the level of saving has a small effect on it. Hence the function K takes low values and in particular smaller than 1. On the contrary, in the opposite case when $var[\tilde{y}] < (s^{\star\star})^2 var[\tilde{R}]$, the variance of second-period income mainly depends on the variance of the return on saving and is more responsive to a change in saving. Hence the function K takes high values and in particular higher than 1. The last circumstance where $var[\tilde{y}] = (s^{\star\star})^2 var[\tilde{R}]$ is the intermediate case. Here, the function K has the constant value 1 which lies between those of the other two cases.

5 Conclusions

In a recent paper, BMM derive the minimum level of partial relative prudence required for positive precautionary saving in the presence of small labor income risk and interest rate risk. This note studies how this level varies when the covariance between the two risks changes. Our results provide a complete characterization of this relationship.

In the present framework, the effect of change in risk correlation on the required level of PRP is always monotonic but, unlike what one might expect, the direction of this monotonicity is different in different cases. The comparison between the variance of labor income and the variance of total return on saving determines this direction, which is increasing when the former is larger than the latter, decreasing in the opposite case, and constant when the two variances are equal. We also show that this occurs, in the three cases described, because we have different effects of a change in covariance on the elasticity with respect to saving of total variance of second-period income.

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