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More tricks with the lorenz curve

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Abstract

The present note is an elementary essay on how one may manipulate the Lorenz curve with a view to obtaining a couple of simple variants of the Gini coefficient of inequality, one of which is anti transfer-sensitive and the other pro transfer-sensitive, and which together, in convex combination, yield up the transfer-neutral Gini coefficient. The emphasis throughout is on the practical concerns of exposition, intuitive plausibility, and the advancement of easily comprehended and readily usable measures of inequality.

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1. Introduction

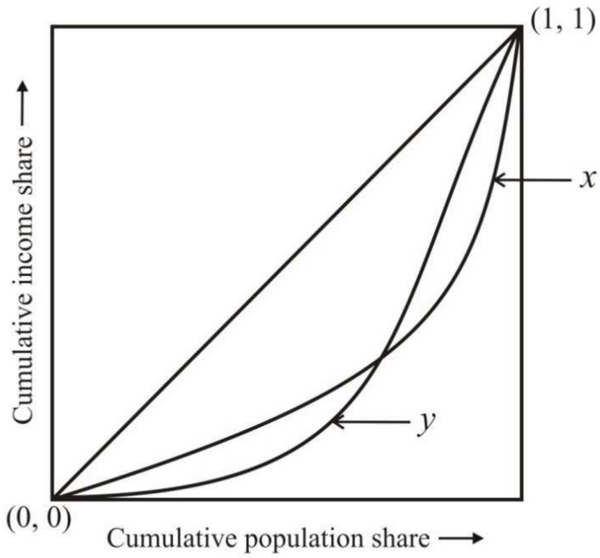
In an earlier paper (Subramanian, 2010), I had presented some simple ideas involving manipulation of the Lorenz curve with a view to generating a class of parametrized versions of the Gini coefficient of inequality such that, over well-defined ranges of the parameter values, the inequality measure would reflect a 'right-wing' orientation by satisfying an 'anti transfer-sensitivity' property, a 'centrist' orientation by satisfying a 'transfer-neutral' property, and a 'left-wing' orientation by satisfying a 'pro transfer-sensitivity' property. These properties, which are inspired by what Kolm (1976) referred to as the 'principle of diminishing transfers', can be described (with some modifications, discussed in Tibiletti and Subramanian, 2014) along the lines undertaken by Foster (1985). Imagine that a given rank-preserving progressive transfer of income takes place between two pairs of individuals such that the individuals in each pair are separated by both a fixed number of individuals and a fixed income. Then, an inequality measure will be said to be anti transfer-sensitive/ transfer-neutral / pro transfer-sensitive depending on whether the diminution in poverty following on the transfer between the poorer pair of individuals is lesser than/ the same as/ greater than the diminution in poverty following on the transfer between the richer pair of individuals.

There is a different but equivalent way of viewing 'right-wing', 'centrist', and 'left-wing' inequality measures, in terms of the skewness of the Lorenz curve. Suppose we have two intersecting Lorenz curves, one of which is skewed toward (1,1) of the unit square (so that it 'bulges at the bottom'), while the other is skewed toward (0,0) of the unit square (so that it 'bulges at the top'), with the areas enclosed by the two Lorenz curves with the diagonal of the unit square being the same. (The income shares of the poorer fractions of the population are lower in the first distribution than in the second.) Then, a 'right-wing', anti transfer-sensitive inequality measure will pronounce inequality to be greater for the Lorenz curve skewed toward (0,0); a 'centrist', transfer-neutral measure - such as the Gini coefficient – will pronounce inequality to be the same for the two Lorenz curves; and a 'left-wing', pro transfer-sensitive measure will pronounce inequality to be greater for the Lorenz curve skewed toward (1,1) (see Figure 1).

In the present paper, as in (Subramanian, 2010), I derive – by resort to some simple-minded twiddling of the Lorenz curve – a couple of elementary variants of the well-known transfer-neutral Gini coefficient, such that these reflect, respectively, right-wing and left-wing orientations. The motivation for the exercise in the present paper remains, regrettably, the same as for the exercise in my earlier paper (Subramanian, 2010; p. 1595), namely, that '… it is difficult to resist the temptation of getting up to tricks of one kind or another in the presence of the seemingly infinite possibilities offered up by the [Lorenz] curve.' The trick, in the present instance (as will become clear in the following section), consists in – so to speak - arranging a marriage between Pythagoras and Lorenz.

Figure 1: Intersecting Lorenz Curves with Opposing Skewness

And the Same Gini Value



2. The Lorenz Curve, Deviation Functions, and Inequality Measures

2.1 Continuous Distribution

x is a random variable designating income, and ranges over the interval $[0,\infty)$. The Lorenz curve is the graph of the function $L(p(x))$ - the income share of the poorest pth fraction of income-earning units (which is the fraction of the population with incomes not exceeding x). Now consider the deviation function – seen as a straightforward measure of the distance between an actual and an equal income share for the pth fraction of the population – that is given by

$$d_C(p) \equiv p - L(p). \quad (1)$$

(See Figure 2; the subscript C on the deviation function will be presently explained.) The familiar Gini coefficient of inequality G is obtained by aggregating the deviation functions $d_C(p)$ in Expression (1) over all values of p in the interval $[0,1]$, and expressing this sum as a ratio of the maximum value the sum can assume (which is one-half):

$$G = 2\int_0^1 d_C(p)dp = 2\int_0^1 (p - L(p))dp. \quad (2)$$

Figures (2) and (3) below enable us to consider a couple of variants of the deviation function $d_C(p)$.

Figure 2 : Diagrammatic Representation of the Functions h_1 (p) and t_1 (p)

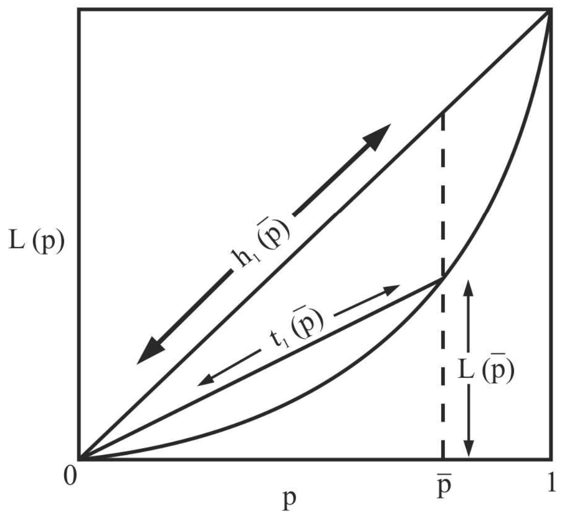
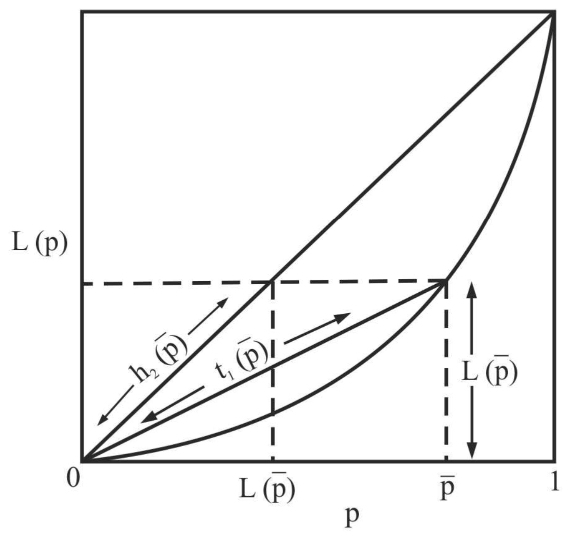


Figure 3 : Diagrammatic Representation of the Functions h_2 (p) and t_1 (p)



In Figure 2, given any $\overline{p} \in [0,1]$, $h_1(\overline{p})$ measures the distance, from the origin, along the diagonal, to the point $(\overline{p}, \overline{p})$ on the diagonal; and $t_1(\overline{p})$ measures the distance of the line-segment connecting the origin to the point $(\bar{p}, L(\bar{p}))$ on the Lorenz curve. Notice that by virtue of Pythagoras' theorem on right-angled triangles,

$$h_1(\bar{p}) = \sqrt{2}\bar{p}, \tag{3a}$$

and

$$t_1(\bar{p}) = (\bar{p}^2 + L^2(\bar{p}))^{\frac{1}{2}}. \tag{3b}$$

Consider the aggregation of the deviation functions $d_R(p) \equiv h_1(p) - t_1(p)$ over all values of p in the interval [0,1], and call the resulting inequality measure I_R :

$$I_R = \int_0^1 d_R(p)dp = \int_0^1 (\sqrt{2}p - (p^2 + L^2(p))^{\frac{1}{2}})dp. \tag{4a}$$

The maximum value of I_R - call it $I_R^{\max}$ - is attained for the distribution in which $L(p) = 0 \forall p \in [0,1)$ and $L(1) = 1$. Then,

$$I_R^{\max} = \int_0^1 (\sqrt{2}p - p)dp = (\sqrt{2} - 1)/2. \tag{4b}$$

From Equations (4a) and (4b) we obtain an inequality measure – call it G_R - which expresses I_R as a proportion of its maximum value:

$$G_R = [2/(\sqrt{2} - 1)]\int_0^1 [\sqrt{2}p - (p^2 + L^2(p))^{\frac{1}{2}}]dp. \tag{5}$$

Next, consider Figure 3. In Figure 3, given any $\bar{p} \in [0,1]$, $h_2(\bar{p})$ measures the distance, from the origin, along the diagonal, to the point $(L(\bar{p}), L(\bar{p}))$ on the diagonal; and $t_1(\bar{p})$ (as before) measures the distance of the line-segment connecting the origin to the point $(\bar{p}, L(\bar{p}))$ on the Lorenz curve. Again, by Pythagoras,

$$h_2(\bar{p}) = \sqrt{2}L(\bar{p}), \tag{6a}$$

and (to repeat Equation 3(b))

$$t_1(\bar{p}) = (\bar{p}^2 + L^2(\bar{p}))^{\frac{1}{2}}. \tag{6b}$$

The aggregation of the deviation functions $d_L(p) \equiv t_1(p) - h_2(p)$ over all values of p in the interval [0,1] yields an inequality measure – call it I_L - which is given by:

$$I_L = \int_0^1 d_L(p)dp = \int_0^1 [(p^2 + L^2(p))^{\frac{1}{2}} - \sqrt{2}L(p)]dp. \tag{7a}$$

The maximum value of I_L – call it $I_L^{\max}$ – would be attained in a situation in which $L(p) = 0 \forall p \in [0,1)$ and $L(1) = 1$. Then,

$$I_L^{\max} = \int_0^1 pdp = 1/2. \tag{7b}$$

From (7a) and (7b) we obtain an inequality measure – call it G_L– which expresses I_L as a proportion of its maximum value:

$$G_L = 2\int_0^1 [(p^2 + L^2(p))^{\frac{1}{2}} - \sqrt{2}L(p)]dp. \tag{8}$$

Finally, note that the plot of $d_C(p)$ against p is (after transformation of the coordinates) just another representation of the Lorenz curve; and we shall call the plots of $d_R(p)$ against p and of $d_L(p)$ against p the *R*-Lorenz curve and the *L*-Lorenz curve respectively. Typical Lorenz, *R*-Lorenz and *L*-Lorenz curves are depicted in Figures 4(a), 4(b) and 4(c). All of these curves are quadratic functions which originate at the point (0,0) and terminate at the point (1,0) on the unit line, first rising, attainining a maximum, and then declining. Lorenz-dominance of one distribution over another obtains whenever the Lorenz curve for the first distribution lies somewhere below and nowhere above the Lorenz curve of the second distribution. *R*-Lorenz-dominance and *L*-Lorenz-dominance can be analogously defined.

2.2 Discrete Distribution

A discrete income distribution is a non-decreasingly ordered, non-zero *n*-vector of incomes $\mathbf{x} = (x_1,...,x_i,...,x_n)$ where x_i is the income of the *i*th poorest person in a community of *n* individuals. The mean of the distribution is designated by μ, and the mean income of the *i* poorest individuals by $\mu_i (\equiv (1/i)\sum_{j=1}^{i} x_j)$. The graph of the Lorenz curve plots the income share of the (i/n)th poorest fraction of the population, and is given by: $L_i \equiv L(\mathbf{x}; i/n) = i\mu_i / n\mu, \forall i = 1,...,n.$ Writing P_i for $i/n,$ the Lorenz curve can be seen to be the plot of L_i against P_i. The discrete analogues of the deviation functions considered in Section 2.2 would be given, respectively, by $d_i^C \equiv P_i - L_i$;

Figure 4 (a) : (Transformed)
Lorenz Curves : Distribution 1
Lorenz - dominates Distribution 2

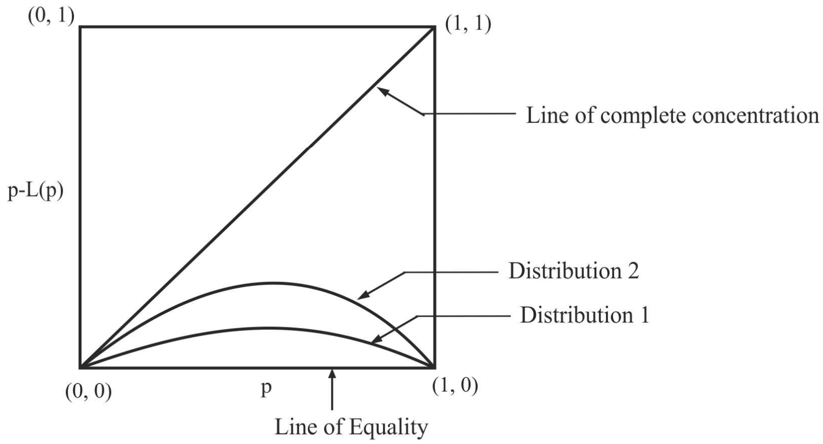


Figure 4 (b) :
R- Lorenz Curves : Distribution 1
R - Lorenz - dominates Distribution 2

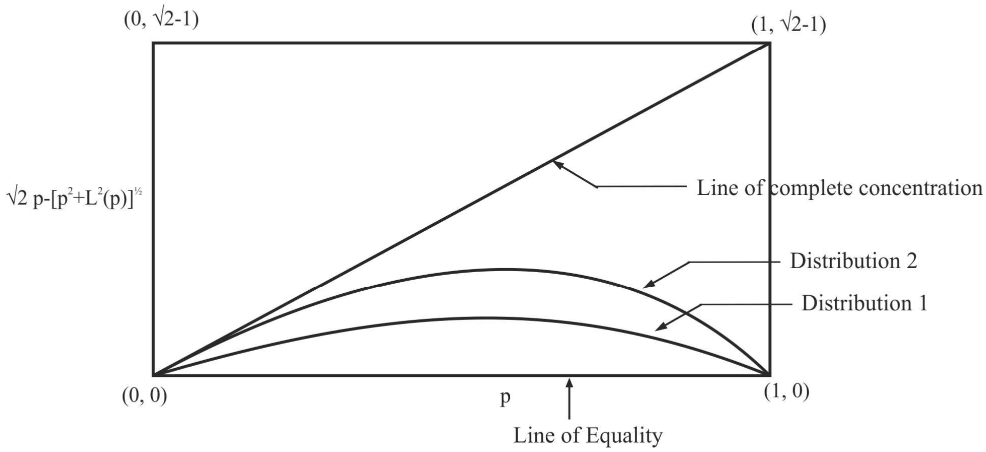
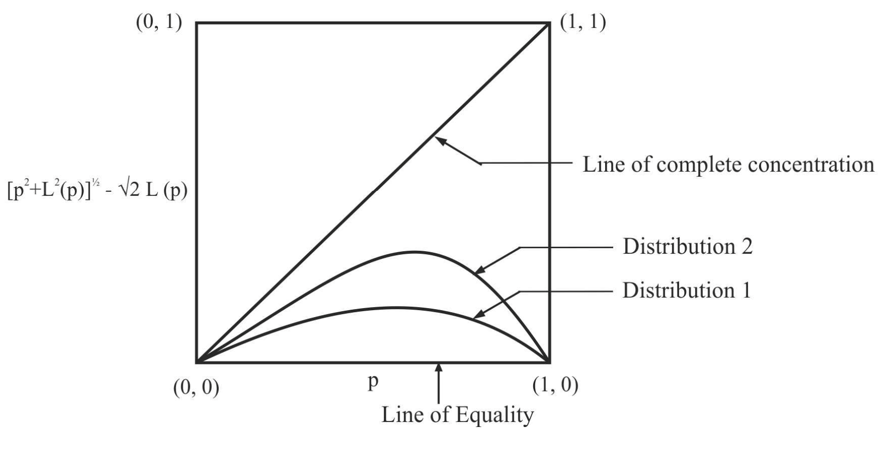


Figure 4 (c) : L - Lorenz Curves:
Distribution 1 L - Lorenz dominates
Distribution 2



$d_i^R \equiv \sqrt{2}P_i - (P_i^2 - L_i^2)^{\frac{1}{2}}$; and $d_i^L \equiv (P_i^2 - L_i^2)^{\frac{1}{2}} - \sqrt{2}L_i$. The plot of d_i^C against P_i is (of course after transformation of the coordinates) simply the (step-function) Lorenz curve, while the plot of d_i^R against P_i is the (step-function) *R*-Lorenz curve, and the plot of d_i^L against P_i is the (step-function) *L*-Lorenz curve (these curves are not here drawn, but are left to the reader's imagination) . The (normalized) areas beneath these respective curves would correspond to the inequality measures which in section 2.2 we have designated by G, G_R and G_L. In the case of the discrete distribution, it can be verified that expressions for these inequality measures can be written in the following comparable forms:

$$G = \left[\frac{2}{n(n-1)\mu}\right]\left[\frac{n(n+1)\mu}{2} - \sum_{i=1}^{n} i\mu_i\right]; \quad (9)$$

$$G_R = \left[\frac{2}{n(n-1)\mu}\right]\left[\frac{n(n+1)\mu}{\sqrt{2}(\sqrt{2}-1)} - \sum_{i=1}^{n} i\frac{(\mu^2+\mu_i^2)^{\frac{1}{2}}}{\sqrt{2}-1}\right]; \text{and} \quad (10)$$

$$G_L = \left[\frac{2}{n(n-1)\mu}\right]\left[\sum_{i=1}^{n} i\{(\mu^2+\mu_i^2)^{\frac{1}{2}} - \sqrt{2}\mu_i\}\right]. \quad (11)$$

3. Centrist, Right-Wing and Left-Wing Inequality Measures.

It is well-known that the Gini coefficient of inequality G is a 'centrist' measure, in the sense that it is transfer-neutral: while it satisfies the Pigou-Dalton transfer axiom, it is not differentially sensitive to transfers at either the lower or the upper end of an income distribution. It will now be claimed that G_R is a 'right-wing' or anti transfer-sensitive measure, while G_L is a 'left-wing' or pro transfer-sensitive measure. To see this more clearly, it can be noted that given the expressions for G, G_R and G_L available in either of the sets of equations {(2), (5), (8)} or {(9), (10), (11)}, it is a simple matter to verify that

$$G = \left(\frac{1}{\sqrt{2}}\right)G_L + \left(1 - \frac{1}{\sqrt{2}}\right)G_R. \quad (12)$$

The Gini coefficient, that is, is a linear (convex) combination of the measures G_R and G_L. This suggests - since we know that G is transfer-neutral - that either both G_R and G_L must be transfer-neutral, or that one of these must be anti transfer-sensitive and the other pro transfer-sensitive. We can satisfy ourselves as to the correct position of affairs by resort to simple arithmetical verification, and it turns out, in the light of such verification, that G_L is pro transfer-sensitive and G_R is anti transfer-sensitive. The subscripts *L*, *C* and *R* employed on the inequality measures and deviation functions now stand explained – '*L*' stands for 'left-wing', '*C*' for 'centrist', and '*R*' for 'right-wing'.

A simple numerical example should clarify the issue. Consider the 5-person distributions **a** = (10,20,30,40,50), **b** = (15,15,30,40,50) and **c** = (10,20,30,45,45). It can be seen that **b** has been derived from **a** by a progressive transfer of 5 income units between two persons one rank and 10 income units apart at the lower end of the distribution; and **c** has been derived from **a** by an identical progressive transfer of 5 income units between two persons the same number of ranks (one) and income units (10) apart, but at the upper end of the distribution. Given an inequality measure *I* which satisfies the Pigou-Dalto transfer axiom, *I* will be transfer-neutral if $I(\mathbf{a}) > I(\mathbf{b}) = I(\mathbf{c})$; *I* will be anti transfer-sensitive if $I(\mathbf{a}) > I(\mathbf{b}) > I(\mathbf{c})$; and *I* will be pro transfer-sensitive if $I(\mathbf{a}) > I(\mathbf{c}) > I(\mathbf{b})$. For the numerical example under review, and given equations (9)-(11), it can be verified that $G(\mathbf{a})$ [= 0.3333]> $G(\mathbf{b}) = G(\mathbf{c})$ [= 0.3162]: the Gini coefficient is transfer-neutral; $G_R(\mathbf{a})$ [= 0.4924] > $G_R(\mathbf{b})$ [= 0.4770] > $G_R(\mathbf{c})$ [= 0.4664] : G_R is anti transfer-sensitive; and $G_L(\mathbf{a})$ [= 0.0.2675] > $G_L(\mathbf{c})$ [= 0.2547] > $G_L(\mathbf{b})$ [= 0.2506]: G_L is pro transfer-sensitive.

The pro transfer-sensitivity of G_L is reminiscent of a similarly 'left-wing' inequality measure derived from the Lorenz curve, and based on the *length* (rather than area, as in the case of the Gini coefficient) of the Lorenz curve: this measure has been advanced by Amato (1968) and Kakwani (1980), the latter of whom also noted its transfer-sensitivity property. There has been a recent revival of interest in this measure, seen from the perspective of an analogue in the physical field of *optics* (Majumder, 2014), and the measure – call it *AK* (after Amato and Kakwani) can be written as:

$$AK = \frac{(1/n\mu)\sum_{i=1}^{n}(\mu^2 - x_i^2)^{\frac{1}{2}} - \sqrt{2}}{2-\sqrt{2}}. \tag{13}$$

I end with a suggestion on a criterion to ensure unanimity of inequality rankings for a class of inequality measures. First, an inequality measure is *symmetric* if it is invariant with respect to any interpersonal permutation of incomes across individuals; it satisfies the *Pigou-Dalton transfer axiom* if - other things remaining equal - its value declines with a progressive rank-preserving transfer of income; it is *scale-invariant* if it is invariant with respect to equiproportionate increases in all incomes; and it is *replication-invariant* if it is invariant with respect to any *k*-fold population replication (*k* being an integer). Let $\mathcal{L}$ be the set of inequality indices which satisfy these four properties. For any pair of income distributions **x** and **y**, and any collection of inequality measures *I*, we shall write

$$\mathbf{x} \succ_I \mathbf{y}$$

to signify that

$I(\mathbf{x}) \le I(\mathbf{y}) \forall I \in \mathcal{I}.$

Foster (1985) has demonstrated the truth of the following result for the class $\mathcal{L}$ of symmetric, transfer-preferring, and scale- and replication-invariant inequality measures (the class of Lorenz-consistent measures, as they are called): for any pair of income distributions **x** and **y**,

$\mathbf{x} \succ_{\mathcal{L}} \mathbf{y}$

if and only if **x** Lorenz-dominates **y**. A subset of the class $\mathcal{L}$ of inequality measures is the class $\mathcal{L}_{\mathbf{R}}$ of symmetric, and scale- and replication–invariant measures which are anti transfer-sensitive; and another subset is the class $\mathcal{L}_{\mathbf{L}}$ of symmetric, and scale- and replication–invariant measures which are pro transfer-sensitive. Presumably, more pairs of distributions are amenable to unambiguous inequality ranking if we were to relax the requirement of Lorenz-dominance to the weaker requirement of *R*-Lorenz-dominance or *L*-Lorenz-dominance. This leads to the suggestion alluded to earlier on a sufficient condition for unambiguous inequality rankings: for any pair of distributions **x** and **y** whose Lorenz curves may intersect not more than once, $\mathbf{x} \succ_{\mathcal{L}_{\mathbf{R}}} \mathbf{y}$

if **x** *R*-Lorenz-dominates **y**, and

$\mathbf{x} \succ_{\mathcal{L}_{\mathbf{L}}} \mathbf{y}$

if **x** *L*-Lorenz-dominates **y**. The extent to which the possibility of unanimous inequality rankings is actually enhanced by the criteria of *R*- and *L*-Lorenz-dominance is an empirical question.

4. Concluding Observations

This note has been concerned with the rather pragmatic issue of advancing alternative variants of the Gini coefficient of inequality which are derivable by simple manipulations of the Lorenz curve apparatus. These manipulations yield an anti and a pro transfer-sensitive version of the transfer-neutral Gini coefficient – which latter, as it turns out, can be written as an uncomplicated convex combination of its two non-neutral variants. For a properly deep and general treatment of transfer sensitivity, or of the inequality rankings of distributions represented by intersecting Lorenz curves, the reader is referred to work such as that by Shorrocks and Foster (1987), or Zoli (2002), or Aaberge (2009). The present paper has been concerned with a more mundanely practical approach to exposition and descriptive measurement.

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