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Long-run effects of capital market integration using Solow's model

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Abstract

The purpose of this paper is to synthesize the three results in the existing literature (and to add a fourth result) in a single unified framework and thus to identify the conditions under which the capital-exporting and capital-importing countries gain from international financial integration. We show that the capital-exporting country wins if it saves a constant fraction of its profits, and that capital-importing country wins if it saves a constant fraction of its wages. In Solow's model for the integration of the capital market to be profitable, it is necessary for savings to be proportional to income, which increases through the integration of the capital market: profit of the lender, and wages of the borrower.

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1. Introduction

In a simple two-country model and in a world without risk, long-run effects of capital market integration, on capital-importing and capital-exporting countries have been studied in several specifics cases. Ruffin (1979) shows that in Solow's model, where each country saves a constant fraction of total income (wages and profits), each country would gain over the long-run in wealth per capita. This theorem, in no way surprising, is analogous to the usual result of gains from trade. More surprising, Quibria (1985) shows that the capital-importing country, if it saves a constant fraction of its profits (asset income), loses in terms of wealth per capita. Darreau and Pigalle (2014) show that the capital-exporting country, if it saves a constant fraction of its wages, loses in terms of stationary utility level. The support of saving (wages or profit) seems the essential ingredient to determine whether the integration of the capital market is profitable or not, in the long term. We construct a model that allows us to generalize this result. We show that the capital-exporting country wins if it saves a constant fraction of its profits, and that capital-importing country wins if it saves a constant fraction of its wages. Generally for the integration of the capital market to be profitable, it is necessary for savings to be proportional to income which increases through the integration of the capital market: wages of the borrower, and profit of the lender.

Section 2 present the model, section 3 determines the steady state of each country, section 4 explores different cases depending on whether the savings is on total income or wages or profits, and section 5 generalizes the result.

2. The model

Consider a world composed of two economies, "Hat" and "Tilde". Denote the relative size of hat country in the world by $\hat{\eta} = \hat{N}/(\tilde{N} + \hat{N})$. To model both the assumptions of a "two-country model" and the assumption of a "small open economy" we assume that the populations are different sizes $\tilde{\eta} \neq \hat{\eta} \in (0, 1)$. Wealth per capita of each country's agents $a = A/N$ is composed of domestic capital and net foreign loans. $k = K/N$ is domestic capital per capita and $e = E/N$ is the Net International Investment Position per capita.

$$\hat{a} = \hat{k} + \hat{e} \qquad \text{and} \qquad \tilde{a} = \tilde{k} + \tilde{e} \tag{1}$$

To obtain a steady state, it is necessary that the rate of population growth n be the same in each country. Assume for simplicity that the two countries have the same level of technology A, the same depreciation rate of capital δ, and the same production function. The production function satisfies constant returns to scale and the usual neoclassical conditions, like in Solow's model :

$$\hat{q} = f\left[\hat{a} - \hat{e}\right] \qquad \text{and} \qquad \tilde{q} = f\left[\tilde{a} - \tilde{e}\right] \tag{2}$$

$$f' > 0; \quad f'' < 0; \quad f'(0) = +\infty; \quad f'(\infty) = 0 \tag{3}$$

Capital market integration implies the equalization of interest rates $R = r + \delta$, and since we have the same production function, the egalization of capital and wage rates. By construction, we have the equality of NIIP ($\hat{E} = -\tilde{E}$) since net lending by one is net borrowing by the other.

$$f'\left[\hat{a}-\hat{e}\right]=R=f'\left[\tilde{a}-\tilde{e}\right] \tag{4}$$

$$\hat{k}=k=\tilde{k} \tag{5}$$

$$f\left[\hat{a}-\hat{e}\right]-R\cdot(\hat{a}-\hat{e})=w=f\left[\tilde{a}-\tilde{e}\right]-R\cdot(\tilde{a}-\tilde{e}) \tag{6}$$

$$\hat{\eta}\hat{e}=-\tilde{\eta}\tilde{e} \tag{7}$$

It is important to distinguish between financial capital a (wealth) and productive capital k and to distinguish GDP q and GNP $y=q+Re=w+Rk+Re=w+Ra$.

Like Ruffin we assume that the dynamic part of the story comes from Solow. Each country saves a constant fraction of its income. But in contrast to Ruffin, we assume that labor income savings and capital income savings can be done at different rates, respectively s^w and s^a, some of which may be zero. Wealth accumulation equations are given by :

$$\begin{cases} D\hat{a}=\hat{s}^w w+\hat{s}^a R\hat{a}-(n+\delta)\,\hat{a} \\ D\tilde{a}=\tilde{s}^w w+\tilde{s}^a R\tilde{a}-(n+\delta)\,\tilde{a} \end{cases} \tag{8}$$

Equations (8) determine the course of the economic system through time. We now determine the wealth of nations at steady state.

3. The steady state

We examine the existence, uniqueness and stability of steady state in a phase diagram. We simply need to represent the $D\hat{a}=0$ and $D\tilde{a}=0$ locus in $(\tilde{a},\hat{a})$ wealth space. For this, we need a point through which passes $D\hat{a}=0$ and $D\tilde{a}=0$ and we need slope of $D\hat{a}=0$ and $D\tilde{a}=0$.

A natural point for the $D\hat{a}=0$ and $D\tilde{a}=0$ locus in $(\tilde{a},\hat{a})$ wealth space is autarky. In autarky, we have $\hat{e}=0$. The converse is not true and the $\hat{e}=0$ curve in $(\tilde{a},\hat{a})$ space, also represents all $(\tilde{a},\hat{a})$ consistent with zero capital movement under free capital mobility. The $\hat{e}=0$ curve comes from equation (4) with $\hat{e}=0$ and is upward sloping, because an increase in $\hat{a}$ must entail an increase in $\tilde{a}$ to eliminate the incentive for capital to migrate. This is a straight line at a 45 degree angle because of our hypothesis of same production function. Below the $\hat{e}=0$ curve, $\hat{e}$ will be positive, the Hat country will be the creditor $\hat{e}>0$ and Tilde country will be the debtor $\tilde{e}<0$.

Back to autarky, we assume that $\hat{a}_{autarky}>\tilde{a}_{autarky}$ and we obtain the two points when $\hat{e}=0$ and in autarky: point B for the Hat country and point C for the Tilde country. In autarky $D\hat{a}$ is independent of $\tilde{a}$ like $D\tilde{a}$ is independent of $\hat{a}$. Thus $D\hat{a}=0$ is vertical and $D\tilde{a}=0$ is horizontal. Thus point A is the steady state in $(\tilde{a},\hat{a})$ wealth space under autarky.

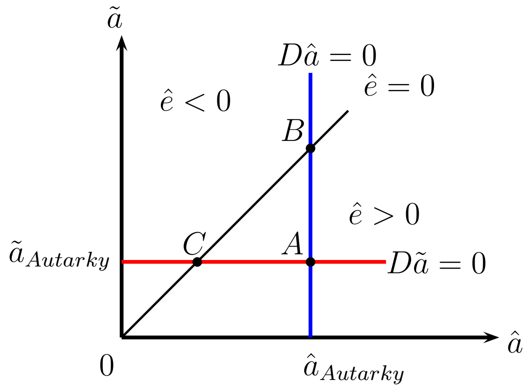

Figure 1: Autarky case

Next, we determine the slope of $D\hat{a} = 0$ and $D\tilde{a} = 0$ under free capital mobility. Implicit differentiation of (8) will yield to :

$$\left.\frac{d\tilde{a}}{d\hat{a}}\right|_{D\hat{a}=0} = -\frac{\frac{\partial D\hat{a}}{\partial \hat{a}}}{\frac{\partial D\hat{a}}{\partial \tilde{a}}} \qquad \text{and} \qquad \left.\frac{d\tilde{a}}{d\hat{a}}\right|_{D\tilde{a}=0} = -\frac{\frac{\partial D\tilde{a}}{\partial \hat{a}}}{\frac{\partial D\tilde{a}}{\partial \tilde{a}}}$$

It is shown in Appendix 1 how the slopes of $D\hat{a} = 0$ and $D\tilde{a} = 0$ are obtained:

$$\left.\frac{d\tilde{a}}{d\hat{a}}\right|_{D\hat{a}=0} = -\frac{\frac{\partial R}{\partial \hat{a}}\left[-\hat{s}^w\left(\hat{a}-\hat{e}\right)+\hat{s}^a\hat{a}\right]+\hat{s}^a R-(n+\delta)}{\frac{\tilde{\eta}}{\hat{\eta}}\frac{\partial R}{\partial \hat{a}}\left[-\hat{s}^w\left(\tilde{a}-\tilde{e}\right)+\hat{s}^a\hat{a}\right]} \tag{9}$$

$$\left.\frac{d\tilde{a}}{d\hat{a}}\right|_{D\tilde{a}=0} = -\frac{\frac{\partial R}{\partial \hat{a}}\left[-\tilde{s}^w\left(\hat{a}-\hat{e}\right)+\tilde{s}^a\tilde{a}\right]}{\frac{\tilde{\eta}}{\hat{\eta}}\frac{\partial R}{\partial \hat{a}}\left[\tilde{s}^a\tilde{a}-\tilde{s}^w\left(\tilde{a}-\tilde{e}\right)\right]+\tilde{s}^a R-(n+\delta)} \tag{10}$$

These equations are the general form of several special cases according the values of s^w and s^a. We now represent phase diagrams in different cases.

4. Different cases

4.1. Ruffin case : $\hat{s}^a = \hat{s}^w = \hat{s}$ and $\tilde{s}^a = \tilde{s}^w = \tilde{s}$

In his seminal paper, Ruffin (1979) supposes as in Solow's model that $\hat{s}^a = \hat{s}^w = \hat{s}$ and $\tilde{s}^a = \tilde{s}^w = \tilde{s}$. Substituting in (9) and (10) we obtain the equations found by Ruffin (23 and 24 pp. 835) :

$$\left.\frac{d\tilde{a}}{d\hat{a}}\right|_{D\hat{a}=0} = -\frac{\hat{s}\left(R+\frac{\partial R}{\partial \hat{a}}\hat{e}\right)-(n+\delta)}{\hat{s}\frac{\tilde{\eta}}{\hat{\eta}}\frac{\partial R}{\partial \hat{a}}\hat{e}} \qquad \text{and} \qquad \left.\frac{d\tilde{a}}{d\hat{a}}\right|_{D\tilde{a}=0} = \frac{\tilde{s}\frac{\partial R}{\partial \hat{a}}\left(\frac{\hat{\eta}}{\tilde{\eta}}\hat{e}\right)}{\tilde{s}\left(R-\frac{\partial R}{\partial \hat{a}}\hat{e}\right)-(n+\delta)}$$

The first equation indicates that $D\hat{a} = 0$ has a negative slope for $\hat{e} > 0$. Indeed for $\hat{e} > 0$ the denominator is negative since $\partial R/\partial \hat{a} < 0$. The numerator is negative as shown in Appendix 2.

The second equation indicates that $D\tilde{a} = 0$ has a positive slope for $\hat{e} > 0$. Indeed for $\hat{e} > 0$ the numerator is negative since $\partial R/\partial \hat{a} < 0$. The denominator is negative as shown in Appendix 2.

We represent in Fig. 2 the curves $D\hat{a} = 0$ and $D\tilde{a} = 0$. Curve $D\hat{a} = 0$ passes through point B (the steady state in autarky for the Hat country), has a negative slope for $\hat{e} > 0$ and a positive slope for $\hat{e} < 0$. Curve $D\tilde{a} = 0$ passes through point C (the steady state in autarky for the Tilde country), has a positive slope for $\hat{e} > 0$ and a negative slope for $\hat{e} < 0$. In the region $\hat{e} > 0$, the two curves have opposites slopes. This guarantees the uniqueness of the steady state S.

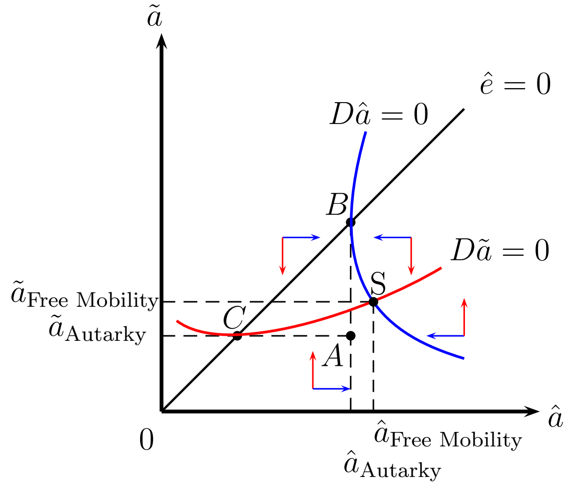

Figure 2: Ruffin case

To understand the dynamic in Fig 2, start from point A (the steady state in the two countries under autarky) and let free capital mobility. Point A is in the area where $\hat{e} > 0$ this implies that capital will migrate to the Tilde country. Point A is below $D\tilde{a} = 0$ which implies that $D\tilde{a} > 0$. Point A is on the left of $D\hat{a} = 0$ which implies that $D\hat{a} > 0$. The global economy converges to point S. Point S is the stable steady state with perfect capital mobility.

The important result of this analysis is the following : Capital market integration led to increase the per capita wealth of both countries under the Ruffin's assumptions.

Theorem 1 *(Ruffin 1979) : When $\hat{s}^a = \hat{s}^w = \hat{s}$ and $\tilde{s}^a = \tilde{s}^w = \tilde{s}$, the steady-state solution for wealth per capita with free capital market, is better than the autarky solution.*

Both countries would gain over the long-run with perfect free capital mobility. But this unsurprising theorem is disputed by Quibria (1986).

Note that under the assumption of a small open economy ($\tilde{\eta} \to 0, \hat{\eta} \to 1$), the curve $D\hat{a} = 0$ tends towards the vertical, the curve $D\tilde{a} = 0$ is steeper[1], and only the small country (Tilde) wins from capital market integration. Under the assumption ($\tilde{\eta} \to 1, \hat{\eta} \to 0$), the curve $D\tilde{a} = 0$ tends towards horizontal, only the small country (Hat) wins from capital market integration.

4.2. Quibria case : $\hat{s}^a = \hat{s}^w = \hat{s}$ and $\tilde{s}^w = 0$

Quibria (1986) suppose that the poor country (the South in his paper, here Tilde) saves only from profits $\hat{s}^a = \hat{s}^w = \hat{s}$ and $\tilde{s}^w = 0$. Substituting in (9) and (10) we obtain the equations (11 and 12 pp. 367) found by Quibria .

[1]To calculate the slope of $D\tilde{a} = 0$ use (11) and (7) in Appendix 1 .

$$\left.\frac{d\tilde{a}}{d\hat{a}}\right|_{D\hat{a}=0} = -\frac{\hat{s}\left(R+\frac{\partial R}{\partial \hat{a}}\hat{e}\right)-(n+\delta)}{\hat{s}\frac{\tilde{\eta}}{\hat{\eta}}\frac{\partial R}{\partial \hat{a}}\hat{e}} \quad \text{and} \quad \left.\frac{d\tilde{a}}{d\hat{a}}\right|_{D\tilde{a}=0} = -\frac{\frac{\partial R}{\partial \hat{a}}\tilde{s}^a\tilde{a}}{\frac{\tilde{\eta}}{\hat{\eta}}\frac{\partial R}{\partial \hat{a}}\tilde{s}^a\tilde{a}+\tilde{s}^aR-(n+\delta)} = -\frac{\hat{\eta}}{\tilde{\eta}}$$

Since in the steady state $\tilde{s}^aR = (n+\delta)$, the second equation shows that $D\tilde{a} = 0$ has a negative slope equal to $-\hat{\eta}/\tilde{\eta}$ (-1 in Quibria which suppose countries of same size).

As in the case of Ruffin, the first equation shows that $D\hat{a} = 0$ has a negative slope for $\hat{e} > 0$. This slope is greater in absolute value than the slope of $D\tilde{a} = 0$ since $\hat{s}(w/\hat{a}) > 0)$ and $\tilde{\eta}\hat{s}\hat{e}R' < 0$.

$$\left.\frac{d\tilde{a}}{d\hat{a}}\right|_{D\hat{a}=0} = -\frac{\hat{\eta}}{\tilde{\eta}}\frac{\hat{s}\left(R+\frac{\partial R}{\partial \hat{a}}\hat{e}\right)-(n+\delta)}{\hat{s}\frac{\partial R}{\partial \hat{a}}\hat{e}} = -\frac{\hat{\eta}}{\tilde{\eta}}+\frac{\hat{\eta}}{\tilde{\eta}}\frac{(n+\delta)-\hat{s}R}{\hat{s}\hat{\eta}R'\hat{e}} = -\frac{\hat{\eta}}{\tilde{\eta}}+\frac{\hat{s}\,(w/\hat{a})}{\tilde{\eta}\hat{s}\hat{e}R'} < -\frac{\hat{\eta}}{\tilde{\eta}}$$

The same analysis as before allows us to represent the Quibria case and unique and stable steady state S.

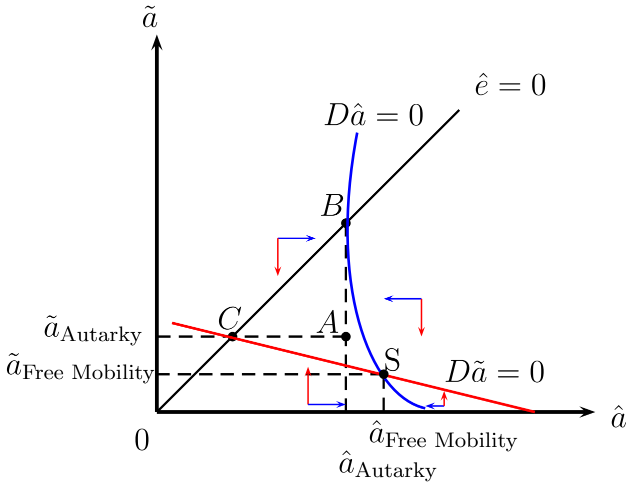

Figure 3: Quibria case

The surprising result of Quibria is that the borrower country (Tilde) lost from opening its capital market : $\tilde{a}_{\text{free mobility}} < \tilde{a}_{\text{autarky}}$.

Theorem 2 *(Quibria 1986) : When $\hat{s}^a = \hat{s}^w = \hat{s}$ and $\tilde{s}^w = 0$, the steady-state solution for wealth per capita with free capital market, is better than the autarky solution for the lender country (Hat), but is worse than the autarky solution for the borrowing country (Tilde).*

4.3. Darreau Pigalle case : $\hat{s}^a = \tilde{s}^a = 0$

Darreau and Pigalle (2014) show that in the overlapping generations model (OLG), both countries increase their stationary utility levels only if the autarky solutions are on opposites sides of the golden rule. When the two countries are in dynamic efficiency, the lender country suffers a decline in its stationary utility level under free capital market compared to autarky. A feature of the OLG model is that savings are a fraction of wages, therefore $\hat{s}^a = \tilde{s}^a = 0$. Substituting in (9) and (10) we obtain :

$$\left.\frac{d\tilde{a}}{d\hat{a}}\right|_{D\hat{a}=0} = -\frac{\frac{\partial R}{\partial \hat{a}}\left[-\hat{s}^w\left(\hat{a}-\hat{e}\right)\right]-(n+\delta)}{\frac{\tilde{\eta}}{\hat{\eta}}\frac{\partial R}{\partial \hat{a}}\left[-\hat{s}^w\left(\tilde{a}-\tilde{e}\right)\right]} = \frac{\frac{w}{\hat{a}}-\frac{\partial w}{\partial \hat{a}}}{\frac{\partial w}{\partial \tilde{a}}}$$

$$\left.\frac{d\tilde{a}}{d\hat{a}}\right|_{D\tilde{a}=0} = -\frac{\frac{\partial R}{\partial \hat{a}}\left[-\tilde{s}^w\left(\hat{a}-\hat{e}\right)\right]}{\frac{\tilde{\eta}}{\hat{\eta}}\frac{\partial R}{\partial \hat{a}}\left[-\tilde{s}^w\left(\tilde{a}-\tilde{e}\right)\right]-(n+\delta)} = \frac{\frac{\partial w}{\partial \hat{a}}}{\frac{w}{\tilde{a}}-\frac{\partial w}{\partial \tilde{a}}}$$

We use (13) and (14) and the condition of steady-state $(n+\delta) = \hat{s}^w\left(w/\hat{a}\right) = \tilde{s}^w\left(w/\tilde{a}\right)$. Both slopes are positive if $\frac{w}{a} > \frac{\partial w}{\partial a}$ which is the case if capital and labor are sufficiently substitutable (see Appendix 3). The two equations have positive slopes, pass through the points B and C, and intersect at point S as in the following figure. The same analysis as before allows us to represent the Darreau-Pigalle case and the unique and stable steady state S.

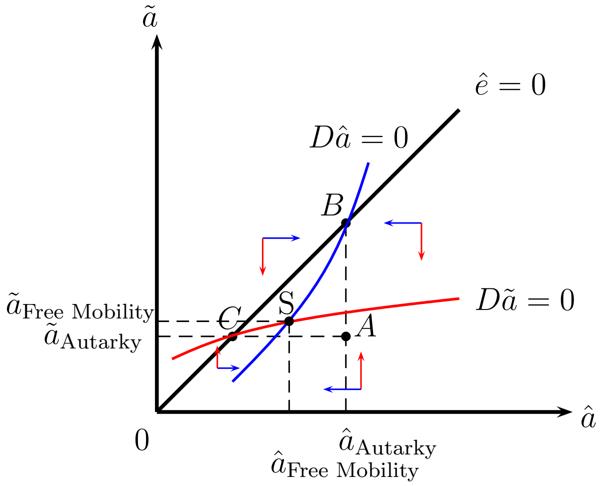

Figure 4: Darreau-Pigalle case

The surprising result of Darreau-Pigalle (2014) is that the lender country (Hat) loses from opening its capital market : $\hat{a}_{\text{free mobility}} < \hat{a}_{\text{autarky}}$.

Theorem 3 *: When $\hat{s}^a = \tilde{s}^a = 0$, the steady-state solution for the wealth of nations with free capital market, is better than the autarky solution for the borrowing country (Tilde) but is worse than the autarky solution for the lender country (Hat).*

Note that under the assumption of a small open economy $(\tilde{\eta} \to 0, \hat{\eta} \to 1)$, the curve $D\hat{a} = 0$ tends towards the vertical and the curve $D\tilde{a} = 0$ is steeper[2]. Only the small country (Tilde) wins from capital market integration. Under the assumption $(\tilde{\eta} \to 1, \hat{\eta} \to 0)$, the curve $D\tilde{a} = 0$ tends towards the horizontal and only the small country (Hat) loses to capital market integration.

4.4. Everyone loses case : $\hat{s}^a = 0, \tilde{s}^w = 0$

The Quibria and Darreau-Pigalle cases suggest combination of them. Substituting $\hat{s}^a = 0, \tilde{s}^w = 0$ in (9) and (10) we obtain:

[2]To calculate the slope of $D\tilde{a} = 0$ use (11) and (7).

$$\left.\frac{d\tilde{a}}{d\hat{a}}\right|_{D\hat{a}=0} = \frac{\frac{w}{\hat{a}} - \frac{\partial w}{\partial \hat{a}}}{\frac{\partial w}{\partial \tilde{a}}} \quad \text{and} \quad \left.\frac{d\tilde{a}}{d\hat{a}}\right|_{D\tilde{a}=0} = -\frac{\hat{\eta}}{\tilde{\eta}}$$

$D\hat{a} = 0$ has a positive slope and $D\tilde{a} = 0$ has a negative slope.

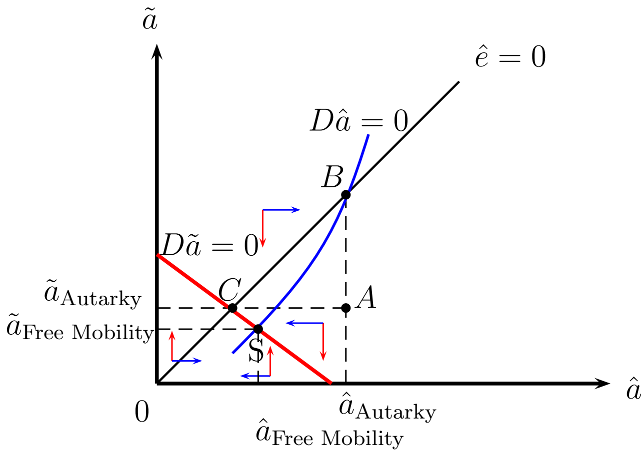

Figure 5: Everyone loses case

In the unique and stable steady state S, the Hat country and the Tilde country are both poorest.

Theorem 4 *: When $\hat{s}^a = 0, \tilde{s}^w = 0$ the steady-state solution for wealth per capita with free capital market, is worse than the autarky solution for the lender country (Hat) and for the borrowing country (Tilde).*

5. Discussion

It has been shown that the capital-importing country wins if it saves a constant fraction of its wages and that the capital-exporting country wins if it saves a constant fraction of its profits. Conversely the capital-exporting country loses if it saves a constant fraction of its wages, and the capital-importing country loses if it saves a constant fraction of its profits.

For the capital-exporting country, it is clear (see a proof in Ruffin 1979, theorem 2) that free capital mobility lowers wages and raises profits (in short and long run) compared to autarky. The contribution of this paper is to show that, if savings are a fraction of wages, savings will decrease for the capital-exporting country, which explains why its wealth decreases. If savings are a fraction of the profits, savings will increase for the capital-exporting country, which explains why its wealth increases.

For the capital-importing country, free capital mobility raises wages and lowers profits (at least in the short run in Quibria and Everyone loses cases) compared to autarky. If savings are a fraction of wages, savings and wealth will increase. If savings are a fraction of profits, savings and wealth will decrease.

The free capital market increases the wealth of nations in previous cases, only if savings are proportional to income which increases through the integration of the capital market: wages of the borrower, and profit of the lender. More generally, since $S = s^w w + s^a Ra$

and $\frac{\partial S}{\partial R} = s^w \frac{\partial w}{\partial R} + s^a a + s^a \frac{\partial a}{\partial R} R = s^w(-k) + s^a a + s^a a \left(\frac{\partial a}{\partial R}\frac{R}{a}\right) = s^w(-k) + s^a a(1+\varepsilon)$, it is a general result that $\frac{\partial S}{\partial R} > 0 \Leftrightarrow \frac{s^a}{s^w} > \frac{k}{a(1+\varepsilon)}$ where ε is the elasticity of wealth relative to the interest rate. Savings and wealth increase as a result of an increase in interest rate if the savings rate on profits is high relative to the savings rate on wages. Savings and wealth increase as a result of an decrease in interest rate if the savings rate on profits is low relative to the savings rate on wages. The free capital market increases the wealth of nations in Solow's model, for lender countries if and only if $\frac{s^a}{s^w} > \frac{k}{a(1+\varepsilon)}$ and for borrowing countries if and only if $\frac{s^a}{s^w} < \frac{k}{a(1+\varepsilon)}$.

6. Conclusion

When the free capital market is not justified by risk sharing as in the financial literature, its effects on the wealth of nations are far from being systematically positive. This elusive gains from international financial integration, to use the expression of Gourinchas and Jeanne (2006), is a curious result, empirically but also theoretically.

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Appendix 1

The $D\hat{a} = 0$ and $D\tilde{a} = 0$ slopes are calculated in the plane $(\tilde{a}, \hat{a})$. For this we need to calculate $\partial Da/\partial a$, and before that $\partial R/\partial a$ and $\partial w/\partial a$.

- **Computing** $\partial R/\partial a$:

  From both side of(4) we have : $\frac{\partial R}{\partial \hat{a}} = R'\left(1 - \frac{\partial \hat{e}}{\partial \hat{a}}\right) = R'\left(-\frac{\partial \tilde{e}}{\partial \hat{a}}\right)$

  From (7) we have $\frac{\partial \hat{e}}{\partial \hat{a}} = -\frac{\tilde{\eta}}{\hat{\eta}}\frac{\partial \tilde{e}}{\partial \hat{a}}$

  We obtain $\frac{\partial \hat{e}}{\partial \hat{a}} = \tilde{\eta}$ thus:

$$\frac{\partial R}{\partial \hat{a}} = \hat{\eta} R' < 0 \tag{11}$$

  With (4): $\frac{\partial R}{\partial \tilde{a}} = R'\left(-\frac{\partial \hat{e}}{\partial \tilde{a}}\right) = R'\left(1 - \frac{\partial \tilde{e}}{\partial \tilde{a}}\right)$ and (7) $\frac{\partial \hat{e}}{\partial \tilde{a}} = -\frac{\tilde{\eta}}{\hat{\eta}}\frac{\partial \tilde{e}}{\partial \tilde{a}}$.

  We obtain $\frac{\partial \tilde{e}}{\partial \tilde{a}} = \hat{\eta}$, thus:

$$\frac{\partial R}{\partial \tilde{a}} = \frac{\tilde{\eta}}{\hat{\eta}}\frac{\partial R}{\partial \hat{a}} \tag{12}$$

- **Computing $\partial w/\partial a$:**

  From (6) we obtain:

$$\frac{\partial w}{\partial \hat{a}} = \left(1 - \frac{\partial \hat{e}}{\partial \hat{a}}\right) R - \left(\frac{\partial R}{\partial \hat{a}}\left(\hat{a} - \hat{e}\right) + R\left(1 - \frac{\partial \hat{e}}{\partial \hat{a}}\right)\right)$$

  thus:

$$\frac{\partial w}{\partial \hat{a}} = -\frac{\partial R}{\partial \hat{a}}\left(\hat{a} - \hat{e}\right) > 0 \tag{13}$$

$$\frac{\partial w}{\partial \tilde{a}} = \left(1 - \frac{\partial \tilde{e}}{\partial \tilde{a}}\right) R - \left(\frac{\partial R}{\partial \tilde{a}}\left(\tilde{a} - \tilde{e}\right) + R\left(1 - \frac{\partial \tilde{e}}{\partial \tilde{a}}\right)\right)$$

  thus:

$$\frac{\partial w}{\partial \tilde{a}} = -\frac{\tilde{\eta}}{\hat{\eta}}\frac{\partial R}{\partial \hat{a}}\left(\tilde{a} - \tilde{e}\right) > 0 \tag{14}$$

- **Computing $\partial Da/\partial a$:**

$$\begin{aligned}
\frac{\partial D\hat{a}}{\partial \hat{a}} &= \hat{s}^w \frac{\partial w}{\partial \hat{a}} + \hat{s}^a \frac{\partial R}{\partial \hat{a}}\hat{a} + \hat{s}^a R - (n + \delta) \\
&= -\hat{s}^w \frac{\partial R}{\partial \hat{a}}\left(\hat{a} - \hat{e}\right) + \hat{s}^a \frac{\partial R}{\partial \hat{a}}\hat{a} + \hat{s}^a R - (n + \delta) \\
&= \frac{\partial R}{\partial \hat{a}}\left(-\hat{s}^w\left(\hat{a} - \hat{e}\right) + \hat{s}^a \hat{a}\right) + \hat{s}^a R - (n + \delta)
\end{aligned} \tag{15}$$

$$\begin{aligned}
\frac{\partial D\hat{a}}{\partial \tilde{a}} &= \hat{s}^w \frac{\partial w}{\partial \tilde{a}} + \hat{s}^a \frac{\partial R}{\partial \tilde{a}}\hat{a} \\
&= -\hat{s}^w \frac{\tilde{\eta}}{\hat{\eta}}\frac{\partial R}{\partial \hat{a}}\left(\tilde{a} - \tilde{e}\right) + \hat{s}^a \frac{\tilde{\eta}}{\hat{\eta}}\frac{\partial R}{\partial \hat{a}}\hat{a} \\
&= \frac{\tilde{\eta}}{\hat{\eta}}\frac{\partial R}{\partial \hat{a}}\left(-\hat{s}^w\left(\tilde{a} - \tilde{e}\right) + \hat{s}^a \hat{a}\right)
\end{aligned} \tag{16}$$

$$\frac{\partial D\tilde{a}}{\partial \hat{a}} = \tilde{s}^w \frac{\partial w}{\partial \hat{a}} + \tilde{s}^a \frac{\partial R}{\partial \hat{a}}\tilde{a} = \frac{\partial R}{\partial \hat{a}}\left(-\tilde{s}^w\left(\hat{a} - \hat{e}\right) + \tilde{s}^a \tilde{a}\right) \tag{17}$$

$$\begin{aligned}
\frac{\partial D\tilde{a}}{\partial \tilde{a}} &= \tilde{s}^w \frac{\partial w}{\partial \tilde{a}} + \tilde{s}^a \frac{\partial R}{\partial \tilde{a}}\tilde{a} + \tilde{s}^a R - (n + \delta) \\
&= -\tilde{s}^w \frac{\tilde{\eta}}{\hat{\eta}}\frac{\partial R}{\partial \hat{a}}\left(\tilde{a} - \tilde{e}\right) + \tilde{s}^a \frac{\tilde{\eta}}{\hat{\eta}}\frac{\partial R}{\partial \hat{a}}\tilde{a} + \tilde{s}^a R - (n + \delta) \\
&= \frac{\tilde{\eta}}{\hat{\eta}}\frac{\partial R}{\partial \hat{a}}\left(\tilde{s}^a \tilde{a} - \tilde{s}^w\left(\tilde{a} - \tilde{e}\right)\right) + \tilde{s}^a R - (n + \delta)
\end{aligned} \tag{18}$$

- **Calculation of slope $D\hat{a} = 0$ and $D\tilde{a} = 0$ in $(\tilde{a}, \hat{a})$ space:**

  Implicit differentiation of (8) will yield to :

$$\left.\frac{d\tilde{a}}{d\hat{a}}\right|_{D\hat{a}=0} = -\frac{\frac{\partial D\hat{a}}{\partial \hat{a}}}{\frac{\partial D\hat{a}}{\partial \tilde{a}}} \quad \text{and} \quad \left.\frac{d\tilde{a}}{d\hat{a}}\right|_{D\tilde{a}=0} = -\frac{\frac{\partial D\tilde{a}}{\partial \hat{a}}}{\frac{\partial D\tilde{a}}{\partial \tilde{a}}}$$

With (15) and (16), the slope of $D\hat{a}=0$ is:

$$\left.\frac{d\tilde{a}}{d\hat{a}}\right|_{D\hat{a}=0} = -\frac{\frac{\partial R}{\partial \hat{a}}\left[-\hat{s}^{w}\left(\hat{a}-\hat{e}\right)+\hat{s}^{a}\hat{a}\right]+\hat{s}^{a}R-(n+\delta)}{\frac{\tilde{\eta}}{\hat{\eta}}\frac{\partial R}{\partial \hat{a}}\left[-\hat{s}^{w}\left(\tilde{a}-\tilde{e}\right)+\hat{s}^{a}\hat{a}\right]}$$

with (17) and (18) the slope $D\tilde{a}=0$ is:

$$\left.\frac{d\tilde{a}}{d\hat{a}}\right|_{D\tilde{a}=0} = -\frac{\frac{\partial R}{\partial \hat{a}}\left[-\tilde{s}^{w}\left(\hat{a}-\hat{e}\right)+\tilde{s}^{a}\tilde{a}\right]}{\frac{\tilde{\eta}}{\hat{\eta}}\frac{\partial R}{\partial \hat{a}}\left[\tilde{s}^{a}\tilde{a}-\tilde{s}^{w}\left(\tilde{a}-\tilde{e}\right)\right]+\tilde{s}^{a}R-(n+\delta)}$$

Appendix 2

$$\hat{y}=f\left[\hat{a}-\hat{e}\right]+R\hat{e}\Rightarrow\frac{\partial\hat{y}}{\partial\hat{a}}=R+\frac{\partial R}{\partial\hat{a}}\hat{e}$$

$$\tilde{y}=f\left[\tilde{a}-\tilde{e}\right]+R\tilde{e}\Rightarrow\frac{\partial\tilde{y}}{\partial\tilde{a}}=R+\frac{\partial R}{\partial\tilde{a}}\tilde{e}=R+\frac{\tilde{\eta}}{\hat{\eta}}\frac{\partial R}{\partial\hat{a}}\left(-\frac{\hat{\eta}}{\tilde{\eta}}\hat{e}\right)=R-\frac{\partial R}{\partial\hat{a}}\hat{e}$$

As with Ruffin, let the marginal social product of labor be positive.

$$\frac{\partial\hat{Y}}{\partial\hat{L}}=\hat{y}-\hat{a}\frac{\partial\hat{y}}{\partial\hat{a}}>0\Rightarrow\hat{y}-R\hat{a}-\frac{\partial R}{\partial\hat{a}}\hat{e}\hat{a}=\frac{(n+\delta)\,\hat{a}}{\hat{s}}-R\hat{a}-\frac{\partial R}{\partial\hat{a}}\hat{e}\hat{a}>0\Rightarrow\;(n+\delta)>\hat{s}\left(R+\frac{\partial R}{\partial\hat{a}}\hat{e}\right)$$

$$\frac{\partial\tilde{Y}}{\partial\tilde{L}}=\tilde{y}-\tilde{a}\frac{\partial\tilde{y}}{\partial\tilde{a}}>0\Rightarrow\tilde{y}-R\tilde{a}+\frac{\partial R}{\partial\hat{a}}\hat{e}\tilde{a}=\frac{(n+\delta)\,\tilde{a}}{\tilde{s}}-R\tilde{a}+\frac{\partial R}{\partial\hat{a}}\hat{e}\tilde{a}>0\,(n+\delta)>\tilde{s}\left(R-\frac{\partial R}{\partial\hat{a}}\hat{e}\right)$$

Appendix 3

We show the condition for positive slope (for $D\hat{a}=0$), on the elasticity of substitution between capital and labor.

A.3.1) Since $w=f(k)-kf'(k)$ we have $\frac{\partial w}{\partial k}=f'(k)-[f'(k)+f''(k).k]=-f''(k).k$

A.3.2) The elasticity of substitution is defined as $\sigma(k)=\frac{d\ln(K/L)}{d\ln(F_L/F_K)}$. Under constant returns to scale this can be written as $\sigma(k)=\frac{F_K F_L}{F_{KL}.F}=-\frac{f'(k).w}{f''(k)kf(k)}$

Thus $-f''(k)k=\frac{f'(k).w}{\sigma(k)f(k)}$ or with our notations $-R'k=\frac{R.w}{\sigma(k).q}$

A.3.3) Equations (11) and (13) imply : $\frac{\partial w}{\partial\hat{a}}=-\frac{\partial R}{\partial\hat{a}}\left(\hat{a}-\hat{e}\right)=-\hat{\eta}R'k$

Thus $\frac{w}{\hat{a}}>\frac{\partial w}{\partial\hat{a}}\Leftrightarrow\frac{w}{\hat{a}}>-\hat{\eta}R'k$, and with (A.3.2) result, $\frac{w}{\hat{a}}>\hat{\eta}\frac{Rw}{\sigma(k).q}\Leftrightarrow\sigma(k)>\hat{\eta}\frac{R\hat{a}}{q}\Leftrightarrow\sigma(k)>\frac{Rk}{q}\frac{\hat{\eta}\hat{a}}{k}\Leftrightarrow\sigma(k)>\frac{Rk}{q}\frac{\hat{\eta}\hat{a}}{\tilde{\eta}\tilde{a}+\hat{\eta}\hat{a}}$

Thus

$$\frac{w}{\hat{a}}>\frac{\partial w}{\partial\hat{a}}\Leftrightarrow\sigma(k)>\alpha(k).\hat{p}(k)$$

The condition (for $D\hat{a}=0$ have a positive slope) is that the elasticity of substitution between capital and labor is higher than the share of capital in GDP multiplied by the share of the Hat country in wealth per capita in the world. The right-hand side is less than one. The left hand side is equal to one ($\sigma=1$) for the Cobb-Douglas production function. The general condition is that capital and labor are sufficiently substitutable. For example, if K and L are complementary ($\sigma=0$), an export of capital will lead to a decrease in L such that k is constant. Then w is constant and the Hat country's savings do not fall.