Economics Bulletin

Volume 35, Issue 4

Nash Equilibria in a Two-Person Discrete All-Pay Auction with Unfair Tie-Break and Complete Information

Hironori Otsubo
Soka University

Abstract

This note provides a full characterization of the set of Nash equilibria in the two-person all-pay auction with discrete strategy space and complete information where bidders are treated differently in case of a tie. There exist a unique symmetric mixed-strategy equilibrium and a continuum of asymmetric mixed-strategy equilibria. In some of these equilibria the bidder who loses in case of a tie is more likely to win the prize than the bidder who wins in case of a tie.

I would like to thank two anonymous reviewers for their constructive comments and suggestions. The usual disclaimer applies.
Citation: Hironori Otsubo, (2015) "Nash Equilibria in a Two-Person Discrete All-Pay Auction with Unfair Tie-Break and Complete Information", *Economics Bulletin*, Volume 35, Issue 4, pages 2443-2454
Contact: Hironori Otsubo - ootsubo@soka.ac.jp.
Submitted: September 09, 2015. **Published:** November 21, 2015.

1 Introduction

In a wide variety of economic, political, and social situations, people often expend their scarce and irrevocable resources (e.g., money, time, effort) in order to secure prizes or influence the chances of winning them. Examples include R&D competitions, arms races, electoral campaigns, lobbying activities, litigations, waiting in line, and athletic competitions. The game that well captures characteristics of these situations is the all-pay auction (Hillman and Samet, 1987; Baye et al., 1996).[1] In this auction, all bidders forfeit their bids, and only the highest bidder wins the prize. The literature traditionally assumes the symmetric tie-breaking rule in the all-pay auction. This means that in case of a tie the prize is either split equally among bidders with the highest bid if the prize is divisible or awarded to one of them with equal probability otherwise. The all-pay auction is fair in that that tie-breaking rule treats bidders equally.

Yet, casual observation questions such fair treatment. For example, favoritism may take place in the workplace. Employers favor one employee over the other equally performed employees in promotion. In many developing countries government officials treat firms differently in government procurement (Lien, 1990). An official's favored firm can still obtain a contract by giving the same amount of bribes to the official as the other firms.

The objective of this note is to characterize the set of Nash equilibria in the two-person all-pay auction with discrete strategy space and complete information in which bidders are treated differently in case of a tie. In this auction, each bidder independently choose one of discrete bids. The winner of the prize is whoever chooses the highest bid. In case of a tie, however, one designated bidder wins the prize. Thus, *one must outbid the other whereas the other only needs a tie.*[2]

This note complements the existing literature of all-pay auctions with discrete strategy space. Bouckaert et al. (1992) and Schep (1994) analyze the two-person all-pay auction in which ties are broken randomly with equal probability. They show that there are multiple equilibria in mixed strategies.[3] Rapoport and Amaldoss (2000) analyze two-person all-pay auctions with budget constraints in which neither bidder wins the prize in case of a tie, and later Dechenaux et al. (2006) characterize the set of Nash equilibria. Cohen and Sela (2007) investigate all-pay auctions with a variety of tie-breaking rules. None of these studies, however, explores the tie-

[1] For a recent review of the literature, see Konrad (2009).

[2] Bornstein et al. (2005) theoretically and experimentally analyze a somewhat related asymmetric contest. Their game is an asymmetric competition between two three-person *groups* over a single prize. Each group member decides whether to contribute her entire endowment or not. The two groups compete with each other in terms of the number of contributors. Asymmetry comes into play when the two groups tie; one group is a pre-selected winning group of the contest in case of a tie. Each contributor forfeits her contribution, regardless of whether her group wins or not. Then, each member of the winning group can enjoy the prize, regardless of whether she contributes or not.

[3] If the strategy space is continuous, there exists a unique equilibrium in which each bidder randomizes uniformly over the closed interval $[0, v]$, where v is the common valuation of the prize (see Baye et al., 1996).

breaking rule that favors one bidder over the other bidder.[4]

This literature is relatively small compared to that of all-pay auctions with continuous strategy space.[5] However, two major reasons make this line of research worthwhile pursuing. First, it is well documented that the differences between continuous and discrete versions of a model often matters. Second, experimental methods have rapidly been gaining popularity in the field of contest theory (Dechenaux et al., 2014), and the implementation of laboratory experiment requires discreteness. Subjecting the continuous version of a model to experimental testing may lead to erroneous conclusions.

The rest of the note proceeds as follows. Section 2 formally presents the model and the set of Nash equilibria. Section 3 concludes.

2 Set of Nash Equilibria

There are two risk-neutral bidders, indexed by 1 and 2, respectively. Hereafter, $i \in \{1, 2\}$ is used to refer to a generic bidder and $-i$ the other bidder. They seek a single, indivisible prize v, the valuation of which is the same for both bidders. Each bidder simultaneously chooses her bid b_i from the discrete strategy space $B = \{0, \epsilon, 2\epsilon, \dots\}$, where $\epsilon > 0$. v is assumed to be multiples of ϵ such that $v \geq 2\epsilon$.[6]

Bidder 1 wins the prize if $b_1 \geq b_2$ and bidder 2 wins otherwise. In other words, bidder 2 has to outbid bidder 1. Formally, bidder 1's contest success function is

$$p_1(b_1, b_2) = \begin{cases} 1 & \text{if } b_1 \geq b_2 \\ 0 & \text{otherwise} \end{cases}$$

and bidder 2's contest success function is $p_2(b_2, b_1) = 1 - p_1(b_1, b_2)$ because the winner of the prize is always determined. Bidder i's preferences are represented by the expected value of the payoff function given by

$$u_i(b_i, b_{-i}) = v \cdot p_i(b_i, b_{-i}) - b_i.$$

It is a well-known result that the all-pay auction with complete information does not possess a pure-strategy equilibrium. Since it is straightforward to show that the same is true even in the current setting, the proof is omitted.

In the mixed extension of the games, denote by (σ_1, σ_2) a profile of mixed strategies, where σ_i is bidder i's mixed strategy, i.e., a probability distribution over B, and $\sigma_i(b)$ is the probability assigned by σ_i to a pure strategy $b \in B$. Then, here is the main result:

[4] One exception is Otsubo (2013), which introduces this asymmetric tie-breaking rule to the same game studied by Rapoport and Amaldoss (2000).

[5] A partial list of the studies of the continuous all-pay auction that consider various tie-breaking rules include Lien (1990), Konrad (2002), Araujo et al. (2008), Feess et al. (2008), and Szech (2015).

[6] If $v = \epsilon$, there exist two pure-strategy Nash equilibria; one in which both bidders choose a bid 0 and the other in which bidder 1 chooses a bid 0 and bidder 2 chooses a bid ϵ.

Theorem. *There exist two types of Nash equilibria in mixed strategies* (σ_1^*, σ_2^*). *The first type is a unique symmetric Nash equilibrium characterized by*

$$\sigma_1^*(b) = \sigma_2^*(b) = \begin{cases} \frac{\epsilon}{v} & \text{if } b \in \{0, \epsilon, \ldots, v-\epsilon\} \\ 0 & \text{if } b \geq v \end{cases} \tag{1}$$

with equilibrium payoffs ϵ *for bidder 1 and* 0 *for bidder 2. The other type is a continuum of asymmetric Nash equilibria characterized by*

$$\sigma_1^*(b) = \begin{cases} \frac{\epsilon}{v} & \text{if } b \in \{0, \epsilon, \ldots, v-\epsilon\} \\ 0 & \text{if } b \geq v \end{cases} \tag{2}$$

and

$$\sigma_2^*(b) = \begin{cases} \frac{c}{v} & \text{if } b = 0 \\ \frac{\epsilon}{v} & \text{if } b \in \{\epsilon, 2\epsilon, \ldots, v-\epsilon\} \\ \frac{\epsilon - c}{v} & \text{if } b = v \\ 0 & \text{if } b > v \end{cases} \tag{3}$$

with equilibrium payoffs c *for bidder 1 and* 0 *for bidder 2, where* c *is a free parameter such that* $0 \leq c < \epsilon$.

Proof. See Appendix. □

This theorem warrants two comments. First, even in the current asymmetric setting there exists a unique symmetric equilibrium, in which both bidders assign equal probability $\frac{\epsilon}{v}$ to each of the bids $\{0, \epsilon, \ldots, v-\epsilon\}$. The same symmetric equilibrium exists in the symmetric setting studied by Bouckaert et al. (1992) and Schep (1994) if v is odd.

Second, there also exist a continuum of asymmetric equilibria. Bidder 1's equilibrium strategy does not differ from the one in the symmetric equilibrium, i.e., randomizes over bids $\{0, \epsilon, \ldots, v-\epsilon\}$ with equal probability. On the other hand, the support of bidder 2's equilibrium mixed strategy varies depending on the value of a free parameter $c \in [0, \epsilon)$. If $c = 0$, bidder 2 randomizes over bids $\{\epsilon, 2\epsilon, \ldots, v\}$ with equal probability. Both bidders earn a payoff of 0. If $c > 0$, bidder 2 randomizes over bids $\{0, \epsilon, \ldots, v\}$, assigning probability $\frac{\epsilon}{v}$ to each of the bids $\{\epsilon, 2\epsilon, \ldots, v-\epsilon\}$ and splitting the remaining probability $\frac{\epsilon}{v}$ between bids 0 and v. A higher value of c implies that bidder 2 puts more probability on 0 than on v. Bidder 2 earns 0 whereas bidder 1 earns c. One can pin down an asymmetric equilibrium for a given c and generate a continuum of asymmetric equilibria by varying c over the half-closed interval $[0, \epsilon)$.

Table 1 summarizes theoretical implications. It is worthwhile noting some highlights.

1. In the symmetric equilibrium there is no difference in the expected bid between bidders. Due to the asymmetric tie-breaking rule, however, the probability of winning differs between them; bidder 2 is more likely to lose the prize. This disadvantage disappears as either $\epsilon \to 0$ or $v \to \infty$.

Equilibrium	Expected Bid		Bidder 2's Prob.
	Bidder 1	Bidder 2	of Winning
Symmetric	$\frac{v-\epsilon}{2}$	$\frac{v-\epsilon}{2}$	$\frac{1}{2}-\frac{\epsilon}{v}$
Asymmetric	$\frac{v-\epsilon}{2}$	$\frac{v-\epsilon}{2}+(\epsilon-c)$	$\frac{1}{2}-\frac{1}{v}\left(c-\frac{\epsilon}{2}\right)$

Table 1: Theoretical implications

2. In the asymmetric equilibria bidder 2 is expected to bid more aggressively than bidder 1. Bidder 2's probability of winning exceeds $\frac{1}{2}$ in the asymmetric equilibria with $c < \frac{\epsilon}{2}$.
3. The auctioneer's expected revenue, i.e., the sum of expected bids, is higher in the asymmetric equilibria than in the symmetric equilibria.

Of particular interest is the existence of asymmetric equilibria in which bidder 2 is more likely to win the prize. Even in an unfair contest like the current setting, the contestant in a disadvantageous position still bids aggressively and wins the prize more often in some equilibria.

3 Concluding Remarks

Many real-world contests do not always treat contestants equally even if their performances are the same. This note sheds light on the tie-breaking rule that favors one bidder over the other in the two-person all-pay auction with complete information and discrete strategy space. The result shows that there are only one symmetric equilibrium and infinitely many asymmetric equilibria and that in some asymmetric equilibria the weaker bidder (bidder 2) is more likely to win the prize than the stronger one (bidder 1).

Appendix

Proof. Let σ_i^* be bidder i's (non-degenerated) mixed strategy in equilibrium. Hereafter, denote by S_i^* the support of σ_i^*, α_i^* its minimum element, and β_i^* its maximum element, respectively.

Let (σ_1^*, σ_2^*) be a mixed-strategy equilibrium for the game and $u_i^* = u_i(\sigma_i^*, \sigma_{-i}^*) = \sum_{b\in B} \sigma_i^*(b)\cdot u_i(b, \sigma_{-i}^*)$ the corresponding equilibrium payoff of bidder i, where $u_i(b, \sigma_{-i}^*)$ denotes bidder i's expected payoff from a bid b given bidder $-i$'s equilibrium mixed strategy σ_{-i}^*. Then, $u_i^* = u_i(b, \sigma_{-i}^*)$ for $b \in S_i^*$ and $u_i^* \geq u_i(b, \sigma_{-i}^*)$ for $b \notin S_i^*$.

Lemmas 1–8 pin down the supports of mixed strategies in equilibrium.

Lemma 1. *In any equilibrium (σ_1^*, σ_2^*), for all b such that $b > v$, $b \notin S_i^*$ for all i, $i \in \{1, 2\}$.*

Proof. Suppose that there exists a bid w $(> v)$ such that $\sigma_i^*(w) > 0$ for some i. Since $u_i(w, \sigma_{-i}^*) < 0$, bidder i wishes to unilaterally deviate from σ_i^* by loading probability 1 on a bid 0, which contradicts $w \in S_i^*$. □

Lemma 2. *In any equilibrium* (σ_1^*, σ_2^*), *(i)* $v \in S_i^*$ *for at most one bidder. Moreover, (ii)* $v \notin S_1^*$.

Proof. (i) Suppose that $v \in S_i^*$ for all i, $i \in \{1, 2\}$. Since $\sigma_1^*(v) > 0$, bidder 2's expected payoff from a bid v is

$$u_2(v, \sigma_1^*) = v \cdot \sum_{b<v} \sigma_1^*(b) - v = v \cdot (1 - \sigma_1^*(v)) - v < 0.$$

Therefore, bidder 2 wishes to unilaterally deviate from σ_2^* by loading probability 1 on a bid 0. This contradicts $v \in S_2^*$.

(ii) Suppose that $v \in S_1^*$. It follows from (i) that $v \notin S_2^*$. Bidder 1's expected payoffs from bids v and β_2^* are

$$u_1(v, \sigma_2^*) = 0$$

and

$$u_1(\beta_2^*, \sigma_2^*) = v - \beta_2^* > 0,$$

respectively. Then, $u_1(v, \sigma_2^*) < u_1(\beta_2^*, \sigma_2^*)$, which contradicts $v \in S_1^*$. □

Lemma 3. *In any equilibrium* (σ_1^*, σ_2^*), *(i)* $0 \in S_i^*$ *for at least one bidder. Moreover, (ii)* $0 \in S_1^*$. *Consequently, (iii)* $u_1^* \geq 0$.

Proof. (i) Suppose that $0 \notin S_i^*$ for all i, $i \in \{1, 2\}$. Then, bidder 1 can get better off choosing a bid 0 with probability 1 if $\alpha_1^* < \alpha_2^*$ and bidder 2 can get better off choosing a bid 0 with probability 1, otherwise. A contradiction.

(ii) Suppose that $0 \notin S_1^*$. It follows from (i) that $0 \in S_2^*$ and therefore $\alpha_1^* > \alpha_2^* = 0$. Then, all b such that $0 < b \leq \alpha_1^*$ are not in the support of bidder 2 because her mixed strategy that assigns probability 1 to a bid 0 strictly dominates these pure strategies. In this case, bidder 1's expected payoffs from bids α_1^* and $0 \notin S_1^*$ are

$$u_1(\alpha_1^*, \sigma_2^*) = v \cdot \sum_{b \leq \alpha_1^*} \sigma_2^*(b) - \alpha_1^* = v \cdot \sigma_2^*(0) - \alpha_1^*$$

and

$$u_1(0, \sigma_2^*) = v \cdot \sigma_2^*(0),$$

respectively. Hence, $u_1(\alpha_1^*, \sigma_2^*) < u_1(0, \sigma_2^*)$, which contradicts $\alpha_1^* \in S_1^*$.

(iii) (i) and (ii) imply that either $0 \in S_2^*$ or $0 \notin S_2^*$ must be true. If $0 \in S_2^*$, $u_1^* > 0$ because

$$u_1^* = u_1(0, \sigma_2^*) = v \cdot \sigma_2^*(0) - 0 > 0.$$

Also, if $0 \notin S_2^*$, $u_1^* = 0$ because

$$u_1^* = u_1(0, \sigma_2^*) = v \cdot 0 - 0 = 0.$$

Therefore, $u_1^* \geq 0$. □

Lemma 4. *In any equilibrium* (σ_1^*, σ_2^*)*, for all* $b \in B$ *such that* $0 < b \le v - \epsilon$*, if* $b \in S_1^*$*, then* $b \in S_2^*$*.*

Proof. Suppose that there exist a bid x ($0 < x \le v-\epsilon$) such that $x \in S_1^*$ and $x \notin S_2^*$. Consider two cases: (i) $\alpha_2^* < x$ and (ii) $\alpha_2^* > x$.

(i) Suppose that $\alpha_2^* < x$. Define $b_x = \max\{b \in S_2^* | \alpha_2^* \le b < x\}$. Bidder 1's expected payoffs from bids x and b_x are

$$u_1(x, \sigma_2^*) = v \cdot \sum_{b \le x} \sigma_2^*(b) - x = v \cdot \sum_{b \le b_x} \sigma_2^*(b) - x$$

and

$$u_1(b_x, \sigma_2^*) = v \cdot \sum_{b \le b_x} \sigma_2^*(b) - b_x,$$

respectively. Then, $u_1(x, \sigma_2^*) < u_1(b_x, \sigma_2^*)$, which contradicts $x \in S_1^*$.

(ii) Suppose that $\alpha_2^* > x$. Then, since $u_1(x, \sigma_2^*) = -x < 0$, bidder 1 can get better off by choosing a bid 0 with probability 1. This contradicts $x \in S_1^*$. □

Lemma 5. *In any equilibrium* (σ_1^*, σ_2^*)*, for all* $b \in B$ *such that* $\epsilon < b \le v$*, if* $b \in S_2^*$*, then* $b - \epsilon \in S_1^*$*.*

Proof. Suppose that there exist a bid y ($\epsilon < y \le v$) such that $y \in S_2^*$ and $y - \epsilon \notin S_1^*$. Bidder 2's expected payoffs from bids y and $y - \epsilon$ are

$$u_2(y, \sigma_1^*) = v \cdot \sum_{b \le y-\epsilon} \sigma_1^*(b) - y = v \cdot \sum_{b \le y-2\epsilon} \sigma_1^*(b) - y$$

and

$$u_2(y-\epsilon, \sigma_1^*) = v \cdot \sum_{b \le y-2\epsilon} \sigma_1^*(b) - (y-\epsilon),$$

respectively. Thus, $u_2(y, \sigma_1^*) < u_2(y-\epsilon, \sigma_1^*)$. This contradicts $y \in S_2^*$. □

Lemma 6. *In any equilibrium* (σ_1^*, σ_2^*)*,* $S_1^* = \{0, \epsilon, 2\epsilon, \dots, \beta_1^*\}$ *and* $\{\epsilon, 2\epsilon, 3\epsilon, \dots, \beta_1^*\} \subseteq S_2^*$*.*

Proof. By definition, $\beta_1^* \in S_1^*$. It follows from Lemma 5 that $\beta_1^* \in S_2^*$. By Lemma 4, $\beta_1^* \in S_2^*$ implies $\beta_1^* - \epsilon \in S_1^*$. By recursively applying Lemmas 4 and 5, it is clear that $\{\epsilon, 2\epsilon, 3\epsilon, \dots, \beta_1^*\} \subseteq S_1^*$ and $\{\epsilon, 2\epsilon, 3\epsilon, \dots, \beta_1^*\} \subseteq S_2^*$. Since $0 \in S_1^*$ by Lemma 3, $S_1^* = \{0, \epsilon, 2\epsilon, \dots, \beta_1^*\}$. □

Lemma 7. *In any equilibrium* (σ_1^*, σ_2^*) *with* $v \notin S_2^*$*,* $S_1^* = S_2^* = \{0, \epsilon, 2\epsilon, \dots, v - \epsilon\}$*. Consequently,* $u_1^* = \epsilon$ *and* $u_2^* = 0$*.*

Proof. Given $v \notin S_2^*$ it is obvious that $\beta_2^* \le v - \epsilon$. First show that (i) $\beta_1^* = \beta_2^*$. Then, show that (ii) $\beta_1^* = \beta_2^* = v - \epsilon$. Finally, show that (iii) $0 \in S_2^*$.

(i) It follows from Lemma 6 that $\beta_1^* \le \beta_2^*$. Suppose that $\beta_1^* < \beta_2^*$. This implies that $u_2(\beta_2^*, \sigma_1^*) = v - \beta_2^* \ge v - (v - \epsilon) = \epsilon > 0$. Then, $0 \notin S_2^*$ must be true because if

$0 \in S_2^*$, $u_2(0, \sigma_1^*) = 0$, which contradicts $u_2(0, \sigma_1^*) = u_2(\beta_2^*, \sigma_1^*)$. Given that $0 \notin S_2^*$, bidder 1's expected payoffs from bids 0 and β_2^* are

$$u_1(0, \sigma_2^*) = v \cdot \sigma_2^*(0) - 0 = 0$$

and

$$u_1(\beta_2^*, \sigma_2^*) = v \cdot \sum_{b \leq \beta_2^*} \sigma_2^*(b) - \beta_2^* = v - \beta_2^* > 0,$$

respectively. Therefore, $u_1(0, \sigma_2^*) < u_1(\beta_2^*, \sigma_2^*)$. This contradicts $0 \in S_1^*$.

(ii) Suppose that $\beta_1^* = \beta_2^* < v - \epsilon$. Bidder 1's expected payoff from a bid β_1^* is

$$u_1(\beta_1^*, \sigma_2^*) = v - \beta_1^* > v - (v - \epsilon) = \epsilon > 0.$$

Then, $0 \in S_2^*$ must be true because if $0 \notin S_2^*$, $u_1(0, \sigma_2^*) = 0$, which contradicts $u_1(0, \sigma_2^*) = u_1(\beta_1^*, \sigma_2^*)$. Given that $0 \in S_2^*$, bidder 2's expected payoffs from bids 0 and $v - \epsilon$ are

$$u_2(0, \sigma_1^*) = 0$$

and

$$u_2(v - \epsilon, \sigma_1^*) = v \cdot \sum_{b < v - \epsilon} \sigma_1^*(b) - (v - \epsilon) = v - (v - \epsilon) = \epsilon > 0,$$

respectively. Then, $u_2(0, \sigma_1^*) < u_2(v - \epsilon, \sigma_1^*)$. This contradicts $0 \in S_2^*$.

(iii) (ii) implies that $u_1^* = \epsilon$, which in turn implies that $0 \in S_2^*$. Thus, $u_2^* = 0$. □

Lemma 8. *In any equilibrium (σ_1^*, σ_2^*) with $v \in S_2^*$, (i) $u_2^* = 0$. Moreover, (ii) $S_1^* = \{0, \epsilon, 2\epsilon, \ldots, v - \epsilon\}$ and $\{\epsilon, 2\epsilon, 3\epsilon, \ldots, v\} \subseteq S_2^*$. Consequently, $0 \leq u_1^* < \epsilon$. Furthermore, (iii) $u_1^* = 0$ implies $S_2^* = \{\epsilon, 2\epsilon, 3\epsilon, \ldots, v\}$, and $u_1^* > 0$ implies $S_2^* = \{0, \epsilon, 2\epsilon, \ldots, v\}$.*

Proof. (i) It immediately follows from $v \in S_2^*$ that $u_2^* = 0$.

(ii) It follows from recursively applying Lemmas 4 and 5 that $v \in S_2^*$ implies that $\{0, \epsilon, 2\epsilon, \ldots, v - \epsilon\} \subseteq S_1^*$ and $\{\epsilon, 2\epsilon, 3\epsilon, \ldots, v\} \subseteq S_2^*$. Since $v \notin S_1^*$ by Lemma 2, $S_1^* = \{0, \epsilon, 2\epsilon, \ldots, v - \epsilon\}$. Since $\beta_1^* = v - \epsilon$ and $\beta_2^* = v$, bidder 1's expected payoff from a bid $v - \epsilon$ is

$$u_1(v - \epsilon, \sigma_2^*) = v \cdot \sum_{e \leq v - \epsilon} \sigma_2^*(e) - (v - \epsilon) = \epsilon - v\sigma_2^*(v) < \epsilon.$$

Since $u_1^* \geq 0$ by Lemma 3, $0 \leq u_1^* < \epsilon$.

(iii) By (ii), it suffices to show that (a) $u_1^* = 0$ implies $0 \notin S_2^*$ and that (b) $u_1^* > 0$ implies $0 \in S_2^*$.

(a) Suppose that $u_1^* = 0$ and $0 \in S_2^*$. Since $0 \in S_1^*$ by Lemma 3, $u_1(0, \sigma_2^*) = 0$. However,

$$u_1(0, \sigma_2^*) = v \cdot \sigma_2^*(0) - 0 > 0.$$

This contradicts $u_1^* = 0$.

(b) Suppose that $u_1^* > 0$ and $0 \notin S_2^*$. Then, since $0 \notin S_2^*$ it is obvious that $u_1(0, \sigma_2^*) = 0$. This contradicts $u_1^* > 0$. □

Lemma 9. *There exists a unique symmetric equilibrium* (σ_1^*, σ_2^*) *with* $v \notin S_2^*$. *In this equilibrium,*

$$\sigma_1^*(b) = \sigma_2^*(b) = \begin{cases} \frac{\epsilon}{v} & \text{if } b \in \{0, \epsilon, \dots, v-\epsilon\} \\ 0 & \text{if } b \geq v \end{cases}$$

with $u_1^* = \epsilon$ *and* $u_2^* = 0$.

Proof. Prove existence by construction. If (σ_1^*, σ_2^*) is an equilibrium with $v \notin S_2^*$, it follows immediately from Lemma 7 that $S_1^* = S_2^* = \{0, \epsilon, \dots, v-\epsilon\}$ and consequently $u_1^* = \epsilon$ and $u_2^* = 0$. Then, for bidder 1, $\sigma_2^*(b)$'s must be the solution to the following system of $\frac{v}{\epsilon}$ equations with $\frac{v}{\epsilon}$ unknowns:

$$\begin{cases} v \cdot \sigma_2^*(0) - 0 & = \epsilon \\ v \cdot [\sigma_2^*(0) + \sigma_2^*(\epsilon)] - \epsilon & = \epsilon \\ v \cdot [\sigma_2^*(0) + \sigma_2^*(\epsilon) + \sigma_1^*(2\epsilon)] - 2\epsilon & = \epsilon \\ \vdots & \\ v \cdot [\sigma_2^*(0) + \sigma_2^*(\epsilon) + \cdots + \sigma_2^*(v-2\epsilon)] - (v-2\epsilon) & = \epsilon \\ v \cdot [\sigma_2^*(0) + \sigma_2^*(\epsilon) + \cdots + \sigma_2^*(v-\epsilon)] - (v-\epsilon) & = \epsilon \end{cases}$$

This system can be expressed as follows:

$$\underbrace{\begin{bmatrix} v & 0 & 0 & \dots & 0 & 0 \\ v & v & 0 & \dots & 0 & 0 \\ v & v & v & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ v & v & v & \dots & v & 0 \\ v & v & v & \dots & v & v \end{bmatrix}}_{\frac{v}{\epsilon} \times \frac{v}{\epsilon}} \underbrace{\begin{bmatrix} \sigma_2^*(0) \\ \sigma_2^*(\epsilon) \\ \sigma_2^*(2\epsilon) \\ \vdots \\ \sigma_2^*(v-2\epsilon) \\ \sigma_2^*(v-\epsilon) \end{bmatrix}}_{\frac{v}{\epsilon} \times 1} = \underbrace{\begin{bmatrix} \epsilon \\ 2\epsilon \\ 3\epsilon \\ \vdots \\ v-\epsilon \\ v \end{bmatrix}}_{\frac{v}{\epsilon} \times 1} \tag{A.1}$$

The leftmost square matrix of (A.1) is nonsingular. Hence, there exists a unique solution of the system. Find $\sigma_2^*(0)$ and then substitute this into the second equation in the system to obtain $\sigma_2^*(\epsilon)$. Recursively applying the substitution method eventually yields the solution for the system. It is straightforward to see that $0 < \sigma_2^*(b) < 1$ for all $b \in S_2^*$ and $\sum_{b \in S_2^*} \sigma_2^*(b) = 1$.

For bidder 2, $\sigma_1^*(b)$'s must solve the following system of $\frac{v}{\epsilon}$ equations with $\frac{v}{\epsilon}$ unknowns:

$$\begin{cases} v \cdot \sigma_1^*(0) - \epsilon & = 0 \\ v \cdot [\sigma_1^*(0) + \sigma_1^*(\epsilon)] - 2\epsilon & = 0 \\ v \cdot [\sigma_1^*(0) + \sigma_1^*(\epsilon) + \sigma_1^*(2\epsilon)] - 3\epsilon & = 0 \\ \vdots & \\ v \cdot [\sigma_1^*(0) + \sigma_1^*(\epsilon) + \cdots + \sigma_1^*(v-2\epsilon)] - (v-\epsilon) & = 0 \\ \sigma_1^*(0) + \sigma_1^*(\epsilon) + \cdots + \sigma_1^*(v-\epsilon) & = 1 \end{cases} \tag{A.2}$$

This system can be expressed in the following matrix form:

$$\underbrace{\begin{bmatrix} v & 0 & 0 & \dots & 0 & 0 & 0 \\ v & v & 0 & \dots & 0 & 0 & 0 \\ v & v & v & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ v & v & v & \dots & v & v & 0 \\ 1 & 1 & 1 & \dots & 1 & 1 & 1 \end{bmatrix}}_{\frac{v}{\epsilon}\times\frac{v}{\epsilon}} \underbrace{\begin{bmatrix} \sigma_1^*(0) \\ \sigma_1^*(\epsilon) \\ \sigma_1^*(2\epsilon) \\ \vdots \\ \sigma_1^*(v-2\epsilon) \\ \sigma_1^*(v-\epsilon) \end{bmatrix}}_{\frac{v}{\epsilon}\times 1} = \underbrace{\begin{bmatrix} \epsilon \\ 2\epsilon \\ 3\epsilon \\ \vdots \\ v-\epsilon \\ 1 \end{bmatrix}}_{\frac{v}{\epsilon}\times 1} \tag{A.3}$$

Just as before, the leftmost square matrix of (A.3) is nonsingular. Hence, there exists a unique solution of the system. The substitution method described before is applied to derive the solution. One can make sure that $0 < \sigma_1^*(b) < 1$ for all $b \in S_1^*$ and $\sum_{b\in S_1^*} \sigma_1^*(b) = 1$. □

Lemma 10. *There exist a continuum of asymmetric equilibria (σ_1^*, σ_2^*) with $v \in S_2^*$. In these equilibria,*

$$\sigma_1^*(b) = \begin{cases} \frac{\epsilon}{v} & \text{if } b \in \{0, \epsilon, \dots, v-\epsilon\} \\ 0 & \text{if } b \geq v \end{cases}$$

and

$$\sigma_2^*(b) = \begin{cases} \frac{u_1^*}{v} & \text{if } b = 0 \\ \frac{\epsilon}{v} & \text{if } b \in \{\epsilon, 2\epsilon, \dots, v-\epsilon\} \\ \frac{\epsilon - u_1^*}{v} & \text{if } b = v \\ 0 & \text{if } b > v \end{cases}$$

with $0 \leq u_1^* < \epsilon$ and $u_2^* = 0$.

Proof. As in Lemma 9, prove existence by construction. By Lemma 8, if (σ_1^*, σ_2^*) is an equilibrium with $v \in S_2^*$, $S_1^* = \{0, \epsilon, \dots, v-\epsilon\}$, $S_2^* = \{\epsilon, 2\epsilon, \dots, v\}$ if $u_1^* = 0$, and $S_2^* = \{0, \epsilon, \dots, v\}$ if $u_1^* > 0$.

Suppose that $u_1^* = 0$. This implies that $S_2^* = \{\epsilon, 2\epsilon, \dots, v\}$ and that $u_2^* = 0$. Then, for bidder 1, $\sigma_2^*(b)$'s must be the solution to the following system of $\frac{v}{\epsilon}$ equations with $\frac{v}{\epsilon}$ unknowns:

$$\begin{cases} v \cdot \sigma_2^*(\epsilon) - \epsilon & = 0 \\ v \cdot [\sigma_2^*(\epsilon) + \sigma_2^*(2\epsilon)] - 2\epsilon & = 0 \\ v \cdot [\sigma_2^*(\epsilon) + \sigma_2^*(2\epsilon) + \sigma_2^*(3\epsilon)] - 3\epsilon & = 0 \\ \vdots & \\ v \cdot [\sigma_2^*(\epsilon) + \sigma_2^*(2\epsilon) + \dots + \sigma_2^*(v-\epsilon)] - (v-\epsilon) & = 0 \\ \sigma_2^*(\epsilon) + \sigma_2^*(2\epsilon) + \dots + \sigma_2^*(v-\epsilon) + \sigma_2^*(v) & = 1 \end{cases}$$

In the matrix form,

$$\underbrace{\begin{bmatrix} v & 0 & 0 & \dots & 0 & 0 & 0 \\ v & v & 0 & \dots & 0 & 0 & 0 \\ v & v & v & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ v & v & v & \dots & v & v & 0 \\ 1 & 1 & 1 & \dots & 1 & 1 & 1 \end{bmatrix}}_{\frac{v}{\epsilon}\times\frac{v}{\epsilon}} \underbrace{\begin{bmatrix} \sigma_2^*(\epsilon) \\ \sigma_2^*(2\epsilon) \\ \sigma_2^*(3\epsilon) \\ \vdots \\ \sigma_2^*(v-\epsilon) \\ \sigma_2^*(v) \end{bmatrix}}_{\frac{v}{\epsilon}\times 1} = \underbrace{\begin{bmatrix} \epsilon \\ 2\epsilon \\ 3\epsilon \\ \vdots \\ v-\epsilon \\ 1 \end{bmatrix}}_{\frac{v}{\epsilon}\times 1} \tag{A.4}$$

Next, suppose that $u_1^* > 0$. This implies that $S_2^* = \{0, \epsilon, \dots, v\}$ and that $u_2^* = 0$. Then, for bidder 1, $\sigma_2^*(b)$'s must be the solution to the following system of $(\frac{v}{\epsilon}+1)$ equations with $(\frac{v}{\epsilon}+1)$ unknowns:

$$\begin{cases} v\cdot\sigma_2^*(0) - 0 & = u_1^* \\ v\cdot[\sigma_2^*(0)+\sigma_2^*(\epsilon)] - \epsilon & = u_1^* \\ v\cdot[\sigma_2^*(0)+\sigma_2^*(\epsilon)+\sigma_2^*(2\epsilon)] - 2\epsilon & = u_1^* \\ \vdots & \\ v\cdot[\sigma_2^*(0)+\sigma_2^*(\epsilon)+\cdots+\sigma_2^*(v-\epsilon)] - (v-\epsilon) & = u_1^* \\ \sigma_2^*(0)+\sigma_2^*(\epsilon)+\cdots+\sigma_2^*(v-\epsilon)+\sigma_2^*(v) & = 1 \end{cases}$$

In the matrix form,

$$\underbrace{\begin{bmatrix} v & 0 & 0 & \dots & 0 & 0 & 0 \\ v & v & 0 & \dots & 0 & 0 & 0 \\ v & v & v & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ v & v & v & \dots & v & v & 0 \\ 1 & 1 & 1 & \dots & 1 & 1 & 1 \end{bmatrix}}_{\left(\frac{v}{\epsilon}+1\right)\times\left(\frac{v}{\epsilon}+1\right)} \underbrace{\begin{bmatrix} \sigma_2^*(0) \\ \sigma_2^*(\epsilon) \\ \sigma_2^*(2\epsilon) \\ \vdots \\ \sigma_2^*(v-\epsilon) \\ \sigma_2^*(v) \end{bmatrix}}_{\left(\frac{v}{\epsilon}+1\right)\times 1} = \underbrace{\begin{bmatrix} u_1^* \\ u_1^*+\epsilon \\ u_1^*+2\epsilon \\ \vdots \\ u_1^*+(v-\epsilon) \\ 1 \end{bmatrix}}_{\left(\frac{v}{\epsilon}+1\right)\times 1} \tag{A.5}$$

The leftmost square matrices of (A.4) and (A.5) are nonsingular. Hence, there exists a unique solution for each case. By the substitution method, one can obtain the solution. It is easy to see that $0 < \sigma_2^*(b) < 1$ for all $b \in S_2^*$ and $\sum_{b\in S_2^*}\sigma_2^*(b) = 1$.

For bidder 2, $\sigma_1^*(b)$'s must be the solution to the following system of $\frac{v}{\epsilon}$ equations with $\frac{v}{\epsilon}$ unknowns:

$$\begin{cases} v\cdot\sigma_1^*(0) - \epsilon & = 0 \\ v\cdot[\sigma_1^*(0)+\sigma_1^*(\epsilon)] - 2\epsilon & = 0 \\ v\cdot[\sigma_1^*(0)+\sigma_1^*(\epsilon)+\sigma_1^*(2\epsilon)] - 3\epsilon & = 0 \\ \vdots & \\ v\cdot[\sigma_1^*(0)+\sigma_1^*(\epsilon)+\cdots+\sigma_1^*(v-2\epsilon)] - (v-\epsilon) & = 0 \\ v\cdot[\sigma_1^*(0)+\sigma_1^*(\epsilon)+\cdots+\sigma_1^*(v-\epsilon)] - v & = 0 \end{cases}$$

This system of equations is in fact identical to (A.2). Thus, there exists a unique solution such that $0 < \sigma_1^*(b) < 1$ for all $b \in S_1^*$ and $\sum_{b \in S_1^*} \sigma_1^*(b) = 1$.

□

Lemmas 1–10 complete the proof of the theorem. □

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