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Misunderestimation: exponential-growth bias and time-varying returns

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Abstract

Exponential-growth bias is the tendency to neglect the compounding of interest. The economics literature has used the fact that a biased agent in many circumstances will underestimate the value of assets that grow according to compound interest. We show that the opposite can also be true. It is always possible to make an agent who underestimates exponential growth to overestimate the value of an asset that grows exponentially. This paradoxical phenomenon arises when interest rates vary over time. This gives rise to the averaging effect of exponential-growth bias, which causes agents to perceive the mean return to exceed the true mean. Consequently, biased agents will strictly prefer assets with time-varying returns over equivalent constant-return assets. With sufficient variation in returns any biased agent will overestimate the true value of an asset for any time horizon.

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1. Introduction

Exponential-growth bias (EGB) is the tendency to neglect the compounding of interest (Wagenaar & Sagaria, 1975; Stango & Zinman, 2009). It has been measured in representative samples and found to be widespread across income and age categories, and a predictor of important economic outcomes (Stango & Zinman, 2011; Goda et al., 2015). Naturally, those who neglect compounding will often underestimate the value of assets. While this effect of EGB has received much attention, there is an additional effect that has been largely ignored. Biased agents will combine interest rates over time in a more linear way, perceiving the mean to be closer to the arithmetic mean than to the geometric mean. For example, an asset with annual returns over the past five years of <40%, 4%, 1%, -60%, 70%> may appear impressive to a biased agent, yielding an arithmetic mean of 11%, when in fact the relevant geometric mean is marginally negative. We call the error that biased agents make when estimating the mean of interest rate vector the *averaging effect of exponential-growth bias.*[1]

The averaging effect leads to two important results. First, biased agents will always perceive the mean rate to be weakly greater than the true mean.[2] Consequently, a biased agent will prefer interest vectors that vary over time relative to those that are equivalent but constant. For example, a fully-biased agent will prefer an asset that increases by 100% in odd periods and 0% in even periods (the geometric mean return is 41.4%) over an asset that increases by 45% in every period. Because EGB operates through perceived payoffs, a biased investor may choose overly risky assets and even dominated assets.

Second, because the averaging effect causes biased agents to overestimate the mean return, the biased agent may actually overestimate the value of an asset. While it is true that the neglect of compounding will drive perceptions of value downward, the biased agent's error when combining rates may dominate. Hence when the interest rate varies sufficiently over time a biased agent will, paradoxically, *overestimate* the value of an asset. We show that this effect can be made sufficiently strong that a biased agent will overestimate the value of an asset for any time horizon and any geometric mean. An exploitative seller of assets can frame returns in various ways that will cause a biased agent to pay *more* for the asset in terms of present value than the true value.

The focus in this paper is to show that EGB can, counterintuitively, cause agents to over-estimate the monetary returns of certain types of assets (rather than under-estimate, as is often assumed). Our results are without reference to the agent's objective function in order to keep the results general, but they will have implications for choice when greater intertemporal wealth is preferred to less intertemporal wealth. The vectors that we analyze are return vectors that directly affect wealth, but they are not consumption vectors. The paper shows that biased agents will distort the expected future value of an asset in predictable ways, and may therefore value assets incorrectly. We omit risk for clarity in focusing on the mean perceived return, but the results naturally extend to risky returns — and would also

[1] One notable paper that relates to this point is Ensthaler et al. (2013), who study how biased agents misperceive the distribution of final returns when an asset receives a random return of either 70% or -60%, each period. They show theoretically that biased agents will severely overestimate the median and underestimate the variance and skew of the distribution, and they find that behavior in their experiment strongly agrees with the prediction.

[2] This formalizes the intuition presented in Levy & Tasoff (2015).

lead to analogous mis-perceptions of the higher moments of the asset.

2. Model

As in Levy & Tasoff (2015), let $p(\vec{\imath}, t)$ be the agent's perception of the period-T value of one dollar invested at time $t \le T$. The correct perception is $e(\vec{\imath}, t) \equiv \prod_{s=t}^{T-1}(1 + i_s)$. However, the agent may exhibit EGB. We assume that the interest-perception function is a twice-differentiable function $p : \mathbb{R}^T \times \{0, 1, \ldots, T\} \to \mathbb{R}$, and that it satisfies four properties.

(A1) (boundary conditions) $p(\vec{0}, t) = 1$ for all t and $p(\vec{\imath}, T) = 1$ for all $\vec{\imath}$

(A2) (compounding) $\partial p(\vec{\imath}, t)/\partial i_k \ge 0$ and $\partial^2 p(\vec{\imath}, t)/\partial i_j \partial i_k \ge 0$. For $k \ge t$, if $i_\tau = 0$ for all $\tau \ne k$, $\tau \ge t$, then $\partial p(\vec{\imath}, t)/\partial i_k = 1$

(A3) (irrelevancy of history) $p(<i_1, \ldots, i_t, i_{t+1}, \ldots, i_{T-1}>, t) = p(<0, \ldots, 0, i_t, i_{t+1}, \ldots, i_{T-1}>, 0)$

(A4) (irrelevancy of order) If for some $k, l \ge t$, $i_k = j_l$, $i_l = j_k$, and for any $\tau \notin \{k, l\}$, $i_\tau = j_\tau$, then $\partial p(\vec{\imath}, t)/\partial i_k = \partial p(\vec{\jmath}, t)/\partial j_l$

An interest perception function p exhibits greater exponential-growth bias than q if:

$$\left[\frac{\partial^2 p(\vec{\imath}, t)}{\partial i_j \partial i_k}\right] \Big/ \left[\frac{\partial p(\vec{\imath}, t)}{\partial i_j}\right] \le \left[\frac{\partial^2 q(\vec{\imath}, t)}{\partial i_j \partial i_k}\right] \Big/ \left[\frac{\partial q(\vec{\imath}, t)}{\partial i_j}\right] \tag{1}$$

where $T > 1$, and the inequality is strict if $j \ne k$. We call a perception function biased if it displays greater exponential-growth bias than the true growth process, $e(\vec{\imath}, T)$. Assumption (A2) is the key assumption for exponential-growth bias, and ensures that perceptions increase at least linearly in interest rate, and the definition of EGB ensures that perceptions increase no more than the true growth rate. Together, assumptions (A1)–(A4) are the weakest conditions on perceptions that admit exponential-growth bias, but which do not impose a particular functional form or extra distortions on perceptions which do not operate through under-appreciation of compounding. For example, an interest vector with a single non-zero interest rate is always perceived correctly, regardless of its length or when the non-zero interest rate occurs.

For simplicity we assume that the interest rate vector $\vec{\imath}$ is certain. Even in this extreme case a biased agent will exhibit a strict revealed "preference" for some interest rate vectors that are equivalent to other interest rate vectors, based on the variation in returns over time. Of course these "preferences", as they may be perceived by an outside observer, are actually errors. Our results extend directly to uncertain interest rates in which the mean return varies over time but is not autocorrelated.

3. Results

The model leads to two nonclassical effects that influence the biased agent's perception of the value of an asset with time-varying returns. The first is the *extrapolation effect of exponential-growth bias*. When the horizon is more than one period in length, the biased agent will underestimate the speed at which assets with positive interest rates will grow, and overestimate the speed at which assets with negative interest rates will shrink. This is the primary focus of most economic research on EGB (see for example Wagenaar & Sagaria, 1975; Goda et al., 2014; Song, 2012). The second effect, introduced here, involves how a biased agent will combine interest rates over time. An agent with accurate perceptions knows that an interest vector is equivalent to the vector of its geometric mean. A biased agent, on the other hand, will overestimate the constant-rate equivalent for a given interest vector. We refer to this as the *averaging effect of exponential-growth bias.*

Proposition 1 (Averaging Effect of EGB) *Define $m(\vec{\imath}; p)$ such that*

$$p(<m(\vec{\imath}; p), \ldots, m(\vec{\imath}; p)>, 0) = p(\vec{\imath}, 0) \text{ for all } \vec{\imath} \in \mathbb{R}^T.$$

Then if p exhibits greater EGB than q, $m(\vec{\imath}; p) \geq m(\vec{\imath}; q)$. The inequality is strict if $\exists j, k$ s.t. $i_j \neq i_k$.

All proofs are in the Appendix. The proposition implies that a biased agent prefers a varying interest vector to an equivalent non-varying interest vector when investing. The premium that the agent places on the varying interest vector is increasing in his bias. The proposition illustrates how the display of annualized stock returns can hide low total returns. A table of individual returns may lead an investor to erroneously conclude that a stock outperformed a risk-free investment, even in cases where the stock lost value overall. This provides an incentive for sellers of assets to present periodic returns instead of their overall returns. The effect goes in reverse when the biased agent considers a loan. This may help to explain the prevalence of fixed-rate loans and mortgages which, in addition to reducing risk and smoothing repayment, also appear to have lower rates to biased agents. A biased agent would actually be willing to incur a premium on his fixed-rate loan rather than have it replaced with a perfectly-forecastable variable rate that is economically equivalent to the original rate.

When a biased agent is presented a vector of varying returns and must determine the value of the asset after some time horizon, both effects influence his assessment. The extrapolation effect and the averaging effect often work in opposite directions. The more biased the agent, the more he will underestimate the compounding of interest, but the more he will overestimate the equivalent rate. For sufficient variation in the returns, however, a biased agent could be made to *overestimate* the value of the asset for *any* time horizon.

Proposition 2 (Overestimation) *Given any $\vec{\imath} \in \mathbb{R}^S$, $T \in \mathcal{N}$, and any biased interest perception function $p(\cdot, \cdot)$, there exists some $\vec{\jmath}$ such that*

i. $m(\vec{\imath}, e) = m(\vec{\jmath}, e)$

ii. $p(<m(\vec{\jmath}, p), ..., m(\vec{\jmath}, p)>, T) > e(<m(\vec{\jmath}, e), ..., m(\vec{\jmath}, e)>, T)$

Proposition 2 shows that given any vector of returns and any extrapolative time horizon, there is always an equivalent vector $\vec{\jmath}$ for which the averaging effect will dominate the extrapolation effect. The averaging effect can be made arbitrarily large by replacing an interest vector with another of equal geometric mean and arbitrarily large variation, eventually mixing both positive and negative returns. This allows the averaging effect to dominate the extrapolation effect for any horizon T.

For Proposition 2 to affect behavior requires that agents may smooth consumption (i.e. the agent may save and borrow in a way that leads to an interior solution in the dynamic consumption problem). If instead the agent must consume each return in the period in which it occurs, EGB does not have any implications since returns do not actually compound. Moreover, diminishing marginal utility would cause the agent to prefer interest vectors with less variation.[3] Proposition 2 is therefore relevant to settings in which the agent may misperceive their intertemporal resource constraint, such as lifecycle savings models — but not, for instance, valuation of an annuity.

The proposition implies that an exploitative principal could in fact trick a biased agent to *overpay* for any asset that exhibits exponential growth. The principal presents an interest vector as a sample for the agent to evaluate. The principal is constrained to keep the geometric mean of the vector to the true level (equivalently, it must be the actual total return) but can otherwise freely vary the interest rates, and by doing so can control the biased agent's perceptions. While this may be unrealistic in many financial matters, the proposition expresses the limit of possible exploitation. Greater constraints on the framing of the evaluation return vector will temper this result but often even the original return vector will suffice.

Example 1: If $T = 1 < S$, then a biased agent will always overestimate. This follows directly from Proposition 1, regardless of whether the true return is positive or negative. For example, a biased agent presented with an annualized return for an asset that compounds faster than annually will always overestimate the return for horizons less than a full year.

Example 2: Let $p(\vec{\imath}, S; \alpha) = \prod_{s=0}^{S}(1 + \alpha i_s) + (1 - \alpha) \sum_{s=0}^{S} i_s$. Note, $\alpha = 1$ corresponds to correct perceptions and $\alpha = 0$ to linear perceptions. Then if $m(\vec{\imath}, p) > m(\vec{\imath}, e)/\alpha$ the agent will always overestimate. For example, if $\vec{\imath} =< -30\%, 50\% >$ the agent will always overestimate, over any horizon T, for $\alpha \in [0.27, 1)$.

4. Conclusion

While most applications of EGB have focused on the lower expectations of biased individuals, we have shown that a separate channel will cause them to overestimate the average return and can even cause them to overestimate the total return. This averaging effect will always dominate over short horizons where simple interest exceeds compound interest, but can dominate for any arbitrary time horizon given sufficient variation in the interest profile.

There are many important implications of our results. In addition to the manipulation implied by Proposition 2, an exploitative seller of assets could leverage the averaging effect and eliminate the extrapolation effect by presenting a time-varying return vector for evaluation, and providing a calculator capable of doing compound interest for a single interest rate. A biased agent would then overestimate the mean return but then correctly compound this return. This leads to a perceived value of the asset greater than the true perception and greater than the biased perception without the calculator. Such a form of exploitation is particularly pernicious as it masquerades as a cure to misperceptions when in fact it

[3] We thank the editor for pointing this out.

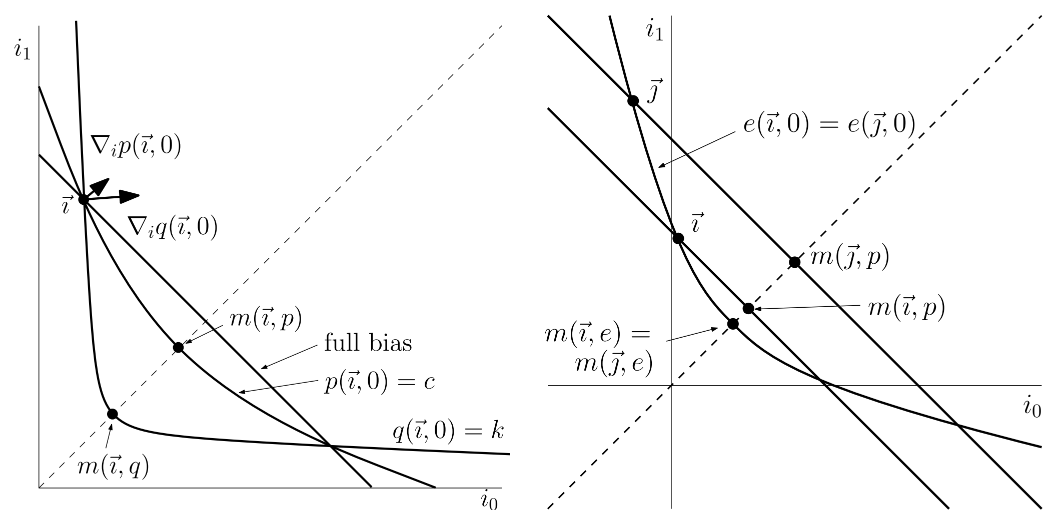
exacerbates them.

A. Appendix: Proofs

Proof of Proposition 1 Consider a contour of the perception function $p(\cdot,0) = p(\vec{\imath},0)$ in interest-rate space $\mathbb{R}^T$, i.e. the set $\{\vec{\jmath} \in \mathbb{R}^T | p(\vec{\jmath},0) = p(\vec{\imath},0)\}$. By construction, $p(\vec{m},0) \equiv p(\vec{\imath},0)$. Along the contour $\sum_{s=0}^{T-1} \frac{\partial p(\vec{\jmath},0)}{\partial i_s} = 0$. From (A4) the contour is symmetric about the 45-degree line, that is it is symmetric about a ray from the origin defined by $\gamma\vec{1}$ where $\gamma > 0$. From (A2) the perception function is quasiconcave in i_t since the function is strictly increasing in i_t and $\frac{\partial^2 p(\vec{\imath},0)}{\partial i_j \partial i_k} \geq 0$ for all j,k. This implies that the contours of the perception function are convex relative to the origin.

The gradient of the perception function $\nabla_i p(\vec{\imath},0)$ is orthogonal to the contour and hence fully determines the contour's shape (see Figure 2(a)). By definition the intersection between the contour and $\gamma\vec{1}$ uniquely determines $m(\vec{\imath};p) = \gamma$. We now wish to show that the contour of $p(\vec{\imath},0) = c$ must intersect the ray at a higher γ then $q(\vec{\imath},0) = k$. By (A4), $\frac{\partial p(\gamma\vec{1},0)/\partial i_j}{\partial p(\gamma\vec{1},0)/\partial i_k} = \frac{\partial q(\gamma\vec{1},0)/\partial i_j}{\partial q(\gamma\vec{1},0)/\partial i_k} = 1$. For $i_j < i_k$, because p is more biased than q, $-\frac{\partial q(\vec{\imath},0)/\partial i_j}{\partial q(\vec{\imath},0)/\partial i_k} < -\frac{\partial p(\vec{\imath},0)/\partial i_j}{\partial p(\vec{\imath},0)/\partial i_k}$. This expression is analogous to a comparison of marginal rates of substitution. This implies that at all points at $\vec{\imath}$, $\nabla_i q(\vec{\imath},0)$ will be directed more towards $\gamma\vec{1}$ than is $\nabla_i p(\vec{\imath},0)$, and the contour $q(\vec{\imath},0) = k$ will curve more steeply towards the origin, and as a consequence $m(\vec{\imath};q) < m(\vec{\imath};p)$.

Figure A.1: Interest Perception Function for Two Interest Rates



(a) Perception function contours for $T = 2$. (b) Equivalent rates and overestimation

Proof of Proposition 2 First, we show that for any T and $R > 0$, if p exhibits EGB then

$$\lim_{\varepsilon \to 0} p(<0,...,0,(1/\varepsilon)R-1,\varepsilon R-1>,T) = \infty.$$

Figure 2(b) shows intuitively that increasing the variance of returns increases $m(\vec{\imath}, p)$. We formally prove that this is unbounded. Since p exhibits EGB, $\partial p(<i_0, i_1>, 2)/\partial i_1 < (1 + i_0)$ for all $i_0 > 0$. Thus given any i_0, $\exists \delta < 1$ s.t. $\partial p(<i_0, i_1>, 2)/\partial i_1 < \delta(1 + i_0) < (1 + i_0)$. Using the fundamental theorem of calculus, we have:

$$
\begin{aligned}
p(<0, ..., 0, (1/\varepsilon)R - 1, \varepsilon R - 1>, T) &= (1/\varepsilon)R - \int_{\varepsilon R-1}^{0} \left[\partial p(<(1/\varepsilon)R - 1, x>, 2)/\partial x\right] dx \\
&> (1/\varepsilon)R - \int_{\varepsilon R-1}^{0} \delta(1/\varepsilon)R dx \\
&= R\left[\frac{1}{\varepsilon}(1-\delta) + \delta R\right] \to_{\varepsilon \to 0} \infty
\end{aligned}
$$

where $\delta \in [0, 1)$ is found for a fixed $\varepsilon < \infty$ and $R > 0$.

Now let $\vec{\jmath}(\vec{\imath}, \varepsilon) = <0, ..., 0, (1/\varepsilon)e(\vec{\imath}, S)^{1/2} - 1, (\varepsilon)e(\vec{\imath}, S)^{1/2} - 1> \in \mathbb{R}^S$. By construction, $e(\vec{\jmath}(\vec{\imath}, \varepsilon), S) = e(\vec{\imath}, S)$ for all $\vec{\imath}$ and ε. Hence $m(\vec{\jmath}(\vec{\imath}, \varepsilon), e) = m(\vec{\imath}, e)$ for all $\vec{\imath}$ and ε as well.

Moreover, given $\vec{\imath}$ and T, $e(<m(\vec{\imath}, e), ..., m(\vec{\imath}, e)>, T)$ is finite and constant in ε. However, $p(< m(\vec{\jmath}(\vec{\imath}, \varepsilon), p), ..., m(\vec{\jmath}(\vec{\imath}, \varepsilon), p) >, T) \to \infty$ as $\varepsilon \to 0$. Thus $\exists\ \varepsilon > 0$ s.t. $p(< m(\vec{\jmath}(\vec{\imath}, \varepsilon), p), ..., m(\vec{\jmath}(\vec{\imath}, \varepsilon), p)) >, T) > e(<m(\vec{\jmath}, e), ..., m(\vec{\jmath}, e)>, T) = e(<m(\vec{\imath}, e), ..., m(\vec{\imath}, e)>, T)$. This completes the proof.

References

Ensthaler, L., Nottmeyer, O., Weizsäcker, G., & Zankiewicz, C. (2013). "Hidden skewness: On the difficulty of multiplicative compounding under random shocks". DIW Berlin Discussion Paper No. 1337.

Goda, G. S., Levy, M. R., Manchester, C. F., Sojourner, A., & Tasoff, J. (2015). "The Role of Exponential-Growth Bias and Present Bias in Retirement Savings Decisions". NBER Working Paper No. 21482.

Goda, G. S., Manchester, C. F., & Sojourner, A. (2014). "What Will My Account Really Be Worth? Experimental Evidence on How Retirement Income Projections Affect Saving". *Journal of Public Economics*, 119, 80–92.

Levy, M. R. & Tasoff, J. (2015). "Exponential-Growth Bias and Lifecycle Consumption". *Journal of the European Economic Association*, Forthcoming.

Song, C. (2012). "Financial Illiteracy and Pension Contributions: A Field Experiment on Compound Interest in China". Working Paper.

Stango, V. & Zinman, J. (2009). "Exponential Growth Bias and Household Finance". *Journal of Finance*, 64, 2807–2849.

Stango, V. & Zinman, J. (2011). "Fuzzy Math, Disclosure Regulation, and Market Outcomes: Evidence from Truth-in-Lending Reform". *Review of Financial Studies*, 24, 506–534.

Wagenaar, W. A. & Sagaria, S. D. (1975). "Misperception of Exponential Growth". *Perception and Psychophysics*, 18, 416–422.