Economics Bulletin

Volume 36, Issue 2

Representation of Epstein-Marinacci derivatives of absolutely continuous TU games

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Abstract

We show that, for some classes of transferable utility (TU) games widely used in Game Theory and Mathematical Economics, Epstein and Marinacci derivatives have a natural representation in terms of a "generalized" Radon-Nikodym derivative. This has a straightforward interpretation in a General Equilibrium context, where marginal contributions can be seen as a fair way to reward each group of agents.

Citation: Francesca Centrone, (2016) "Representation of Epstein-Marinacci derivatives of absolutely continuous TU games", *Economics Bulletin*, Volume 36, Issue 2, pages 1149-1159
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Submitted: March 12, 2014. **Published:** June 11, 2016.

1. Introduction

Set functions, not necessarily additive, are extensively used in Cooperative Game Theory, where they model the notion of transferable utility (TU) game. Given a set I and a σ-algebra Σ on I, interpreted respectively as a set of players and a collection of possible coalitions, a TU *game* is a function $\nu : \Sigma \to \mathbb{R}$ such that $\nu(\emptyset) = 0$. For each $S \in \Sigma$, the number $\nu(S)$ represents the total payoff that coalition S can obtain for its members, if it forms.

The use of differential techniques in Cooperative Game Theory dates back to Aumann and Shapley (1974): in their treatise they worked out a notion of derivative related to games which allowed to translate and maintain the original meaning of the Shapley value as a weighted average of each player's *marginal contribution* to the worth of all possible coalitions in which he/she may participate, in the context of non-atomic games. With an analogous spirit, Epstein and Marinacci (2001) developed and applied to the study of the core of large TU games a notion of differentiability (firstly introduced by Epstein (1999) in a Decision Theory context) which goes as follows.

Let Λ be a directed set. For any $S \in \Sigma$, let $\{S^{j,\lambda}\}_{j=1}^{n_\lambda}$ be a finite partition of S and let $\{S^{j,\lambda}\}_{\lambda\in\Lambda}$ be the *Moore-Smith sequence* of all finite partitions of S ordered by refinements, that is, $\lambda' > \lambda$ implies that the partition corresponding to λ' refines the one corresponding to λ.

Definition 1. A game ν is called *outer differentiable* at a set $E \in \Sigma$ if there exists a bounded and strongly non-atomic finitely additive measure $\delta^+\nu(\cdot; E)$ on Σ such that

$$\sum_{j=1}^{n_\lambda} |\nu(E \cup F^{j,\lambda}) - \nu(E) - \delta^+\nu(F^{j,\lambda}; E)| \underset{\lambda}{\longrightarrow} 0,$$

for all $F \subset E^c$.

$\delta^+\nu(\cdot; E)$ is termed as *outer derivative* of ν at E.

By $\underset{\lambda}{\longrightarrow}$ we mean the limit in the Moore-Smith sense: for every $\varepsilon > 0$, there exists a finite partition of measurable sets $\{F^{j,\lambda}\}_{j=1}^{n_\lambda}$ of F such that, for every $\lambda' > \lambda$,

$$\sum_{j=1}^{n_{\lambda'}} |\nu(E \cup F^{j,\lambda'}) - \nu(E) - \delta^+\nu(F^{j,\lambda'}; E)| < \varepsilon.$$

The *inner derivative* at $E \in \Sigma$, denoted by $\delta^-\nu(\cdot; E)$, must verify

$$\sum_{j=1}^{n_\lambda} |\nu(E - G^{j,\lambda}) - \nu(E) + \delta^-\nu(G^{j,\lambda}; E)| \underset{\lambda}{\longrightarrow} 0,$$

for all $G \subset E$.

Then, if a game ν admits both outer and inner derivative at E, the *derivative* of ν at E can be defined as $\delta\nu(\cdot; E) := \delta^+\nu(\cdot; E) + \delta^-\nu(\cdot; E)$.
For every $F \subset E^c$ and finite partition $\{F^{j,\lambda}\}_{j=1}^{n_\lambda}$ of F, $\nu(E \cup F^{j,\lambda}) - \nu(E)$ represents the *marginal value* of $\{F^{j,\lambda}\}$ relative to E. Hence, in the limit, $\delta\nu(\cdot; E)$ represents the *total marginal value* of F relative to the base coalition E. An analogous interpretation holds for the inner derivative.
Throughout the paper we shall call this derivative *derivative by refinements.*
The above approach is bounded to a non-atomic context: anyway we point out that, more recently, Montrucchio and Semeraro (2008) extended it to games defined on algebras of sets, removed the non-atomicity assumption and thus considerably widened the class of differentiable games.
Epstein (1999) introduced and applied a notion of differentiability where the idea of smallness is not captured by a refinement process but, as is more familiar in calculus, through the limit of a difference quotient. We state it below.

Definition 2. Given a game ν and a finitely additive non-negative measure μ, ν is called *μ-differentiable* at $E \in \Sigma$ if:

(i) $\mu(N \Delta N') = 0$, for $N, N' \in \Sigma$, implies $\nu(F \cup N) = \nu(F \cup N')$, for every $F \in \Sigma$;

(ii) there exists a bounded and strongly non-atomic finitely additive measure $\Delta\nu(\cdot; E)$ on Σ such that, for all $F \subset E^c$ and $G \subset E$ and $\mu(F \cup G) > 0$,

$$\lim_{\mu(F \cup G) \to 0} \frac{|\nu(E \cup F - G) - \nu(E) - \Delta\nu(F; E) + \Delta\nu(G; E)|}{\mu(F \cup G)} = 0.$$

Property **(i)** is easily seen to be equivalent (see Centrone and Martellotti, 2012) to property:

(i$'$) if for $N \in \Sigma$, $\mu(F \cup N) = \mu(F)$ for every $F \in \Sigma$, then $\nu(F \cup N) = \nu(F)$ for every $F \in \Sigma$.

The two above types of derivatives coincide for μ-differentiable games (Epstein, 1999), but μ-differentiability is indeed stronger than differentiability by refinements (for an example, see Centrone and Martellotti, 2012).
At this point, we observe that a notion very close to Definition 2 above, already appeared in the paper of Artzner and Ostroy (1983) in a General Equilibrium context. There, the authors introduce the notion of "*product-exhaustion*" for a non-atomic production economy (I, Σ, w, f), where (I, Σ) is the measurable space of agents, and w is a non-atomic vector measure on the σ-algebra of coalitions Σ such that, for every $S \in \Sigma$, $w(S) \in \mathbb{R}^l_+$ is the endowment vector of resources available to coalition S. Here f is a production function, and $f \circ w : \Sigma \longrightarrow \mathbb{R}$ (defined by $(f \circ w)(S) = f(w(S))$, which is the production of coalition S), is an example of a so called *vector measure game* in the terminology of Aumann and Shapley (1974). In Artzner and Ostroy's framework, a

non-atomic production economy for which there exists a measure η satisfying Definition 2, for $E = I$, $S = G$, when the norm $\|w(S)\|$ (representing the "size" of S) tends to zero, is said to exhibit product exhaustion. Hence, interpreting $\eta(S)$ has the payment of a coalition S, the differentiability condition tells us that, in the limit, this payment is close to the "marginal" product $f(w(I)) - f(w(I - S))$, as according to the classical marginal productivity theory of distribution. Then, the authors also show the link between the product exhaustion notion and the price-taking behavior of small group of agents arising from the concept of Walrasian equilibrium.

In the present note we introduce an extension to games of the measure theoretic construction of the Radon-Nikodym derivative and, under suitable hypotheses, we show that this can be used to "represent" Epstein and Marinacci derivatives for some classes of games relevant to Cooperative Game Theory and Mathematical Economics that is, *vector measure games* and *absolutely continuous games.* Indeed, as pointed out in Neyman (2002),"games arising in applications are often either vector measure games or approximated by vector measure games". Market exchange economies of finite type or, as we have already anticipated, some production models provide meaningful examples of this kind of games (see Aumann and Shapley (1974), Einy *et al.* (1999) and Hart and Neyman (1988)). As for the space of absolutely continuous games, denoted by AC, it contains many interesting games such as, for instance, the ones forming the space pNA, i.e., the closure in a suitable topology of the space generated by polynomials in measures (under suitable assumptions, exchange economies of finite type can be expressed as measure games belonging to this space).

We prove that, under customary hypotheses on the underlying players space, for *absolutely continuous games* which are μ- differentiable, and for *vector measure games* (not necessarily absolutely continuous) the derivatives of Epstein and Marinacci can be expressed by means of their *density* with respect to an appropriate measure. To support the economic interpretation, in the model of Artzner and Ostroy (1983), this integral representation tells us that the marginal productivity of a coalition S, and hence its reward, is precisely given by the "sum" of the contributions of "*infinitesimal*" individuals dS.

We also point out how the interest for an integral representation of derivatives of games, arises once more from the work of Aumann and Shapley (1974): for example, on pNA where a value is known to exist, it can be expressed as an integral of a Frechét-type derivative. Our representation result goes in this direction and could thus provide an alternative vision, if the equality between the Aumann and Shapley value and Epstein and Marinacci derivatives could be proven in some relevant cases. Anyway, this topic appears not to be a trivial one.

In the sequel, for the sake of brevity, we recall just the essential definitions and refer the reader to Aumann and Shapley (1974), Bhaskara Rao and Bhaskara Rao (1983) and Marinacci and Montrucchio (2004) for all the other game and measure theoretic related notions.

2. A Radon-Nikodym derivative for AC games

Let I be a topological space and $\mathcal{B}$ its Borel σ-algebra. Suppose that I is the union of a countable family $\mathcal{R}_1$ of disjoint $A_1^1, \dots, A_j^1, \dots \in \mathcal{B}$, called the sets of *the first rank*. Suppose moreover that every set A_j^1 *of the first rank* is itself the union of a countable family of disjoint Borelians $A_{j1}^2, \dots, A_{jk}^2, \dots$ called the sets *of the second rank*, and let $\mathcal{R}_2$ denote the family of all such sets over all j's. Repeating the process for every $n \in \mathbb{N}$ leads at each step to a family $\mathcal{R}_n$ of disjoint Borel sets, called *of the n-th rank*, whose union is the original set I. The family

$$\mathcal{R} = \bigcup_{n=1}^{\infty} \mathcal{R}_n$$

of all sets of finite rank is called a *net* (Shilov and Gurevich (1966)).
Henceforth, according to Aumann and Shapley (1974), we shall denote by NA the space of all non-atomic bounded measures on $\mathcal{B}$, and by NA^+ the cone of all non-negative measures in NA.

Definition 3. (Munroe (1971)) A net $\mathcal{R}$ is called *regular* with respect to $\mu \in NA^+$ if for every $E \in \mathcal{B}$ and for every $\varepsilon > 0$, there are countably many sets $A_1, A_2, \dots$ of $\mathcal{R}$ such that

$$E \subset \bigcup_{n=1}^{\infty} A_n, \qquad \mu(\bigcup_{n=1}^{\infty} A_n) < \mu(E) + \varepsilon. \tag{1}$$

Definition 4. (Munroe (1971)) Let I be endowed with a metric d: a net $\mathcal{R}$ is called *indefinitely fine* if for each $x \in I$ and each $\varepsilon > 0$ there exists a set A_n from one of the $\mathcal{R}_n$ which contains x and has diameter (i.e. $\sup\{d(y,z), y, z \in A_n\}$) less than ε.

Example 1. *If $I = [0,1]$ is endowed with the euclidean metric, the net $\mathcal{R}$ where the set of the n-th rank are the intervals $\left[0, \frac{1}{2^n}\right], \left(\frac{1}{2^n}, \frac{2}{2^n}\right], \dots, \left(1 - \frac{1}{2^n}, 1\right]$ is indefinitely fine and regular with respect to the Lebesgue measure.*

The following theorem expresses the key idea to our purposes.

Theorem 1. (De Possel) (Shilov and Gurevich (1966)) Let $\mathcal{R}$ be a net, regular w.r.t. the non-negative measure μ, and let $\theta \in NA$ be absolutely continuous with respect to μ. Then the derivative $D_{\mathcal{R}}\theta(\cdot)$ of θ with respect to $\mathcal{R}$, i.e. the quantity

$$D_{\mathcal{R}}\theta(x) := \lim_{n \to \infty} \psi_n(x), \tag{2}$$

where

$$\psi_n(x) = \begin{cases} \dfrac{\theta(A_n(x))}{\mu(A_n(x))} & \text{if } \mu(A_n(x)) \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

and $A_n(x)$ is the unique set of the n-th rank containing x, exists on a set of full μ-measure and is integrable. Moreover

$$\theta(S) = \int_S D_{\mathcal{R}}\theta(x)d\mu, \qquad \forall S \in \mathcal{B} \tag{4}$$

i.e. $D_{\mathcal{R}}\theta(\cdot)$ coincides with the Radon-Nikodym derivative of θ with respect to μ.

In particular, this theorem implies that the derivative does not depend on $\mathcal{R}$.

Henceforth, unless otherwise specified, I will be a *Polish metric space,* i.e. it will be supposed to be endowed with a metric d with respect to which I is separable and complete.
Now let ν be a TU game, μ a non-negative measure on $\mathcal{B}$ and $\mathcal{R}$ an indefinitely fine net. Consider a set $E \in \mathcal{B}$ such that its topological boundary ∂E is of μ-null measure. In the sequel we shall indicate by E^{est} the set of topologically external points of E, i.e. the set of points of E^c having a neighborhood contained in E^c, while E° shall denote the topological interior of E.
We introduce the following sequence: for each $n \in \mathbb{N}$, let

$$\phi_{E,n}(x) = \begin{cases} \dfrac{\nu(E \cup A_n(x)) - \nu(E)}{\mu(A_n(x))} & \text{if } x \in E^{est},\ \mu(A_n(x)) \neq 0 \\ \dfrac{\nu(E) - \nu(E - A_n(x))}{\mu(A_n(x))} & \text{if } x \in E^\circ,\ \mu(A_n(x)) \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

where, for every x, $A_n(x)$ denotes the only element of the n-th rank containing x, and let

$$\phi_E(x) = \begin{cases} \lim_{n\to+\infty} \phi_{E,n}(x) & \text{if the limit exists} \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

ϕ_E is measurable and, having supposed $\mu(\partial E) = 0$, it is defined μ-a.e.

Remark 1. *ϕ_E extends expression (2) to the case of non-additive set functions and can thus be interpreted as a generalized Radon-Nikodym derivative.*
Indeed note that, if $x \in E^{est}$, for n sufficiently large we have $A_n(x) \subset E^c$ (and, analogously, if $x \in E^o$ $A_n(x) \subset E$). In fact, as $x \in E^{est}$, there exists a neighborhood $B = B(x, \delta)$ of x such that $B \subset E^c$. Now take $0 < \varepsilon < \delta$: there exists a set $A_n(x)$ of the net with $\operatorname{diam}(A_n(x)) < \varepsilon$ and $x \in A_n(x)$. Hence by construction $A_n(x) \subset B \subset E^c$. The same argument can be applied to $x \in E^o$. Therefore, when ν is a measure, for μ-almost all $x \in I$ there exists $\overline{n}(x) \in \mathbb{N}$ such that $\phi_{E,n}(x) = \psi_n(x)$ for every $n \geq \overline{n}(x)$.

Remark 2. *Intuitively, ϕ_E recalls the density of the measure $\delta\nu(\cdot; E)$ with respect to μ, i.e., for each $S \in \mathcal{B}$,*

$$\delta\nu(S; E) = \int_S \phi_E(x)d\mu, \tag{7}$$

Indeed, consider an interval $F \subset E^c$, and let $\{F^{j,\lambda}\}_\lambda$ be the net of all finite partitions of F. If $\lim_{n\to+\infty} \frac{\nu(E\cup A_n(x))-\nu(E)}{\mu(A_n(x))}$ is approximated by the step functions $\phi_{E,\lambda}$ defined by $\frac{\nu(E\cup F^{j,\lambda})-\nu(E)}{\mu(F^{j,\lambda})}$ on $F^{j,\lambda}$, in the sense of the L^1 norm on F, i.e.

$$\int_F |\phi_{E,\lambda} - \phi_E| d\mu \underset{\lambda}{\longrightarrow} 0, \tag{8}$$

then

$$\begin{aligned}
\sum_{j=1}^{n_\lambda} |\nu(E \cup F^{j,\lambda}) - \nu(E) - \int_{F^{j,\lambda}} \phi_E d\mu| &= \sum_{j=1}^{n_\lambda} |\int_{F^{j,\lambda}} \phi_{E,\lambda} d\mu - \int_{F^{j,\lambda}} \phi_E d\mu| \\
&\le \sum_{j=1}^{n_\lambda} \int_{F^{j,\lambda}} |\phi_{E,\lambda} - \phi_E| d\mu \\
&= \int_F |\phi_{E,\lambda} - \phi_E| d\mu \underset{\lambda}{\longrightarrow} 0.
\end{aligned}$$

Hence the measure $\int_\cdot \phi_E d\mu$ is precisely Epstein-Marinacci outer derivative.

We are now going to show that, for an important class of games, when μ-differentiability is assumed the intuition expressed in Remark 2 is correct, and so both the Epstein-Marinacci and the Epstein derivative can be represented in terms of (7). First, we recall two definitions from Aumann and Shapley (1971).

Definition 5. *A chain C* is a family of sets $\emptyset = S_0 \subset S_1 \subset \cdots \subset S_n = I$.
A *link* of a chain is a pair of consecutive elements $\{S_{i-1}, S_i\}$. A *subchain* of a chain is any set of links.

A chain will be identified with the subchain consisting of all the links. Given a game ν and a subchain Λ of a chain C, the variation of ν over Λ is defined as

$$\|\nu\|_\Lambda := \sum |\nu(S_i) - \nu(S_{i-1})|,$$

where the sum ranges over all indexes i such that $\{S_{i-1}, S_i\}$ is a link in the subchain.

Definition 6. If ν and w are two games defined on $\mathcal{B}$, ν is said to be *absolutely continuous with respect to w in the sense of Aumann and Shapley ($\nu \ll_{AS} w$)* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every chain C and every subchain Λ of C,

$$\|w\|_\Lambda \le \delta \Longrightarrow \|\nu\|_\Lambda \le \varepsilon.$$

A game ν is said to be *absolutely continuous in the sense of Aumann and Shapley* if there exists $\mu \in NA^+$ such that $\nu \ll_{AS} \mu$.
The space of absolutely continuous games in the sense of Aumann and Shapley will be, as it is customary, indicated by AC.

Lemma 1. *Suppose μ is a regular measure on $\mathcal{B}$. Then there exists an indefinitely fine net, and every indefinitely fine net is regular w.r.t. μ.*

Proof. Since I is a separable metric space, there exists an indefinitely fine net (Saks (2005), pag. 152). Furthermore, being μ regular, for every $E \in \mathcal{B}$ and every $\varepsilon > 0$, there exists an open set $O \supset E$: $\mu(O) < \mu(E) + \varepsilon$. Moreover O is a countable union of open balls $B(x, \delta)$. Let $\mathcal{R}$ be an indefinitely fine net; we claim that each ball $B(x, \delta)$ can be written as a countable union of sets A_i from the net $\mathcal{R}$. Indeed, take $y \in B(x, \delta)$ and let $0 < \overline{\delta} < \delta - d(x, y)$. Hence $B(y, \overline{\delta}) \subset B(x, \delta)$. Fix $0 < \varepsilon < \overline{\delta}$: as there exists A_i from $\mathcal{R}$ such that $\mathrm{diam(A_i)} < \varepsilon$ and $y \in A_i$, this implies $A_i \subset B(y, \overline{\delta})$, hence $\mathcal{R}$ is regular. □

We can now state our main result.

Theorem 2. *Let $\nu \in AC$, namely there exists $\mu \in NA^+$ such that $\nu \ll_{AS} \mu$. Suppose that ν satisfies* (**ii**) *of* **Definition 2** *at a set $E \in \mathcal{B}$. Then the function ϕ_E in (6) is in $L^1(I)$ and we have*

$$\Delta\nu(S; E) = \delta\nu(S; E) = \int_S \phi_E(x) d\mu. \tag{9}$$

Proof. Firstly, note that $\Delta\nu(\cdot; E)$ is absolutely continuous (with respect to μ) and is a measure. In fact, take $\varepsilon > 0$. Being $\nu \ll_{AS} \mu$, there exists $\delta_1 > 0$ such that,

$$|\nu(E \cup F) - \nu(E)| < \varepsilon \tag{10}$$

for every $F \in \mathcal{B}_{|E^c}$ with $0 < \mu(F) < \delta_1$ (take the chain $\emptyset \subset E \subset E \cup F \subset I$, and the subchain $\{E, E \cup F\}$).
Furthermore, there exists $\delta_2 > 0$ such that,

$$|\nu(E \cup F) - \nu(E) - \Delta\nu(F; E)| < \varepsilon\mu(F) < \varepsilon\mu(I) \tag{11}$$

for every $F \in \mathcal{B}_{|E^c}$ with $0 < \mu(F) < \delta_2$.
Combining (10) and (11) we get a $\delta^* > 0$ such that

$$|\Delta\nu(F; E)| \leq |\nu(E \cup F) - \nu(E)| + \varepsilon\mu(F) < \varepsilon(\mu(I) + 1) \tag{12}$$

for every $F \in \mathcal{B}_{|E^c}$ with $0 < \mu(F) < \delta^*$.
If otherwise $\mu(F)$=0, then $\mu(F^{j,\lambda}) = 0$ for each finite measurable partition $\{F^{j,\lambda}\}_{j=1,\dots,n_\lambda}$ of F. As $\nu \ll_{AS} \mu$ obviously implies property (**i**$'$) (and hence (**i**)) of Definition 2, we have $\nu(E \cup F^{j,\lambda}) = \nu(E)$ and ν is μ-differentiable. Hence, as $\sum_{j=1}^{n_\lambda} |\Delta\nu(F^{j,\lambda})| \geq |\Delta\nu(F)|$, differentiability by refinements (which is implied by μ-differentiability) yields $|\Delta\nu(F)|$=0. Concluding, $\Delta\nu(\cdot; E) \ll \mu$ on $\mathcal{B}_{|E^c}$. Analogously, $\Delta\nu(\cdot; E) \ll \mu$ on $\mathcal{B}_{|E}$. Moreover $\Delta\nu(\cdot; E)$ is countably additive, being absolutely continuous w.r.t. a measure (see Bhaskara Rao and Bhaskara Rao (1983), pag.163). As μ is regular, being finite on a Polish metric space (see Aliprantis and Border (1999)), by Lemma 1 there exists an indefinitely fine

regular net $\mathcal{R}$ and, by Theorem 1, the Radon-Nikodym derivative of $\Delta\nu(\cdot;E)$ with respect to $\mathcal{R}$ has the form expressed in (2), i.e.

$$D_{\mathcal{R}}\Delta\nu(\cdot;E)(x) = \lim_{n\to\infty}\psi_n(x), \tag{13}$$

μ-a.e. in I, where $A_n(x)$ is, as before, the only element of the n-th rank containing x; therefore

$$\Delta\nu(S;E) = \int_S \lim_{n\to\infty}\psi_n(x)d\mu, \tag{14}$$

for all $S \in \mathcal{B}$.
Now, as ν is μ-differentiable at E, choosing $\overline{n}$ sufficiently large in order to have $A_{\overline{n}}(x) \subset E^c$ we get

$$\lim_{n\to+\infty}|\phi_{E,n}(x) - \psi_n(x)| = 0, \tag{15}$$

for μ-almost all $x \in E^{est}$.
In fact, if $\mu(A_{\overline{n}}(x)) = 0$, then $\mu(A_n(x)) = 0$ for each $n \geq \overline{n}$, from where $|\phi_{E,n}(x) - \psi_n(x)| = 0$ for each $n \geq \overline{n}$. If otherwise $\mu(A_{\overline{n}}(x)) > 0$, then anyway $\mu(A_n(x)) \to 0$ for $n \to +\infty$, and hence by μ-differentiability

$$\lim_{n\to\infty}|\phi_{E,n}(x) - \psi_n(x)| = \lim_{\mu(A_n(x))\to 0}|\phi_{E,n}(x) - \psi_n(x)| = 0.$$

Indeed, let $n \geq \overline{n}$. Then, being the net indefinitely fine, there exists $m_n \in \mathbb{N}$: $\text{diam}A_{m_n}(x) < \dfrac{1}{n}$. Notice that $\displaystyle\bigcap_{n\geq\overline{n}} A_{m_n}(x) = \{x\}$, as if $y \in \displaystyle\bigcap_{n\geq\overline{n}} A_{m_n}(x)$ then $d(x,y) < \frac{1}{n}$ for every $n \geq \overline{n}$ and hence $x = y$. Therefore $\displaystyle\bigcap_{n\geq\overline{n}} A_n(x) = \{x\}$.
Hence, by continuity and by the non-atomicity of μ it turns out $\mu(A_n(x)) \to \mu(\{x\}) = 0$.
We claim that $\bigcap_{n\geq\overline{n}} A_{m_n}(x) = \{x\}$ implies $\bigcap_{n\geq\overline{n}} A_n(x) = \{x\}$. Take an element $y \in \bigcap_{n\geq\overline{n}} A_n(x)$ (this intersection is nonempty as it contains x). W.l.o.g. we can suppose that $m_1 \leq m_2 \leq \dots$, so that $A_{m_1} \supseteq A_{m_2} \supseteq \dots$. If there exists $n^* \geq \overline{n}$ such that $m_{n^*} \geq \overline{n}$ then $y \in A_{m_n}(x)$ for every $m_n \geq m_{n^*}$. But y belongs also to all the $A_{m_n}(x)$ with $m_n < m_{n^*}$ as $A_{m_n}(x) \supseteq A_{m_{n^*}}(x)$ for all $m_n < m_{n^*}$. Then $y \in \bigcap_{n\geq\overline{n}} A_{m_n}(x)$ and hence $y = x$. If, otherwise, for every $n \geq \overline{n}$ one has $m_n \leq \overline{n}$, then $A_n(x) \subseteq A_{m_n}(x)$ for every $n \geq \overline{n}$. Hence $\bigcap_{n\geq\overline{n}} A_n(x) = \{x\}$.
Therefore $\displaystyle\lim_{n\to+\infty}\phi_{E,n}(x)$ exists in $\mathbb{R}$ by (13) and (15), for almost all $x \in E^{est}$, and it holds:

$$\lim_{n\to+\infty}\phi_{E,n}(x) = \lim_{n\to+\infty}\psi_n(x). \tag{16}$$

Hence, from (14) and (16) we have

$$\Delta\nu(F;E) = \int_F \lim_{n\to+\infty}\phi_{E,n}(x)d\mu = \int_F \phi_E(x)d\mu, \tag{17}$$

for every $F \in \mathcal{B}_{|E^c}$. Proceeding analogously for $G \subset E$, we get the thesis. $\square$

The following result shows that, at least for the important class of measure games, formula (7) holds, also without the μ-differentiability assumption.

Definition 7. A game ν is called a *vector measure game* if there exists a vector measure $P = (P_1, \dots, P_N)$, with $P(I) \neq 0$ and each $P_i \in NA$, and a real valued function g defined on the range of P, with $g(0) = 0$, such that $\nu = (g \circ P)$.

We recall that the differentiability hypothesis on g is not sufficient to assure that $\nu \in AC$ (see for example Aumann and Shapley (1974), Tauman (1982)), hence the following result can not be derived from the previous one even assuming μ-differentiability.

Proposition 1. *Let $\nu = (g \circ P) : \mathcal{B} \to \mathbb{R}$ be a vector measure game ($P = (P_1, \dots, P_N)$), with each $P_i \in NA^+$) and set $\widehat{P} := P_1 + \dots + P_N$. If g is differentiable at $P(E)$ (with $P_i(\partial E) = 0$, $\forall i$), then the function ϕ_E in (6) is well defined, and for every $S \in \mathcal{B}$ it holds:*

$$\int_S \phi_E(x) d\widehat{P} = \delta\nu(S; E) = \nabla g(P(E)) \cdot P(S),$$

Proof. First, we arbitrarily introduce the measure $\widehat{P} = P_1 + \dots + P_N$, as each of the P_i's is absolutely continuous with respect to $\widehat{P}$: hence we can apply apply the Radon-Nikodym Theorem in the form of Theorem 1, with $\theta_i = P_i$, and $\mu = \widehat{P}$, for each $i = 1, \dots, N$. Furthermore, as $\widehat{P}$ is regular on $\mathcal{B}$, by Lemma 1 there exists an indefinitely fine net $\mathcal{R}$, regular w.r.t. $\widehat{P}$. Take $E \in \mathcal{B}$ with $\widehat{P}(\partial E) = 0$ and such that g is differentiable at $P(E)$. By the additivity of the integral and of the P_i's, it suffices to prove the result for a Borelian $F \subset E^c$. For every $x \in F$, let $A_n(x)$ be the only element of the n-th rank of $\mathcal{R}$ containing x. For n sufficiently large $E \cap A_n(x) = \emptyset$, therefore we have:

$$\begin{aligned} \nu(E \cup A_n(x)) - \nu(E) &= g(P(E \cup A_n(x)) - g(P(E)) \\ &= g(P(E) + P(A_n(x))) - g(P(E)). \end{aligned} \tag{18}$$

Hence, (18) and the differentiability of g at $P(E)$ imply:

$$\begin{aligned} \nu(E \cup A_n(x)) - \nu(E) = \nabla g(P(E)) \cdot P(A_n(x)) + o(\|P(A_n(x))\|), \\ \text{for} \quad \|P(A_n(x))\| \to 0 \end{aligned} \tag{19}$$

(where $\|P(A_n(x))\| = \widehat{P}(A_n(x))$ and it is assumed to be non null). Thus

$$\frac{\nu(E \cup A_n(x)) - \nu(E)}{\widehat{P}(A_n(x))} = \frac{\nabla g(P(E)) \cdot P(A_n(x)) + o(\|P(A_n(x))\|)}{\widehat{P}(A_n(x))}, \tag{20}$$

that is

$$\frac{\nu(E \cup A_n(x)) - \nu(E)}{\widehat{P}(A_n(x))} = \sum_{i=1}^{N} g_i(P(E)) \frac{P_i(A_n(x))}{\widehat{P}(A_n(x))} + \frac{o(\|P(A_n(x))\|)}{\widehat{P}(A_n(x))} \tag{21}$$

(where g_i denotes the i-th partial derivative of g). Now, as in (3) and (5), define ψ_n^i for each $i = 1, \cdots, N$, and $\phi_{E,n}$. Then, by Theorem 1, $\lim_{n\to+\infty} \psi_n^i(x)$ exists on a set of full $\widehat{P}$-measure, and it holds:

$$P_i(F) = \int_F \lim_{n\to+\infty} \psi_n^i(x) d\widehat{P}. \tag{22}$$

Therefore, (21) and the existence of the limit of the ψ_n^i imply that $\lim_{n\to+\infty} \phi_{E,n}(x)$ exists $\widehat{P}$- almost everywhere in I and

$$\begin{aligned} \int_F \lim_{n\to+\infty} \phi_{E,n}(x) d\widehat{P} &= \sum_{i=1}^N g_i(P(E)) \int_F \lim_{n\to+\infty} \psi_n^i(x) \\ &= \nabla g(P(E)) \cdot P(F). \end{aligned} \tag{23}$$

The result follows now from Epstein and Marinacci (2001), Lemma 3.2. □

As pointed out in **Remark 1**, so far it remains unclear whether the representation result can be extended to more general classes of games than vector measure ones, without the μ-differentiability assumption. Also the link with Aumann and Shapley derivative (see Aumann and Shapley (1974)) is still an open task.

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