Economics Bulletin

Volume 36, Issue 2

Loss Aversion and Ruinous Optimal Wagering in the Markowitz Model of Non-Expected Utility

David Alan Peel
University of Lancaster Dept.Economics

David Law
Bangor

Abstract

The purpose in this note is to demonstrate that the non-expected utility model of Markowitz implies that agents can obtain maximum expected utility from wagering all of their wealth on actuarialy unfair high probability outcomes. In order to remove this property it is necessary to assume that loss aversion tends to infinity as stake size as a proportion of wealth approaches unity.

Citation: David Alan Peel and David Law, (2016) "Loss Aversion and Ruinous Optimal Wagering in the Markowitz Model of Non-Expected Utility", *Economics Bulletin*, Volume 36, Issue 2, pages 688-695
Contact: David Alan Peel - d.peel@lancaster.ac.uk, David Law - dal@aber.ac.uk.
Submitted: September 21, 2015. **Published:** April 14, 2016.

1 Introduction

Markowitz (1952) in a seminal paper proposed a model of non-expected utility that could explain wagering at actuarially unfair odds. He assumed that from a reference point, such as an agent's customary or normal level of wealth, agents are initially risk-loving then risk-averse over gains whilst initially risk-adverse then risk-seeking over losses. The value functions over gains and losses were assumed to be bounded from above and below to avoid the St. Petersburg paradox (Markowitz (1952) p154). Markowitz also assumed that the representative agent is loss-averse[1] and does not exhibit probability distortion. Conlisk (1993) stated that the Markowitz model of utility had been relatively ignored by economists and that is arguably still the case even though the paper now has 1444 citations on Google Scholar. Like Cumulative Prospect Theory (CPT) of Kahneman and Tversky[2] (1979), widely regarded as the major alternative to expected utility theory, the Markowitz model implies the fourfold attitude to risky choice exhibited in numerous experimental studies. The fourfold attitude towards risk implies that when payoffs are large, individuals are risk averse in gains and risk seeking in losses and when the payoffs are small individuals are risk seeking in gains and risk averse in losses. See e.g. Scholten and Read (2014) for recent evidence.

One motivation of Markowitz for his new model of utility was to remove the property of the Friedman and Savage (1948) model of expected utility that individuals when in the risk seeking segment of their utility function derive positive expected utility from potentially ruinous bets by wagering all of their wealth at actuarially unfair odds. This implication runs counter to common observation.

Our purpose in this note is to demonstrate that Markowitz's original specification of non-expected utility also embodies the ruinous wagering property. We show that in order to remove this property it is necessary to make the additional assumption that the degree of loss aversion approaches infinity as stake size as a proportion of wealth approaches unity.

---

[1] Markowitz wrote, 'Generally people avoid symmetric bets. This suggests that the curve falls faster to the left of the origin than it rises to the right. We may assume that $|U(-X)| > U(X), X > 0$ where X = 0 is customary wealth' (Markowitz, 1952, pp. 154-155). This definition of loss aversion was also employed by Kahmenan and Tversky (1979) in their model of Cumulative Prospect Theory.

[2] Cumulative Prospect Theory of Kahnerman and Tversky (1979) and Tversky and Kahnerman (1992) embody the reference point hypothesis of Markowitz and the assumption of loss aversion. However it is assumed that subjects are solely risk-averse over gains and solely risk-loving over loses. To explain gambling on actuarially unfair outcomes it is assumed in CPT that agents subjectively distort the probabilities of events, *via* an inverted s-shaped probability weighting function, so that low probabilities are over-estimated by the representative agent and high probabilities are under estimated.

The remainder of the note is structured as follows. In section 2 we set out a parametric formulation of the Markowitz model and illustrate the ruinous gambling feature. We then present a parametric version of a Markowitz model that eliminates the ruinous wagering property. Section 3 is a brief conclusion.

Section 2 Ruinous Wagers and Loss Aversion

Expected utility or value, EV, for a one outcome wager is given in the Markowitz model by

$$EV = pU^{g}(G) - (1-p)kU^{l}(L) \qquad \textbf{(1)}$$

Where p is the probability of gain, G is the gain and L is the loss. $U^{g}(G)$ Is the value function over gains, $U^{l}(L)$ the value function over losses and k is a positive constant.

The assumption of Markowitz that the value functions are bounded from above and below is captured by imposing the restrictions on the value functions that when gains and losses are infinite $U^{g}(\infty)=1, U^{l}(\infty)=1$ and $U^{g}(0)=0$ and $U^{l}(0)=0.$ This implies the value of gains lay between zero and unity and value of losses between zero and minus k. In the context of wagering on a one outcome gamble or lottery the gain, G, is equal to so and Loss, L, is equal to s, where s is stake size and o=odds.

To illustrate the ruinous property assume for simplicity that stake size is infinite.

From equation (1) this implies that

$$EV = p - (1-p)k \qquad (2)$$

This further implies that expected value, EV, is positive when

$$p > \frac{k}{1+k} \qquad (3)$$

The condition (3) implies the Markowitz agent would accept an actuarially unfair wager of any magnitude if the probability of winning, p, exceeds $\frac{k}{1+k}$.

The degree of loss aversion, LA, is defined by Markowitz (1952) and Tversky and Kahneman (1992) as the ratio of the absolute value of loss of expected value to the gain in expected value from a symmetric gamble. In the context of wagering this implies that loss aversion for stake size, s, is given by

$$LA = \frac{kU^{l}(s))}{U^{g}(s)} \qquad (4)$$

For infinitely large stakes with symmetric gains and losses the boundedness assumptions for the value functions imply from (4) that loss aversion, A, will be equal to k. Consequently it follows from (3) that unless k approaches infinity as stake size approaches infinity agents will obtain positive expected value from wagers with high probability of win regardless of how actuarially unfair[3].

Of course individuals do not wager infinite amounts so we now consider how practically relevant this point is employing a parametric model of the Markowitz model.

As noted by Abdellaoui et al (2007) a parametric form of the Markowitz model can be obtained based on the expo-power function of Saha (1993). For this value function expected utility or value, EV, for a wager with one payoff is given by

$$EV = p(1-e^{-r\lambda(so)^n}) - (1-p)k(1-e^{-\lambda s^n}) \qquad (5)$$

Where r, λ and n are positive constants. When the exponent of the value functions n is greater than unity we obtain the form of value functions hypothesized by Markowitz. The agent is initially risk-loving then risk-averse over gains, as $\frac{(n-1)}{r\lambda n}$ is either greater or less than $(so)^n$ and initially risk-averse then risk-loving over losses, as $\frac{(n-1)}{\lambda n}$ is greater or less than s^n.

Employing the expo-power value functions in (5) the degree of loss aversion as defined by Markowitz, LA, for symmetric gains and losses of size, s, (i.e. an even money wager) is given by

$$LA = \frac{k(1-e^{-\lambda s^n})}{(1-e^{-r\lambda s^n})} > 1 \qquad (6)$$

From (6), employing L'Hopital's Rule, we find that as stake size approaches zero the degree of loss aversion, LA, is given by the lower limit of $\frac{k}{r}$ and as stake size approaches infinity by the upper limit of k. Consequently for the expo-power value functions with the parameter k assumed constant loss aversion lies between the upper and lower limits of

$$k > \frac{k}{r} > 1 \qquad (7)$$

In order to ensure that the marginal value of a loss always exceeds the marginal value of a gain, for symmetric gains and losses of size s, we differentiate (6) with respect to stake size and deduce that the constant r must be greater than unity.[4]

[3] Peel and Law (2008) noted this property but did not provide a solution.

[4] It is important to note that qualitatively the same properties of the expo-power value specifications of the Markowitz value functions are also obtained in the only two other

We can determine the stake that maximizes expected value for the expo-power value functions by differentiating (6) with respect to stake size. (See Peel and Law (2009)) We obtain the stake as

$$s=\left(\frac{\ln[\frac{rpo^n}{k(1-p)}]}{\lambda(ro^n-1)}\right)^{\frac{1}{n}} \text{ Or } s=\left(\frac{\ln[\frac{r(1+\mu)o^n}{k(o-\mu)}]}{\lambda(ro^n-1)}\right)^{\frac{1}{n}} \qquad (8)$$

The second order condition $ro^n - 1 > 0$ ensures a local maximum.
The expected return to a one unit stake is defined by $\mu = po - (1-p)$.
We note that since n>1, the assumption in the Markowitz model, for a given negative expected return, μ, and fixed degree of loss aversion over tiny stakes, $\frac{k}{r}$, the numerator in (8) will always become positive for large enough odds, o. Consequently the Markowitz agent will always obtain positive expected value by wagering on actuarially unfair outcomes that have large enough odds.
However as we will illustrate below the stake in (8) will not be a global maximum if the agent's wealth is sufficiently large.
Consider a Markowitz agent with the expo-power value functions defined by (5) with parameter values n= 1.25, r= 2, k= 4.5 and $\lambda = 0.00001$. Loss aversion for small stakes, $\frac{k}{r}$, with these parameter values is approximately 2.25 the degree of loss aversion assumed by Tversky and Kahneman (1992).
This agent would obtain positive expected value of approximately $Eu = \frac{0.06}{10000}$ by wagering \$8 on one number at US roulette with o=35, p=1/38 and negative expected return to a unit stake of -0.0526. This is illustrated in Figure1.

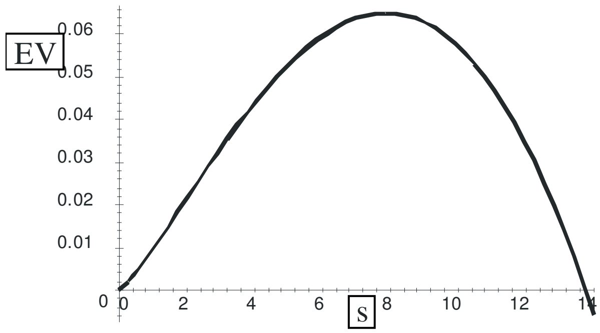
---

specifications of the Markowitz value functions we are aware of. For example the double exponential form specification of the Markowitz value functions, (see Cain et al (2003), where we employ the same constants, though with usually different values, for ease of interpretation, is given by

$$EV = p(1-e^{-r\lambda so} - r\lambda so e^{-r\lambda so}) - (1-p)k(1-e^{-\lambda s} - \lambda s e^{-\lambda s})$$

These value functions have a lower limit for loss version of $\frac{k}{r^2}$, and an upper limit of k with $k > \frac{k}{r^2} > 1$ and r>1.

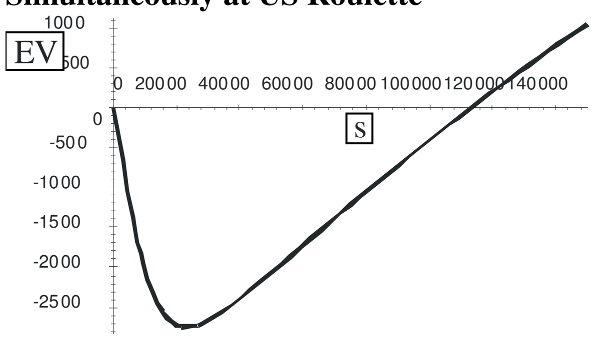
Figure1 Expected Value for a Wager of Size (s) on One Number at US Roulette



EV is multiplied by 10000.

However the same agent could obtain maximum expected value, which is up to approximately 34,000 times greater than the expected value of the one number wager, by wagering all their wealth on 35 numbers simultaneously at US roulette if their wealth exceeded \$113105. For this wager on US roulette p=35/38, o=1/35 and expected return to a unit wager is -0.0526. The expected value of this wager is illustrated in Figure2.

Figure 2 Expected Value for a Wager (s) on Thirty Five Numbers Simultaneously at US Roulette



EV is multiplied by 10000

It is important to note that the parameter values assumed for the value functions in this example are not special. In fact any parameter values that imply that wagers on low probability long shots have positive expected value, a key prediction of the Markowitz model, will also imply potentially ruinous wagers on high probability outcomes because of property (3).

We assume that individuals with large wealth levels would not obtain expected value from engaging in potentially ruinous optimal wagering. Consequently it is an unreasonable prediction of the Markowitz model[5]. To remove the ruinous

[5] Another example with the parameters n=2, r=2, k=4.5 and $\lambda = 0.0002$ an individual would maximize expected utility or value by wagering all their wealth when p=0.85

wagering property it appears necessary to assume that the parameter k, that determines the higher limit for loss aversion, is dependent on stake size as a proportion of wealth becoming infinite when stake size is equal to total wealth[6]. It would be nice if we could modify the specification of the value function over losses, $U^1(L)$, and embody the assumption that loss aversion becomes infinite as loss approaches infinity. However it appears that no modifications of the value function over losses exist which preserve the properties that individuals are initially risk-averse and then risk-seeking over losses and also that loss aversion is always increasing in loss size.

One example of a function for k which captures the desired loss aversion property is

$$k = \frac{k_0(e^{\beta} - 1)}{e^{\beta} - e^{\beta\frac{s}{w}}} \qquad (9)$$

Where k_0 and β are constants with $k_0 > 0$ and $\beta < 0$ and $\frac{s}{w}$ is stake size as a proportion of wealth.

For example with $k_0 = 4.5$ and $\beta = -5$ we obtain values of

$k = 4.73,\ 7.45, 20.65$ and 1022.6 when $\frac{s}{w}$ =0.01, 0.1, 0.3 and 0.9 respectively.

Larger values of the parameter β imply that loss aversion rises less rapidly as $\frac{s}{w}$ increases.[7] Since the function (9) can exhibit a low degree of loss aversion when stake size is a small proportion of wealth individuals will still wager at actuarially unfair odds but no longer obtain positive expected utility from engaging in potentially ruinous wagers.

To illustrate with parameter values of n= 1.25, r= 2, $\lambda = 0.00001$, the same parameter values as employed in first example, and with $k_0 = 4.5$, $\beta = -5$ and initial wealth of $200000 an individual would optimally stake approximately $8 on one number on US roulette as in the example above. However the

---

o=0.00005 and expected return to a unit bet of -0.15, but only when stake size is greater than $397849.

[6] Assuming loss aversion depends on stake size as a proportion of wealth appears in the spirit of the Markowitz model. Unlike in CPT Markowitz assumed an agent's wealth level would impact on the curvature of the value functions. In particular Markowitz assumed the distance between the first and third inflexion points of the value functions was not independent of the agent's wealth level which he assumed were closer to the reference point for a less wealthy individual. (Markowitz p154-155.)

[7] For example with $\beta = -0.5$ k=57 when $\frac{s}{w}$ =0.9

individual will now obtain negative expected value from wagering all their wealth on 35 numbers simultaneously[8].

3 Conclusion

One motivation for Markowitz's specification of a non expected utility model was to remove the counterfactual property of the Friedman and Savage model of expected utility that individuals, when in the risk seeking segment of the utility function, could obtain maximum expected utility by wagering all their wealth on actuarially unfair gambles. We showed that the Markowitz model also embodied this property. To remove this property we showed that it is necessary to assume that loss aversion tends to infinity as stake size as a proportion of wealth tends to unity.

References

Abdellaoui, M. Barrios, C. and Wakker, P. P. (2007) "Reconciling introspective utility with revealed preference: Experimental arguments based on prospect theory", *Journal of Econometrics*, 2, 356-78.

Cain, M., Law, D. and Peel, D.A. (2003) ``the Favourite-Longshot Bias and the Gabriel and Marsden Anomaly: An Explanation Based on Utility Theory." In The Economics of Gambling, edited by L. Vaughan Williams, 2-13. London: Routledge.

Conlisk, J. (1993) "The Utility of Gambling". *Journal of Risk and Uncertainty*, 6:255-275.
Friedman, M. and Savage, L.J. (1948) "The Utility Analysis of Choices Involving Risk". *The Journal of Political Economy"*, 56, 279-304.

Kahneman, D. and Tversky, A. (1979) "Prospect theory: an analysis of decision under risk". *Econometrica*, 47, 2, 263-91.

Markowitz, H. (1952) "The Utility of Wealth". *The Journal of Political Economy*, 60, 151-158.

Peel, D.A. and Law, D. (2008) "Further analysis of the Markowitz Model of Utility with a small degree of probability distortion". *The Journal of Gambling, Business and Economics*, 2, 71-83.

[8] Note that we can solve the Markowitz model for the optimal stake to wealth ratio rather than the level of stake size by a simple transformation in the expo-power model (5) if we divide the stake by the level of existing wealth we can define a new constant $\lambda^1 = \lambda w^n$.

Peel, D. A. and Law, D, (2009) “A More General Non-expected Utility Model as an Explanation of Gambling Outcomes for Individuals and Markets”, *Economica*, 76, 251-263.

Saha, A. (1993) “Expo-Power Utility: A Flexible Form for Absolute and Relative Risk Aversion”. *American Journal of Agricultural Economics*, (75), 905-13.

Scholten, M. and D. Read, (2014) “Prospect theory and the 'forgotten' fourfold pattern of risk preferences,” *Journal of Risk and Uncertainty,* 48, 67-83.

Tversky, A. and Kahneman, D. (1992) “Advances in prospect theory: cumulative representation of uncertainty”. *Journal of Risk and Uncertainty*, vol. 5, 297–323

.