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The Good Candidate

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Abstract

A Good Candidate is an alternative which is top ranked for at least half of the agents. Meanwhile, a Majority Winner is the alternative that is preferred by a majority of agents by pair-wise comparison to any other available alternative. Hence, to have a majority winner, the social majority relation must be acyclic, while in order to find a good candidate this condition is not necessary. In this note, we highlight the fact that there is a close relationship between the good candidate and a majority winner. In particular, they coincide for triples of alternatives whenever we work with a reduced profile of preferences where mutually exclusive preferences are removed. If the number of alternatives increases, then a condition by triples should be accomplished: an alternative should be the good candidate for any triple where it is. Furthermore, we propose a way to compute a majority winner via triples of alternatives.

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1. Introduction

If there exists an alternative which is the most preferred by a majority of people, then it is desirable to be selected for at least half of the society. We call this alternative, the *good candidate*. Hence, our analysis applies to social choice problems for which it can be taken as axiomatic that an alternative should be selected if a majority of voters declare it to be their most preferred alternative. Any social choice rule selecting this alternative is efficient (Moulin, 1988). On the other hand, a *majority winner* is never defeated in pairwise comparison to any other available alternative. A wide variety of social choice rules selects this alternative, such as Majority voting, Copeland or Simpson. Furthermore, these rules are not only efficient but also not manipulable at those profiles where a majority winner exists. Our note is not intended to clarify the conditions in the preference profile to ensure the existence of such alternative. Instead, we highlight the fact that there is a close relationship between a good candidate and a majority winner. We consider the case when there are strictly more than two alternatives. As it is known by May (1952), in the case where there are two candidates, these two concepts coincide and majority voting will be the best rule to make a social decision. Furthermore, we propose a way to compute a majority winner.

Economics has studied the problem to ensure the existence of a majority winner finding the conditions that preference profiles must satisfy. Sen and Pattanaik (1969) laid the groundwork to approach this problem. In particular, they proposed the Value Restriction as a sufficient condition for the existence of a majority winner. This restriction is defined as follows: for each *triple* of alternatives, there must exist an alternative which all agents agree it is not the worst, or agree it is not the best, or agree it is not the medium. The most used preference profiles with this property are single-peaked preferences and single-dipped preferences. These too, have been characterized in terms of triples of alternatives, as in Ballester et. al (2006). Xefteris (2012) demonstrated, using triples of alternatives, that the majority winner will exist if the preference profile is not balanced; that is, there is an alternative such that for a majority of people it is considered the best or the worst given a simplified profile of preferences.

We start this note with the definition of a good candidate. To follow the structure these authors and others have taken, we then define a majoritarian alternative and then propose an alternative definition of a Condorcet winner under triples of alternatives. In section 2 we give the basic notation, then we follow with the presentation of our results in section 3. We close this note with some final remarks and an extension of our main result.

2. Notation

Let A be a finite set of alternatives with cardinality three, a triple; and let N be the set of agents with cardinality n. Agents' preference relations are linear orderings on A. For any agent i, let $\mathcal{P}_i$ be the set of all linear orderings of agent i. We denote as P_i, the preferences of agent i relative to A and $t(P_i)$ as the most preferred alternative of agent i at P_i. For any $N \subset \mathbb{N}_+$, preference profiles of the agents in N are elements of $\times_{i \in N} \mathcal{P}_i$ and they are denoted by $P = (P_1, .., P_n)$.

For any $N \subset \mathbb{N}_+$, $P \in \times_{i \in N} \mathcal{P}_i$, and $x \in A$, let $F^x(P)$ be the set of agents that have alternative x as their most preferred alternative at profile P, i.e. $F^x(P) =$

$\{i \in N : x = t(P_i)\}$. Let $f^x(P)$ be number of agents that regard alternative x as the best alternative at profile P; namely, $f^x(P) = \#F^x(P)$.

DEFINITION 1. For any $P \in \times_{i\in N}\mathcal{P}_i$, $x \in A$ is a **Good Candidate** relative to N if and only if $f^x(P) \geq \frac{n}{2}$.

DEFINITION 2. For any $P \in \times_{i\in N}\mathcal{P}_i$, $x \in A$ is the **Best Candidate** relative to N if and only if $f^x(P) > \frac{n}{2}$.

These are the most preferred alternatives for at least half of the agents; hence, a good candidate.

For any $P \in \times_{i\in N}\mathcal{P}_i$ and any $x, y \in A$, let $N(x,y) = \{i \in N : xP_iy\}$ be the set of agents that strictly prefer x over y. We say that x is weakly preferred by majority to y (we may also say that x is not defeated by y), xR_my, if and only if $N(x,y) \geq N(y,x)$. And x is strictly preferred by majority to y (we may also say that x defeats y), xP_my, if and only if $N(x,y) > N(y,x)$. We refer to R_m as the **majority rule relation**.

Looking at a set of alternatives and a preference profile, we can find two different types of cycles in the majority rule relation:

DEFINITION 3. For any $P \in \times_{i\in N}\mathcal{P}_i$, R_m exhibits a **strict cycle** at P if xP_my, yP_mz, and zP_mx.

DEFINITION 4. For any $P \in \times_{i\in N}\mathcal{P}_i$, R_m exhibits a **weak cycle** at P if zI_mx, xP_my, and yP_mz .

Given a majority rule relation, we call the alternative that is not defeated the *majority winner alternative* relative to A. If R_m does not exhibit a **strict cycle,** then the majority alternative exists relative to A. In particular, if xP_my, yP_mz, and $(zP_mx)^\urcorner$, then x is majority winner alternative and we say that R_m is *strictly acyclic*. But if R_m exhibits a **weak cycle** then, there will not be any strong majority winner, but there exists an alternative which is never defeated. Hence, if zI_mx, xP_my, and yP_mz then, even if x does not defeat by majority all the rest of the alternatives, x never looses as it ties with z and wins over y and we will say that R_m is *weakly acyclic*. Hence, as long as x is not defeated, we will say it is a majority winner. We will denote as $MW(P)$ the set of the majority winners.

It should be noticed that to find a good candidate, we are not imposing the existence of an acyclic majority rule relation.

3. Results

As it should be clear by now, there is a close relationship between a good candidate and a majority winner. Using an analysis via triples of alternatives, in this section we will show the link between them and we will propose a different way to compute the majority winner without using a pairwise comparison.

For any agent i and any $P_i \in \mathcal{P}_i$, there are six possible linear orderings relative to A. Among them, we can find pairs of preferences that are opposite to each other. The identification of such pairs of preferences will help us to compute the majority decisions, as they become irrelevant in the comparison between agents prefering one alternative to another.

Definition 5. For any $P \in \times_{i \in N} \mathcal{P}_i$ and A, we say that agents $i, j \in N$ have **mutually exclusive** preferences P_i and P_j at profile P relative to A if, for any $x, y \in A$, $xP_iy \Longleftrightarrow yP_jx$.

Example 1. Let $N = \{1, 2, 3, 4, 5, 6, 7\}$ and consider the following preference profile P

Table I. Preference profile P

$P_1 = P_2 = P_3$	$P_4 = P_5$	P_6	P_7
z	x	y	y
y	z	x	z
x	y	z	x

Agents 4 and 7 have *mutually exclusive* preference relations.

For any $P \in \times_{i \in N} \mathcal{P}_i$, let M be a subset of N that contains all the agents that have *non-mutual exclusive* preference relations relative to A at profile P. Hence, M is constructed by eliminating pairs of mutually exclusive preference relations. We refer to $\widehat{P}$ as the **net profile** of P relative to A and to M as the **mutually exclusive set of agents** at P relative to A. Likewise, $\widehat{N}(x, y) = \{i \in N : x\widehat{P}_iy\}$.

The reduction of the preference profile from P to $\widehat{P}$ keeps the same set of *majority alternative winners* relative to A. We use Example 2 to clarify these concepts.

Example 2. Consider N and P as in Example 1. Notice that $M = \{1, 2, 3, 5, 6\}$. Hence, $\widehat{P}$ is given in the following table

Table II. Preference profile $\widehat{P}$

$\widehat{P_1} = \widehat{P_2} = \widehat{P_3}$	$\widehat{P_5}$	$\widehat{P_6}$
z	x	y
y	z	x
x	y	z

Note that zP_mx, yP_mx, and zP_my, hence there is no cycle. Furthermore, $MW(P) = MW(\widehat{P}) = \{z\}$ and $f^z(\widehat{P}) \geq \frac{\#M}{2}$.

Our main observation relies on the fact that under the preference profile $\widehat{P}$, $MW(\widehat{P}) \neq \emptyset$ if and only if there exists x such that $f^x(\widehat{P}) \geq \frac{\#M}{2}$ in the set of alternatives A. Furthermore, $MW(P) = MW(\widehat{P})$. Hence, it is enough to look at the top alternative of the agents within the net profile. The good candidate for the agents M will coincide with the non-looser alternative for the set of agents N, a majority winner. Therefore, if there are only three alternatives, the good candidate relative to M and the majority winner relative to N, coincide whenever it exists. Xefteris (2012) worked at the condition to have a majority winner in triples, and started by the reduction of the preference profile using the mutually exclusive set of agents as well. Hence, his results reinforce the fact that the majority winner will not change by this reduction and we can use his results to determine whether it exists or not. The following lemma and proposition support the assertion.

LEMMA 1. *For any $P \in \times_{i\in N}\mathcal{P}_i$ and any $x \in A$, there exists $y \in A\backslash\{x\}$ so that yP_jx for all $j \in M$ such that $x \neq t(P_j)$.*

Proof. Let $P \in \times_{i\in N}\mathcal{P}_i$ and $x \in A$. If there is $j \in M$ such that $x \neq t(P_j)$, then either yP_jzP_jx, or zP_jyP_jx, or yP_jxP_jz, or zP_jxP_jy. Since any pair of agents in M have non-mutual exclusive preference relations relative to P, either there is no $j \in M$ such that yP_jxP_jz or no $j \in M$ such that zP_jxP_jy. In the first case, zP_jx for all $j \in M$. In the second case, yP_jx for all $j \in M$. ∎

Hence, if no two individuals have opposite preferences, then for any alternative x there is an alternative y that ranks above x in the ordering of any individual who does not rank x first.

PROPOSITION 1. *For any $P \in \times_{i\in N}\mathcal{P}_i$, R_m is strictly acyclic if and only if there exists a good or the best candidate for the mutually exclusive set of agents.*

Proof. Let $N \subset \mathbb{N}_+$, $P \in \times_{i\in N}\mathcal{P}_i$, and $x \in A$ be such that $f^x(\widehat{R}) \geq \frac{\#M}{2}$. We show that R_m is *strictly acyclic* relative to A. Since $f^x(\widehat{P}) \geq \frac{\#M}{2}$, we have that $\widehat{N}(x,y) \geq \widehat{N}(y,x)$ and $\widehat{N}(x,z) \geq \widehat{N}(z,x)$. Since for any $a,b \in A$, $N(a,b) \geq N(b,a)$ if and only if $\widehat{N}(a,b) \geq \widehat{N}(b,a)$, then we have that xR_my and xR_mz. Therefore, in either case where $N(z,y) \geq N(y,z)$ or $N(y,z) \geq N(z,y)$, R_m is *strictly acyclic* relative to A.
Let $P \in \times_{i\in N}\mathcal{P}_i$, be such that R_m is *strictly acyclic* relative to A. We have to show that for some $x \in A$, $f^x(\widehat{R}) \geq \frac{\#M}{2}$. Suppose, to get a contradiction, that for all $y \in A$, $f^y(\widehat{R}) < \frac{\#M}{2}$. By Lemma 1 for all $x \in A$, there exists $y \in A\backslash\{x\}$ such that yP_mx. Hence, without loss of generality, yP_mx, xP_mz and zP_mx contradicting that R_m is strictly acyclic. ∎

Given Proposition 1, there is an equivalence between the good candidate, given the net preference profile $\widehat{P}$, and the majority alternative using the preference profile, P. In other words, when there are three alternatives, $x \in A$ is a Good Candidate relative to M if and only if it is a majority alternative relative to N.

But the observation goes beyond. We can see that we are talking about the so called, Condorcet Winner. Hence, this statement is a different way of defining a Condorcet Winner based on triples of alternatives.

DEFINITION 6. For any $R \in \times_{i\in N}R_i$, $x \in A$ is a **Condorcet winner** if and only if it is a *good candidate* relative to A and $M \subseteq N$.

DEFINITION 7. For any $P \in \times_{i\in N}\mathcal{P}_i$, $x \in A$ is a **strong Condorcet winner** if and only if it is the best candidate relative to A and $M \subseteq N$.

4. Final Remarks

In this note we focused only in the case where the total number of alternatives is three, a triple. However, the result can be extended to any cardinality of the set of alternatives. If $\#A > 3$ then, there are two conditions to find the equivalence between a majority winner relative to P and a good candidate relative to the net preference profile. The first one is that there exists an alternative such that it is the majority alternative relative to every triple, $T \subset A$. The second is that if there is an alternative $x \in A$ such that $xP_m y$, but y is a majority alternative in T', then x should be a majority alternative relative to every $T \subset T' \cup \{y\}$. We give the following example to clarify the idea.

EXAMPLE 3. Let $N = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{w, x, y, z, s\}$, and $P \in \times_{i \in N} \mathcal{P}_i$ be as in the following table

Table III. Preference profile P

P_1	$P_2 = P_3 = P_4$	$P_5 = P_6$	$P_7 = P_8$
x	z	w	s
s	w	x	y
w	s	y	x
y	y	s	w
z	x	z	z

We have 10 triples: $\{w, x, y\}$, $\{w, x, z\}$, $\{w, x, s\}$, $\{w, y, z\}$, $\{w, y, s\}$, $\{w, z, s\}$, $\{x, y, z\}$, $\{x, y, s\}$, $\{x, z, s\}$, and $\{y, z, s\}$.

Start with $T = \{s, w, z\}$. The preference profile relative to $T = \{s, w, z\}$ is given by

Table IV. Preference profile P relative to T

P_1^T	$P_2^T = P_3^T = P_4^T$	$P_5^T = P_6^T$	$P_7^T = P_8^T$
s	z	w	s
w	w	s	w
z	s	z	z

The set of *non-mutually exclusive* agents relative to T is $M^T = \{5, 6\}$, $F^s(\widehat{P^T}) = \emptyset$, $F^w(\widehat{P^T}) = \{5, 6\}$, $F^z(\widehat{P^T}) = \emptyset$, and $f^w(\widehat{P^T}) > \frac{\#M^T}{2}$. Therefore, alternatives s and z cannot be majority winners as they violate the condition we explained above, they can not be defeated in any triple. Hence, there is only one triple of alternatives to check, $T' = \{w, x, y\}$. The preference profile relative to T' is given by

Table V. Preference profile P relative to T'

$P_1^{T'}$	$P_2^{T'} = P_3^{T'} = P_4^{T'}$	$P_5^{T'} = P_6^{T'}$	$P_7^{T'} = P_8^{T'}$
x	w	w	y
w	y	x	x
y	x	y	w

The set of *non-mutually exclusive* agents relative to T' is $M^{T'} = \{1, 2, 3, 4\}$, $F^x(\widehat{P^{T'}}) = \{1\}$, $F^y(\widehat{P^{T'}}) = \emptyset$, $F^w(\widehat{P^{T'}}) = \{2, 3, 4\}$, and $f^w(\widehat{P^{T'}}) = 3 > \frac{\#M^{T'}}{2}$. Then, good candidate for T' at the net preference profile $\widehat{P}$ corresponds to the unique majority winner at the preference profile P. That is, alternative $w \in A$ is a *strong Condorcet winner.*

Furthermore, if we work with weak preferences, the results hold when we break the indifference in an accurate way previous to the creation of the net preference profile. In particular, given a triple of alternatives and a preference relation, if an agent i has as preferences $xI_i yP_i z$, then, we break the indifference with the creation of a clone agent i' such that for i, $xP_i yP_i z$, and for i', $yP_i xP_i z$. After such procedure, we obtain the net preference profile as before and the previous results hold.

Xfteris (2012) and our work converges in the fact that both highlight that if there exists an alternative such that it is top ranked by a majority of agents, then the preference profile will not show any cycle; hence, it is a strictly acyclic majority rule relation and there exists a majority winner. However, we go one step forward showing that such alternative will also be a good candidate and that the result can be extended in the case where there are more than three alternatives.

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