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Robust inequality comparisons based on ordinal attributes with Kolm-independent measures

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Abstract

The literature on health inequality with ordinal attributes is benefiting from the development of inequality measures, which are useful in any wellbeing assessment involving ordinal variables (e.g. subjective wellbeing). Lv, Wang, and Xu ("On a new class of measures for health inequality based on ordinal data", Journal of Economic Inequality, 2015) recently characterized a new class of this type of inequality measures axiomatically. In addition to their appealing functional forms, these measures are the only ones in the literature satisfying a property of independence, inspired by Kolm ("Unequal inequalities I", Journal of Economic Theory, 1976). As acknowledged by the authors, it is reasonable to be concerned about the robustness of inequality comparisons with ordinal attributes to the several alternative suitable measures within the class. This note derives the stochastic dominance condition whose fulfilment guarantees that all inequality measures within the class rank a pair of distributions consistently; thereby providing an empirically implementable robustness test for this class of measures.

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1 Introduction

The long-lasting, and ongoing, concern for health inequality has stimulated a burgeoning literature on ordinal inequality measurement, which has provided several classes of measures in the last few years. Prominent examples include Allison and Foster (2004), Apouey (2007), Abul Naga and Yalcin (2008), Erreygers (2009), Reardon (2009), Lazar and Silber (2013), and more recently Lv, Wang, and Xu (2015). The contribution of Lv et al. (2015) is particularly interesting because their proposed class of measures cardinalises the distances between ordinal categories, and yet the indices behave well, fulfilling key properties like aversion to median-preserving spreads. Moreover, the functional forms of the measures by Lv et al. (2015) also bear other appealing traits, including ease of computation (e.g. not requiring the use of the median). Last but not least, this class of indices is the only one fulfilling an independence property inspired by Kolm (1976, property 7, p. 426). To be succinct, we call it Kolm-independence. Basically, in the measurement framework of Lv et al. (2015) the inequality indices are weighted sums of functions mapping from the modulus of the absolute distance between pairs of ordinal categories. The sum is conducted over every possible pair of categories; and the weights, in turn, are the products of the relative frequencies of each pair of categories. In this context, the independence property introduced by Lv et al. (2015) requires that the change in total inequality due to the change in the relative frequency of an ordinal category be independent of the initial level of that frequency. Essentially, *if we wanted to impose this property when measuring inequality with ordinal attributes, then the axiomatic characterization provided by Lv et al. (2015) implies that we should only use measures from their class.*

However, even if we restricted ourselves to this class of Kolm-independent measures, we could still choose among several equally suitable measures. Lv et al. (2015) provide examples of such measures, including one which is basically a Gini index based on the modulus of the differences between the ordinal categories of the variable, cardinalised with natural numbers. Hence we could be naturally concerned by the robustness of pairwise inequality comparisons with ordinal attributes to alternative choices of equally appropriate inequality measures. Referring (perhaps confusingly) to the different functional forms available to cardinalise the differences between pairs of ordinal categories as "weights", Lv et al. (2015, p. 467) echo this concern succinctly: "It may be noted that the choice of weights, wijs, in the construction of a health inequality index f in our context is not unique. From the above discussion, those weights reflect our value judgments about how to deal with health "inequalities" from any two further apart health statuses in the construction of an overall health inequality index f. The choice of a particular set of weights may cause some concerns for researchers and for policy makers when our intuition about such weights is blurry or when we have some conflicting intuitions about exactly what set of weights should be chosen and used."

Addressing this issue, this paper derives the first-order stochastic dominance condition whose fulfilment guarantees that all inequality measures within the Kolm-independent class rank a pair of distributions consistently. Hence the paper provides an empirically implementable robustness test for this class of measures. The condition requires comparing across populations, or samples, their cumulative distributions of products of probabilities, which measure the likelihoods of finding pairs of individuals featuring specific differences between their reported categories (e.g. of self-reported health, life-satisfaction responses, educational levels, etc.). Intuitively, societies with higher probabilities of finding pairs of people with narrower differences (between their category-values) and lower probabilities of finding pairs with wider differences, will tend to be robustly less unequal than others, according to the measures of the class axiomatically characterized by Lv et al. (2015).

The rest of the note proceeds as follows. Section 2 provides the notation and a description of the class of ordinal inequality measures proposed by Lv et al. (2015). Section 3 provides the dominance proposition, together with its respective proof. Then the paper ends with some concluding remarks.

2 Preliminaries

2.1 Notation

Let x be an ordinal variable with c increasingly ordered categories. For example, a question on self-reported health of the form "In general, how would you rate your health today" with answer options: "very bad", "bad", "moderate", "good", and "very good" (Subramanian, Huijts, and Avendano, 2010). Each category is assigned a natural number from 1 to c. The respective discrete probability distribution is given by the vector: $P := [p(1), p(2), ..., p(c)]$, where $p(i) \equiv \Pr[x = i]$. With subscripts we refer to the probabilities, and other statistics, of a specific population or sample. Hence, for example, $p_A(1)$ is the relative frequency of people reporting the lowest category in society A.

Later, in the next section, we will also need statistics which are specific sums of probability products (e.g. $p(1)p(2)$). In particular we define the following functions:

$$\pi(\delta) \equiv \sum_{i=1}^{c-\delta} p(i)p(i+\delta), \ \forall \ \delta = 0, 1, ..., c-1 \tag{1}$$

As it will become apparent below, δ measures the modulus of the difference between two values of the variable, e.g. i and j, where each category has been cardinalised using natural numbers in the range $[1, c]$. Henceforth we refer to these absolute values as "gaps".

Examples of 1 include: $\pi(0) = \sum_{i=1}^{c}[p(i)]^2$, $\pi(1) = p(1)p(2)+p(2)p(3)+...+p(c-1)p(c)$, and $\pi(c-1) = p(1)p(c)$. Note, importantly, that: $\pi(0) + 2\sum_{\delta=1}^{c-1}\pi(\delta) = 1$. These probabilities give us the likelihood of finding two people in the population whose gaps between their reported ordinal categories is equal to δ, assuming that the likelihood of appearance of a person with a value of i is independent from the likelihood of appearance of a person with a value of j. For instance, consider $c = 3$ and $p(1) = 0.2, p(2) = 0.3, p(3) = 0.5$. Then we could ask: what is the probability of randomly sampling two people from this population whose gap between their reported ordinal categories is equal to 0 (essentially, the probability of drawing two people who gave the same answer). The probability is: $\pi(0) = p(1)p(1)+p(2)p(2)+p(3)p(3) = 0.38$. We can also define a cumulative version of definition (1), which will be very useful for the derivation of the dominance condition in the next section:

$$\Pi(\delta) \equiv \pi(0) + 2\sum_{i=1}^{\delta} \pi(i), \ \forall \ \delta = 0, 1, ..., c-1 \tag{2}$$

Clearly, the vector $\Pi := [\Pi(0), \Pi(1), ..., \Pi(c-1)]$ is a discrete cumulative probability distribution, with $\Pi(c-1) = 1$. Each element gives us the probability of finding pairs of people with gaps of δ or lower.

Finally we define, for instance, $\Delta\Pi(\delta) \equiv \Pi_A(\delta) - \Pi_B(\delta)$ in order to denote differences between two populations or samples. Thus, likewise, we apply Δ to other statistics.

2.2 The class of Kolm-independent ordinal inequality measures

Lv et al. (2015) axiomatically characterize the following class of ordinal inequality measures:

$$\mathcal{M} := \{O(P)|O(P) = \sum_{i=1}^{c}\sum_{j\neq i}^{c} g(|i-j|)p(i)p(j)\}, \tag{3}$$

where g is a function mapping from the gaps of cardinalised categories to the non-negative segment of the real line, and $g(1) < g(2) < ... < g(c-1)$.[1] As shown by Lv et al. (2015, proposition 1), the class (3) is the only one satisfying properties of normalization, aversion to median-preserving spreads, invariance to parallel shifts, additivity and independence. Yet different choices of $g(|i-j|)$ are possible, including $g(|i-j|) = |i-j|$ and $g(|i-j|) = 2\alpha^{c-1-|i-j|}$ with $0 < \alpha < 1$ (Lv et al., 2015, p. 469). Hence it is worth inquiring under which situations the choice of g (among all admissible functional forms satisfying $g(1) < g(2) < ... < g(c-1)$) will not affect the inequality ranking of A versus B.

3 The stochastic dominance condition for the class of Kolm-independent ordinal inequality measures

The dominance condition is the following:

Proposition 1. $O(P_A) < O(P_B) \ \forall O(P) \in \mathcal{M}$ *if and only if* $\Delta\Pi(\delta) \geqslant 0 \ \forall \delta = 0, 1, 2, ..., c-1 \wedge \exists \delta \in [0, 1, 2, ..., c-1] | \Delta\Pi(\delta) > 0$.

Proof. First, note what ΔO looks like in terms of the relative frequencies:

$$\begin{aligned}\Delta O = g(1)2[p_A(1)p_A(2) + p_A(2)p_A(3) + ... + p_A(c-1)p_A(c) - p_B(1)p_B(2) - ... - p_B(c-1)p_B(c)] \\ +... + g(c-1)2[p_A(1)p_A(c-1) - p_B(1)p_B(c-1)]\end{aligned} \tag{4}$$

Combining the definitions of π in expression (1) with expression (4), we can define ΔO as:

[1] These functions g are a subclass of the "weights", wij, described by Lv et al. (2015, p. 467).

$$\Delta O \equiv \sum_{\delta=1}^{c-1} g(\delta)\Delta 2\pi(\delta) \quad (5)$$

Applying summation by parts to 5 using Abel's lemma (Guenther and Lee, 1988),[2] we get the following expression:

$$\Delta O = -\sum_{\delta=1}^{c-2}[g(\delta+1)-g(\delta)]\Delta\Pi(\delta) + g(c-1)\Delta\Pi(c-1) - g(1)\Delta\Pi(0) \quad (6)$$

$$= -\sum_{\delta=1}^{c-2}[g(\delta+1)-g(\delta)]\Delta\Pi(\delta) - g(1)\Delta\Pi(0) \quad (7)$$

Note that we moved from line (6) to line (7) because $\Delta\Pi(c-1) = 0$. Now, we know that $g(i) > 0 \ \forall i \geqslant 1$ and that $g(\delta+1) - g(\delta) > 0 \ \forall \delta \geqslant 1$ (since we previously stated that g is a monotonically increasing function). Therefore, from immediate inspection of line (7) we can conclude that $\Delta O < 0$ for all possible choices of g (given the specified constraints on its properties, i.e. $0 < g(1) < g(2) < ... < g(c-1)$) if and only if $\Delta\Pi(\delta) \geqslant 0$ for all $\delta = 0, 1, 2, ..., c-1$ and there is at least one value $\delta \in [0, 1, 2, ..., c-1]$ such that $\Delta\Pi(\delta) > 0$.

■

Basically, proposition (1) states that A is robustly less unequal than B, i.e. according to all members of the class (3), if and only if the cumulative distribution of probability products, i.e. the cumulative distributions of category gaps, is never lower in A than in B, and at least once strictly higher. Roughly, A has higher cumulative proportions of low gaps and lower cumulative proportions of high gaps, vis-a-vis B. In order to implement the condition we first need to compute the cumulative distributions following the instructions of the preliminaries' section.

By way of further illustration of the condition, we discuss the shapes of the cumulative probability vectors, Π, corresponding to the benchmark situations of minimum and maximum inequality with ordinal attributes. In the case of minimum inequality the requirement is that there is one category i such that $p(i) = 1$, i.e. the whole population is in the same category. In that case, we will have $\pi(0) = 1$ and $\pi(\delta) = 0 \ \forall \delta = 1, 2, .., c-1$. Hence $\Pi(0) = \Pi(1) = ... = \Pi(c-1) = 1$. Clearly, with such cumulative distribution, no other distribution (unless there is a category i such that $p(i) = 1$) can exhibit less inequality, since their cumulative distributions of gaps must lie somewhere below. Likewise, all different distributions characterized by having one category i with $p(i) = 1$ are bound to be ranked as having the same level of inequality by all members of the class (3).

Meanwhile, in the case of maximum inequality it turns out that the benchmark commonly used in the literature (e.g. Apouey, 2007, Abul Naga and Yalcin, 2008) and characterized by $p(1) = p(c) = 0.5$, i.e. half of the population in the bottom category and half in the top, does not hold in the case of the inequality measures belonging to class (3). For instance, consider the case $c = 3$ and the following two discrete probability distributions: $A = (0.2, 0.5, 0.3)$ (where $p_A(1) = 0.2$, and so on); and $B = (0.5, 0, 0.5)$. Is it the

[2] This lemma establishes the discrete-setting version of integration by parts.

case that the benchmark distribution B is robustly more unequal than A? If we choose $g = (|i-j|)^{0.1}$, we get $O(A) = 0.628613 > 0.535887 = O(B)$. By contrast, with $g = (|i-j|)^2$ we get $O(A) = 0.98 < 2 = O(B)$. That is, not only does the ranking depend on the choice of admissible inequality index (note that both choices of g are admissible according to Lv et al. (2015, proposition 1, p. 471)), but certainly B cannot represent the situation of maximum inequality when we rely on class (3). However, this should not be surprising once we implement the robustness test based on proposition (1) above, and find that Π_A and Π_B actually *cross*.[3]

4 Conclusion

Proposition (1) provides a partial answer to the concern put forward by Lv et al. (2015) regarding the robustness of inequality comparisons to alternative choices of inequality members from the same Kolm-independent class. When the condition based on the cumulative distributions of gaps holds, the comparison is robust to any choice of index within that class. Otherwise, the ranking between A and B will crucially depend on the particular choice of ordinal inequality index.

Future research in this particular area could look into statistical inference techniques for this robustness condition, which could be useful especially when comparing samples. Secondly, given that there are bound to be empirical situations in which the condition of proposition (1) will not be fulfilled, it is worth deriving higher-order dominance conditions for narrower subclasses of inequality indices within class (3).[4] Finally, it might be worth inquiring into the existence of simpler dominance conditions relying more directly on the cumulative discrete probability distribution, as opposed to the cumulative discrete distribution of probability products.

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[3]Notably, in their axiomatic characterization of class (3), Lv et al. (2015) consider a normalization axiom which *only* takes into account the benchmark of minimum inequality.

[4]I would like to thank an anonymous referee for this interesting suggestion.

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