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On the Production Efficiency of Full Employment under Production Externalities

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Abstract

We address whether it is efficient for all factors of production to be fully employed, even in the presence of production externalities, by considering two general formulations of production externalities categorized by source: output-generated production externalities and input-generated ones. Our answer is affirmative for the former case, while it could be negative for the latter.

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1 Introduction

Full employment—the full use of factors of production—is a starting point for many economic models, often as the consequence of the widely held assumption of the inelastic supply of factors. If there is no externality, full employment is, as is well known, the necessary condition for production efficiency—for operating on the production possibility frontier (PPF).[1] If there are production externalities, however (i.e., if external effects arise from some producers and benefit or harm other producers), does production efficiency still require full employment?

In this note, we address this basic yet essential question. The question is essential in its theoretical significance because it rethinks the relationship between production efficiency and full employment. The literature has elaborated the shape of the PPF under production externalities, especially the conditions under which the concavity or the convexity of the PPF can be guaranteed (e.g., Herberg and Kemp, 1969; Baumol and Bradford, 1972; Panagariya, 1981; Herberg et al., 1982; Wong, 1995, Ch.5; Dalal, 2006; Li, 2015). To our knowledge, however, no formal analysis of the production efficiency of full employment in the presence of production externalities has yet been conducted.

Our question is also essential in its policy implications because production externalities are such common phenomena. For example, waste flowing into a lake from a chemical factory can harm fishermen living nearby, and the relatively high level of air pollution in a city can reduce the productivity of its workers.[2] Our question leads to a consideration of the assumption of the inelastic supply of factors, and of whether regulations on the use of factors are advisable.

We consider two general formulations of production externalities: output-generated production externalities and input-generated ones. The former is formulated as being generated only from outputs; therefore, the extent of external effects can be measured only by outputs. By contrast, the latter is formulated as being generated only from inputs; therefore, the extent is measured only by inputs. We demonstrate that full employment is the necessary condition for production efficiency if production externalities are output-generated, whereas it can be inefficient if production externalities are input-generated.

2 Production Externalities: Output-generated and Input-generated

In general, production externalities can come from outputs, inputs, or both. Consider m goods and n factors. The production technology in the presence of production externalities can be written, in an implicit form, as

$$F_j\left(v_{-j}, x_{-j}, v_j, x_j\right) = 0, \;\; j = 1, ..., m, \tag{1}$$

where $v_j \equiv (v_{1j}, ..., v_{nj})' \in \mathbb{R}^n_+$ is the vector of inputs in good j, $x_j \in \mathbb{R}_+$ the output of good j, $v_{-j} \equiv (v_1, ..., v_{j-1}, v_{j+1}, ..., v_m) \in \mathbb{R}^{n\times(m-1)}_+$ the vector with all input vectors as the components except for v_j, and $x_{-j} \equiv (x_1, ..., x_{j-1}, x_{j+1}, ..., x_m) \in \mathbb{R}^{m-1}_+$ the vector with all outputs as the components except for x_j. The presence of v_{-j} and x_{-j} in the production of good j indicates the presence of production externalities.

[1] Another requirement for the statement is that all factors are marginally productive; otherwise (in the case of Leontief technology, for example), it is not necessary to use all factors to be on the PPF.

[2] The main contribution of this study is in answering the question concerning negative externalities because, under positive externalities, the PPF can always expand by using more factors of production in any production process. Thus, if existing externalities are all positive, production efficiency requires the full employment of factors of production.

Eq. (1) contains two types of production externalities: output-generated production externalities, under which the extent of externalities depends on the outputs, and input-generated production externalities, under which the extent of externalities depends on the inputs (e.g., Kemp, 1955; Meade, 1952). In the following analysis, we use simpler alternative formulations of production externalities in order to focus on each type of externalities. We denote the production function of good j by f_j in an explicit form; the output-generated production externalities can be expressed by

$$x_j = f_j\left(x_{-j}, v_j\right), \quad j = 1, ..., m, \tag{2}$$

and the input-generated production externalities can be expressed by

$$x_j = f_j\left(v_{-j}, v_j\right), \quad j = 1, ..., m. \tag{3}$$

Studies on output-generated production externalities range from general theoretical considerations (e.g., Herberg and Kemp, 1969) to specific contexts, especially trade theory (e.g., Chang, 1981; Ishikawa, 1994) and environmental issues (e.g., Copeland and Taylor, 1999; Brander and Taylor, 1998; Rus, 2016), while investigations of input-generated production externalities can be found in public economics studies, especially those on public intermediate goods (e.g., Manning and McMillan, 1979; Tawada, 1980; Tawada and Abe, 1984) and public goods subject to congestion (e.g., Uzawa, 2005).[3]

3 Production Efficiency of Full Employment

In this section, we present Theorem 1, our main result. To derive the theorem, we impose the following four assumptions:

(A1) f_j is continuous in all arguments.

(A2) f_j is non-decreasing, and strictly increasing if $f_j > 0$, in v_{kj} for any $k \in \{1, ..., n\}$.

(A3) $f_j(\cdot, 0) = 0$.

(A4) The production possibility set (PPS), Ω, is bounded.

Assumption (A2) requires that the inputs be productive when the output is positive, thereby excluding the Leontief technology and sector-specific factors. Assumption (A3) formulates the idea of no free lunch, which also implies that the origin belongs to the PPS in the space of output, which contains all feasible output bundles satisfying the factor constraint

$$E - v \cdot e \geq 0, \tag{4}$$

where $E \equiv (E_1, ..., E_n)' \in \mathbb{R}^n_+$ is the vector of factor endowments, $v \equiv (v_1, ..., v_m) \in \mathbb{R}^{n\times m}_+$ is the vector of input vectors, and $e \equiv (1, ..., 1)' \in \mathbb{R}^m$ is the unitary vector with all components being one. Assumption (A4) ensures the existence of the PPF, defined as the boundary of the PPS as follows:

Definition. PPF $\equiv \{x \in \Omega; \forall \delta \in \mathbb{R}_{++}, \exists u$ such that $x + \delta u \notin \Omega\}$, where $u = (u_1, ..., u_m)' \in \mathbb{R}^m$ satisfies that $u_j = 0$ if $x_j = 0$.

[3] The intra-industrial externalities are included, though not explicitly, in (2) and (3). In the literature, intra-industrial externalities are also extensively discussed and applied (e.g., Jones, 1968; Swanson, 1999; Liu and Turnovsky, 2005).

Intuitively, the definition posits that, given any point in the PPS, if there exists a direction along which any movement staring from that point results in a departure from the PPS, then that point is a point on the PPF.[4]

With the assumptions and the definition in hand, we can derive the following theorem:

Theorem 1. *Given (A1)–(A4), full employment on the PPF*

(i) holds under output-generated production externalities (2), but

(ii) may not hold under input-generated production externalities (3).

Proof of part (i). Instead of proving directly that, under output-generated production externalities (2), operating on the PPF implies full employment, we prove its contraposition—that not full employment implies not operating on the PPF.

Let $x^* = (x_1^*, ..., x_m^*)' \in \mathbb{R}_+^m$ be any output bundle satisfying $x^* \neq 0$ and produced with input $v^* = (v_1^*, ..., v_m^*) \in \mathbb{R}_+^{n\times m}$, where $v_j^* = \left(v_{1j}^*, ..., v_{nj}^*\right)' \in \mathbb{R}_+^n$ is the input vector for producing good j. Assume that certain factor $k \in \{1, ..., n\}$ is not fully used: $\sum_{j=1}^m v_{kj}^* < E_k$. Then, for the purpose of the proof, we shall show that x^* is not on the PPF, which is equivalent (using the definition of the PPF) to showing that, for any $u = (u_1, ..., u_m)' \in \mathbb{R}^m$ satisfying

$$u \neq 0 \text{ and } u_j = 0 \text{ if } x_j = 0, \tag{5}$$

there is $\delta \in \mathbb{R}_{++}$ such that $x^* + \delta u$ can be produced without violating the factor constraint. Let $v^{**} = (v_1^{**}, ..., v_m^{**}) \in \mathbb{R}_+^{n\times m}$ denote the new input; then it holds that $v^{**} \cdot e \leq E$ and

$$x_j^* + \delta u_j = f_j\left(x_{-j}^* + \delta u_{-j}, v_j^{**}\right), \;\; j = 1, ..., m, \tag{6}$$

where $u_{-j} \in \mathbb{R}^{m-1}$ is obtained from u by dropping its j-th component. To find such δ and v^{**}, we go through the three steps below.

Step 1. We rewrite the problem.

Define, for convenience, $\Delta = (\Delta_1, ..., \Delta_m) \equiv v^{**} - v^* \in \mathbb{R}^{n\times m}$. The factor constraint gives

$$(v^* + \Delta) \cdot e \leq E. \tag{7}$$

Subtracting $x_j^* = f_j\left(x_{-j}^*, v_j^*\right)$ from (6) and using $v_j^{**} \equiv v_j^* + \Delta_j$, we can rewrite (6) as[5]

$$A_j\left(\delta u_{-j}, \Delta_j\right) = \delta u_j - B_j\left(\delta u_{-j}\right), \;\; j = 1, ..., m, \tag{8}$$

where $A_j\left(\delta u_{-j}, \Delta_j\right) \equiv f_j\left(x_{-j}^* + \delta u_{-j}, v_j^* + \Delta_j\right) - f_j\left(x_{-j}^* + \delta u_{-j}, v_j^*\right)$ and $B_j\left(\delta u_{-j}\right) \equiv f_j\left(x_{-j}^* + \delta u_{-j}, v_j^*\right) - f_j\left(x_{-j}^*, v_j^*\right)$. Therefore, it is sufficient to show that, for any $x^* \neq 0$ and any u satisfying (5), there exist δ and Δ satisfying (7) and (8).

Step 2. We show that there exist δ and Δ such that (8) holds.

In (8), we have m equations in total, from which we shall determine δ and Δ_1 to Δ_m. This is a system of equations with $(n-1)\,m + 1$ degree of freedom since $\Delta_j \in \mathbb{R}^n$. Moreover, note that Δ_j enters only the corresponding j-th equation in (8). The two features suggest that we may determine δ and Δ by, first, taking δ as given and, second,

[4]Two points about the definition are worth mentioning. First, it requires the j-th component of u to be zero if the output of good j is zero and thus takes into account the degenerated case where some goods are not produced. Second, in the presence of production externalities, it is not necessary for the output of some goods to decline when the output of others rises on the PPF. In the case of two goods, for example, the PPF may have positively sloping intervals. Once we allow for free disposal of outputs, however, these positive sloping intervals, if there are any, disappear. Note that Theorem 1 holds irrespective of whether free disposal is available.

[5]Note that $\delta u_j = f_j\left(x_{-j}^* + \delta u_{-j}, v_j^* + \Delta_j\right) - f_j\left(x_{-j}^*, v_j^*\right) = A_j\left(\delta u_{-j}, \Delta_j\right) + B_j\left(\delta u_{-j}\right)$.

focusing on each equation one by one in (8) to see if the corresponding Δ_j can be solved out from the equation. The details are as follows.

For any good j, consider two cases: case (a) $x_j^* = 0$ and case (b) $x_j^* > 0$.

In case (a), we have, by (A3), $v_j^* = 0$ and, by (5), $u_j = 0$. Given any δ, simply let $\Delta_j = 0$ and the j-th equation in (8) holds.

In case (b), we focus on factor k, which (as assumed) is not fully used, and consider two sub-cases: case (b.1) $v_{kj}^* > 0$ and case (b.2) $v_{kj}^* = 0$.

In case (b.1), let $\Delta_j = t_j e_k$, where $t_j \in \mathbb{R}$ and $e_k \in \mathbb{R}^n$ is the vector with the k-th component being one and others zero. Thus, the new input vector in good j, $v_j^* + \Delta_j = v_j^* + t_j e_k$, adjusts only the use of factor k in the original input. By doing so, (8) becomes

$$A_j\left(\delta u_{-j}, t_j e_k\right) = \delta u_j - B_j\left(\delta u_{-j}\right). \tag{9}$$

By (A1), $A_j\left(\delta u_{-j}, t_j e_k\right)$ is a continuous function of t_j and δ, and $\delta u_j - B_j\left(\delta u_{-j}\right)$ is a continuous function of δ. We have then $\lim_{t_j \to 0} A_j\left(\delta u_{-j}, t_j e_k\right) = 0$ and $\lim_{\delta \to 0}\left(\delta u_j - B_j\left(\delta u_{-j}\right)\right) = 0$. On the other hand, by (A2), $A_j\left(\delta u_{-j}, t_j e_k\right)$ is strictly increasing in t_j. This suggests that, given any δ small enough such that $x_j^* + \delta u_j > 0$ (i.e., good j is still produced), there exists t_j satisfying (9). The properties of $A_j\left(\delta u_{-j}, t_j e_k\right)$ and $\delta u_j - B_j\left(\delta u_{-j}\right)$ also suggest that t_j can be solved from (9) as a continuous function of δ: $t_j = t_j(\delta)$, which satisfies $\lim_{\delta \to 0} t_j(\delta) = 0$. Note that $t_j(\delta)$ can be of either sign, depending on that of $\delta u_j - B_j\left(\delta u_{-j}\right)$.[6] This also requires δ to be small enough to ensure the non-negativity of the use of factor k in good j, $v_{kj}^* + t_j(\delta)$.

In case (b.2), since $v_{kj}^* = 0$, it is impossible to use less of factor k in good j. Furthermore, consider two sub-cases by focusing on the sign of $\delta u_j - B_j\left(\delta u_{-j}\right)$: case (b.2.1) $\delta u_j - B_j\left(\delta u_{-j}\right) \geq 0$ and case (b.2.2) $\delta u_j - B_j\left(\delta u_{-j}\right) < 0$.

In case (b.2.1), let $\Delta_j = t_j e_k$ as in case (b.1); then the argument in case (b.1) applies. Therefore, given a δ small enough such that $x_j^* + \delta u_j > 0$, there exists a function of δ, $t_j(\delta)$, satisfying $\lim_{\delta \to 0} t_j(\delta) = 0$ such that (9) holds for $t_j = t_j(\delta)$. Since $\delta u_j - B_j\left(\delta u_{-j}\right) \geq 0$, we have $t_j(\delta) \geq 0$, and the use of factor k in good j, $v_{kj}^* + t_j(\delta) = t_j(\delta)$, is non-negative.

In case (b.2.2), it follows from $x_j^* > 0$ and (A3) that $\exists k' \in \{1, ..., n\} \neq k$ such that $v_{k'j}^* > 0$. Let $\Delta_j = t_j e_{k'}$. Thus, the new input vector in good j, $v_j^* + \Delta_j = v_j^* + t_j e_{k'}$, adjusts only the use of factor k' in the original input. It follows from (8) that

$$A_j\left(\delta u_{-j}, t_j e_{k'}\right) = \delta u_j - B_j\left(\delta u_{-j}\right). \tag{10}$$

By (A1) and (A2), $A_j\left(\delta u_{-j}, t_j e_{k'}\right)$ is continuous in t_j and δ, strictly increasing in t_j, and satisfying $\lim_{t_j \to 0} A_j\left(\delta u_{-j}, t_j e_{k'}\right) = 0$. Given any δ small enough such that $v_{k'j}^* + t_j > 0$, there exists a function of δ, denoted also by $t_j(\delta)$ to save notation, such that (10) holds for $t_j = t_j(\delta)$. Clearly, $\lim_{\delta \to 0} t_j(\delta) = 0$ and, given that $\delta u_j - B_j\left(\delta u_{-j}\right) < 0$, $t_j(\delta) < 0$. Moreover, as long as δ is small enough, the non-negativity constraint $v_{k'j}^* + t_j(\delta) \geq 0$ holds.

The discussion above suggests that, given any δ small enough, we can obtain Δ_j for any $j \in \{1, ..., m\}$ such that the j-th equation in (8) holds. This simply means that we can choose a δ small enough and then obtain $\Delta = (\Delta_1, ..., \Delta_m)$ satisfying (8).

Step 3. We check if the new input $v^* + \Delta$ satisfies the factor constraint (7).

For any factor $i \neq k, k'$, the total use of the factor in the new input, $v^* + \Delta$, is the same as before, which (by assumption) satisfies the factor constraint.

[6] Recalling that $\lim_{t_j \to 0} A_j\left(\delta u_{-j}, t_j e_k\right) = 0$ and that $A_j\left(\delta u_{-j}, t_j e_k\right)$ is strictly increasing in t_j, this means that t_j and $A_j\left(\delta u_{-j}, t_j e_k\right)$, and consequently $\delta u_j - B_j\left(\delta u_{-j}\right)$ by (9), are of the same sign.

For factors k and k', let M_1 denote the index set that belongs to cases (b.1) and (b.2.1) and M_2 the index set that belongs to case (b.2.2). Then, the total use of factor k in the new input is $\sum_{j=1}^{m} v_{kj}^{*} + \sum_{j \in M_1} t_j(\delta)$. Recall that $\lim_{\delta \to 0} t_j(\delta) = 0$ and, as is assumed, $\sum_{j=1}^{m} v_{kj}^{*} < E_k$; it thus holds that $\sum_{j=1}^{m} v_{kj}^{*} + \sum_{j \in M_1} t_j(\delta) \leq E_k$ as long as δ is small enough. The total use of factor k' in the new input is $\sum_{j=1}^{m} v_{k'j}^{*} + \sum_{j \in M_2} t_j(\delta)$. Since $t_j(\delta) < 0$ for $j \in M_2$, it holds that $\sum_{j=1}^{m} v_{k'j}^{*} + \sum_{j \in M_2} t_j(\delta) \leq E_{k'}$.

Therefore, as long as δ is small enough, the new input, $v^{*} + \Delta$, where Δ is obtained in Step 2, satisfies the factor constraint (7).

From the three steps above, we show that, if there is a factor not fully used, for any $u \in \mathbb{R}^m$ satisfying (5), there exists $\delta \in \mathbb{R}_{++}$ such that $x^{*} + \delta u$ can be produced without violating the factor constraint. This simply means that x^{*} is not on the PPF. We thus prove the contraposition of part (i). □

Proof of part (ii). To show that full employment may fail to hold on the PPF under input-generated production externalities (3), it is sufficient to provide such an example.

Example. There are two goods and one factor of production. The factor endowment $E > 1$ and the technology satisfies

$$x_1 = \begin{cases} (1 - v_2) v_1 & \text{if } v_2 \leq 1, \\ 0 & \text{if } v_2 > 1, \end{cases} \qquad x_2 = \begin{cases} (1 - v_1) v_2 & \text{if } v_1 \leq 1, \\ 0 & \text{if } v_1 > 1, \end{cases}$$

where $v_1, v_2 \in \mathbb{R}_+$ are the inputs in good 1 and good 2.

To derive the PPF, consider three regions in the output space: x_1–axis where $x_2 = 0$, x_2–axis where $x_1 = 0$, and the first quadrant where $x_1, x_2 > 0$. On x_1–axis, clearly the optimal inputs are $v_1^{*} = E$ and $v_2^{*} = 0$, and the corresponding output bundle is $(E, 0)$. Similarly, on x_2–axis, the optimal inputs are $v_1^{*} = 0$ and $v_2^{*} = E$, and the corresponding output bundle is $(0, E)$. In the first quadrant, let $v_1 + v_2 = v$ to obtain $x_1 = 1 + x_2 - \sqrt{(v-1)^2 + 4x_2}$.[7] For any given x_2, note that x_1 achieves the maximum when $v = 1$. This implies that $v_1^{*} + v_2^{*} = 1$ holds on the PPF in the first quadrant, which can be written as $x_1 = 1 + x_2 - 2\sqrt{x_2}$.

To summarize, the example of input-generated production externalities above yields the PPF consisting of two points on the axes and a curve in the first quadrant. Full employment does not hold on the latter since there $v_1^{*} + v_2^{*} = 1 < E$. □

4 Discussion

In this section, we discuss two related issues. First, using specific examples, we show that output-generated production externalities are not equivalent to input-generated ones. Second, we provide the necessary condition under which full employment does not hold on the PPF under input-generated production externalities.

4.1 Nonequivalence between Output-generated and Input-generated

One may think that output-generated production externalities (2) can be regarded as a special case of input-generated production externalities (3), since outputs are produced from inputs. Doing so misses the point, however, that, under output-generated production externalities, the outputs are produced not only from inputs but also "from" other outputs,

[7] Using $v_1 + v_2 = v$, we can obtain $x_1 = (1 - v_2)(v - v_2) = v - (1 + v) v_2 + v_2^2$ and $x_2 = (1 - v + v_2) v_2 = (1 - v) v_2 + v_2^2$. Solving the second equation for v_2 and plugging it into the first equation gives the result.

which are further produced from other inputs and, again, "from" some other outputs, and so on. In some cases, we can express output-generated production externalities equivalently in the form of input-generated production externalities. For example,

$$x_1 = v_1, \quad x_2 = \begin{cases} (1 - x_1) v_2 = (1 - v_1) v_2 & \text{if } v_1 \leq 1, \\ 0 & \text{if } v_1 > 1, \end{cases}$$

where $v_1, v_2 \in \mathbb{R}_+$ are the inputs in good 1 and good 2.

However, this is not always the case. For example,

$$x_1 = f_1(x_2, v_1), \quad x_2 = f_2(x_1, v_2).$$

Substitute the second equation into the first for x_2 and obtain $x_1 = f_1(f_2(x_1, v_2), v_1)$. The implicit function theorem suggests that, given that both f_1 and f_2 are differentiable, x_1 cannot be written as a function of (v_1, v_2) if there exists (x_1, v_1, v_2) such that $1 - (\partial f_1/\partial x_2)(\partial f_2/\partial x_1) = 0$. Another example is as follows:[8]

$$x_1 = \begin{cases} (1 - x_2) v_1 & \text{if } x_2 \leq 1, \\ 0 & \text{if } x_2 > 1, \end{cases} \quad x_2 = \begin{cases} (1 - x_1) v_2 & \text{if } x_1 \leq 1, \\ 0 & \text{if } x_1 > 1, \end{cases}$$

where v_1 and v_2 are also scalars. Letting $(v_1, v_2) = (2, 2)$, it is easy to verify that three output bundles are feasible: $(x_1, x_2) = (2, 0)$, $(2/3, 2/3)$, $(0, 2)$.[9] Since an input bundle can produce only a unique output bundle under input-generated production externalities (3), the example above cannot be expressed in the input-generated form equivalently.

4.2 The Necessary Condition for Inefficiency of Full Employment

Theorem 1 provides an answer to the "Yes/No" question about the production efficiency of full employment in the presence of production externalities, which suggests that full employment may be inefficient (i.e., not holding on the PPF) if production externalities are input-generated. It is then pertinent to ask under what conditions this happens. To obtain an answer, assume the differentiability so that we can apply calculus to analyze the problem more precisely:

(A1′) f_j is continuously differentiable in all arguments.

The PPF can be characterized by maximizing, without loss of generality, the output of good 1, x_1, given the outputs of other goods, x_{-1}, subject to the constraints of technology and factor endowment. The Lagrangian can be written as

$$\mathcal{L} = x_1 - \sum_{j=1}^{m} p_j (x_j - f_j(v_{-j}, v_j)) + \sum_{i=1}^{n} \lambda_i \left(E_i - \sum_{j=1}^{m} v_{ij} \right) + \sum_{i=1}^{n} \sum_{j=1}^{m} \mu_{ij} v_{ij},$$

[8] In the case of output-generated production externalities, the timing of production and occurrence of externalities should be considered with care. For example, it would be more plausible to consider the current production affected by the outputs in the previous period:

$$x_{1,t} = \begin{cases} (1 - x_{2,t-1}) v_{1,t} & \text{if } x_{2,t-1} \leq 1, \\ 0 & \text{if } x_{2,t-1} > 1, \end{cases} \quad x_{2,t} = \begin{cases} (1 - x_{1,t-1}) v_{2,t} & \text{if } x_{1,t-1} \leq 1, \\ 0 & \text{if } x_{1,t-1} > 1. \end{cases}$$

Although our static framework cannot incorporate such dynamic aspects, one possible interpretation is that our model looks at the steady state of the corresponding dynamic model, where the same output levels are repeatedly achieved.

[9] We can verify that, given the dynamic process in Fn. 8, the three output bundles are steady states, among which $(2, 0)$ and $(0, 2)$ are locally stable while $(2/3, 2/3)$ is unstable.

where p_j, λ_i, μ_{ij} are the Lagrange multipliers corresponding to the technology of producing good j, the constraint of factor i, and the non-negativity of inputs. Suppose that factor l' is not fully used on the PPF, namely that $E_{l'} > \sum_{j=1}^{m} v_{l'j}$. It follows from the first-order conditions that $\lambda_{l'} = 0$ and[10]

$$\sum_{j=1}^{m} \frac{\partial G}{\partial v_{l'k}} = -\mu_{l'k} \leq 0, \quad k = 1, ..., m, \tag{11}$$

where $G \equiv \sum_{j=1}^{m} p_j x_j = \sum_{j=1}^{m} p_j f_j (v_{-j}, v_j)$ can be interpreted as the GDP measured using good 1 since p_j is the shadow price of good j and $p_1 = 1$.[11] This produces the result below.

Proposition 2. *Given (A1′) and (A2)–(A4), if a factor is not fully used on the PPF, its marginal contribution to the GDP by using it in any production process is non-positive.*

The intuition is made clearer by partitioning the derivatives in (11) to obtain

$$\frac{\partial G}{\partial v_{l'k}} = p_k \frac{\partial f_k}{\partial v_{l'k}} + \sum_{j=1}^{m(j \neq k)} p_j \frac{\partial f_j}{\partial v_{l'k}},$$

which gives two effects on the GDP from the use of factor l' in good k. The first term on the right-hand side measures its contribution to the output of good k as an input in the production, which is non-negative (or positive if the output of good k is already positive) according to (A1). On the other hand, the second term measures its external effects on the production of other goods. Note that, in a model without externalities, the second term is absent, and (11) fails to hold, implying that all factors should be fully used on the PPF. Under input-generated production externalities, the external effects arise, and Proposition 2 suggests that if a factor is not fully used on the PPF, the aggregate of its external effects must be negatively large enough to overcome its positive contribution as a factor of production.

5 Conclusion

Focusing on two general and widely applied formulations of production externalities—output-generated and input-generated—we have shown that full employment is the necessary condition for production efficiency if production externalities are output-generated but could be inefficient if production externalities are input-generated. This result has a significant policy implication. If production externalities are input-generated, we must be careful about whether inputs are being overused; if this is occurring, the use of factors of production should be regulated. Contrariwise, if production externalities are output-generated, a policy of full employment is always advantageous. It remains to be determined how to identify production externality types; this task is left to future research.

[10] The first-order conditions are $\partial \mathcal{L}/\partial x_1 = 1 - p_1 = 0$, $\partial \mathcal{L}/\partial v_{lk} = \sum_{j=1}^{m} p_j \partial f_j / \partial v_{lk} - \lambda_l + \mu_{lk} = 0$, and the Kuhn–Tucker conditions $\lambda_l \geq 0$, $\lambda_l \left(E_l - \sum_{j=1}^{m} v_{ij} \right) = 0$, and $\mu_{lk} \geq 0$, $\mu_{lk} v_{lk} = 0$ ($l = 1, ..., n$, $k = 1, ..., m$).

[11] Here, the GDP is calculated using the shadow prices, which measure the tradeoff between goods in production. In a strict sense, the GDP as a statistic should be calculated using the realized market prices that take account of the consumer as well as the producer side. Although the two price systems may coincide—under perfect competition with externalities appropriately internalized, for example—in many cases they do not.

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