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Characterizing SW-Efficiency in the Social Choice Domain

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Abstract

Recently, Dogan, Dogan and Yildiz (2016) presented a new efficiency notion for the random assignment setting called SW (social welfare)-efficiency and characterized it. In this note, we generalize the characterization for the more general domain of randomized social choice.

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1. Introduction

The *random assignment setting* captures the scenario in which n agents express preferences over n objects and the outcome is a probabilistic assignment. For the the setting, two interesting efficiency notions are ex post efficiency and SD (stochastic dominance)-efficiency [1, 3, 5, 6, 8, 10]. The assignment setting can be considered as a special case of voting where each deterministic assignment can be viewed as a voting alternative [2, 4, 7].

Recently, Dogan et al. [9] presented a new notion of efficiency called SW (social welfare)-efficiency for the random assignment setting. They characterize SW-efficiency. In this note, we generalize the characterization to the more general voting setting.

2. Preliminaries

Consider the social choice setting in which there is a set of agents $N = \{1, \dots, n\}$, a set of alternatives $A = \{a_1, \dots, a_m\}$ and a preference profile $\succsim = (\succsim_1, \dots, \succsim_n)$ such that each $\succsim_i$ is a complete and transitive relation over A. We write $a \succsim_i b$ to denote that agent i values alternative a at least as much as alternative b and use $\succ_i$ for the strict part of $\succsim_i$, i.e., $a \succ_i b$ iff $a \succsim_i b$ but not $b \succsim_i a$. Finally, $\sim_i$ denotes i's indifference relation, i.e., $a \sim_i b$ iff both $a \succsim_i b$ and $b \succsim_i a$. The alternatives in A could be any discrete structures: voting outcomes, house allocation, many-to-many two-sided matching, or coalition structures. A utility profile $u = (u_1, \dots, u_n)$ specified for each agent $i \in N$ his utility for $u_i(a)$ for each alternative $a \in N$. A utility profile is *consistent* with the preference profile $\succsim$, if for each $i \in N$ and $a, b \in A$, $u_i(a) \geq u_i(b)$ if $a \succsim_i b$. Two alternatives $a, b \in A$ are *Pareto indifferent* if $a \sim_i b$ for all $i \in N$. For any alternative $a \in A$, we will denote by $D(a)$ the set $\{b \in A : \exists i \in N, a \succ_i b\}$. An alternative $a \in A$ is *Pareto dominated* by $b \in A$ if $b \succsim_i a$ for all $i \in N$ and $b \succ_i a$ for some $i \in N$. An alternative is *Pareto optimal* if it is not Pareto dominated by any alternative.

We will also consider randomized outcomes that are lotteries over A. A lottery is a probability distribution over A. We denote the set of lotteries by $\Delta(A)$. For a lottery $p \in \Delta(A)$, we denote by $p(a)$ the probability of alternative $a \in A$ in lottery p. We denote by support $supp(p)$ the set $\{a \in A : p(a) > 0\}$. A lottery p is *interesting* if there exist $a, b \in supp(p)$ such that there exist $i, j \in N$ such that $a \succ_i b$ and $b \succ_j a$. A lottery is *degenerate* if it puts probability one on a single alternative.

Under *stochastic dominance (SD)*, an agent prefers a lottery that, for each alternative $x \in A$, has a higher probability of selecting an alternative that is at least as good as x. Formally, $p \succsim_i^{SD} q$ iff $\forall y \in A$: $\sum_{x \in A: x \succsim_i y} p(x) \geq \sum_{x \in A: x \succsim_i y} q(x)$. It is well-known that $p \succsim^{SD} q$ iff p yields at least as much expected utility as q for any von-Neumann-Morgenstern utility function consistent with the ordinal preferences [4, 8]. A lottery is *SD-efficient* if it is Pareto optimal with respect to the SD relation. A lottery is *ex post efficient* if each alternative in the support is Pareto optimal.

3. SW-efficiency

We now consider SW-efficiency as introduced by Dogan et al. [9]. Although Dogan et al. [9] defined SW-efficiency in the context of random assignment, the definition extends in a straightforward manner to the case of voting.

Definition 1 (SW-efficiency)**.** *A lottery p is* SW-efficient *if there exists no other lottery q that SW dominates it. Lottery q SW dominates p if for any utility profile for which p maximizes welfare, q maximises welfare, and there exists at least one utility profile for which q maximises welfare but p does not.*

We prove a series of lemmas which will help us obtain a characterization of SW-efficiency.

Lemma 1. *For a preference profile $\succsim$, consider a Pareto optimal alternative $a \in A$ and a non-empty set $D(a) = \{b \in A : \exists i \in N, a \succ_i b\}$. Then, there exists a utility profile u consistent with $\succsim$ such that $\sum_{i\in N} u_i(a) > \sum_{i\in N} u_i(b)$ for all $b \in D(a)$.*

Proof. We can construct the required utility function profile u consistent with $\succsim$ as follows. Whenever $a \succ_i b$, make the difference $u_i(a) - u_i(b)$ huge. Whenever $b \succ_j a$, make the difference $u_j(b) - u_j(a)$ arbitrarily small. Hence the value $u_i(a)-u_i(b)$ is large enough that it makes up for all j for which $u_j(b)-u_j(a) > 0$. Hence $\sum_{i\in N}(u_i(a) - u_i(b)) > 0$. □

Lemma 2. *SW-efficiency implies SD-efficiency, which implies ex post efficiency.*

Proof. It is well-known that SD-efficiency implies ex post efficiency [4].

Consider a lottery p that is not SD-efficient. Then there exists another lottery q that SD-dominates it. Hence p does not maximize welfare for any consistent utility profile because q yields more utility for each utility profile. □

Lemma 3. *An interesting lottery is not SW-efficient.*

Proof. If an interesting lottery p is not SD-efficient, we are already done because by Lemma 2, p is not SW-efficient. So let us assume p is SD-efficient and hence ex post efficient. Since p is interesting, there exists at least one $a \in supp(p)$ such that $a \succ_i b$ for some $b \in supp(p)$ and $i \in N$. Note that a is Pareto optimal. By Lemma 1, there exists a utility profile u such that $\sum_{i\in N} u_i(a) > \sum_{i\in N} u_i(b)$ for all $b \in D(a)$ where $D(a) \cap supp(p) \neq \emptyset$. Hence, there exists a utility profile u such that $\sum_{i\in N} u_i(a) > \sum_{i\in N} u_i(b)$ for all $b \in supp(p) \cap D(a)$. This means that $\sum_{i\in N} u_i(a) > \sum_{i\in N} u_i(p)$. Hence for the lottery q that puts probability 1 on alternative a, $\sum_{i\in N} u_i(q) > \sum_{i\in N} u_i(p)$. Also note that for any utility profile for which p maximizes welfare, q maximizes welfare as well since $a \in supp(p)$. Thus q SW dominates p. □

Lemma 4. *An uninteresting lottery over Pareto optimal alternatives is SW-efficient.*

Proof. An uninteresting lottery p over Pareto optimal alternatives is SD-efficient. Assume that there is another lottery q that SW-dominates p. Then $supp(q)$ contains one alternative b that is not Pareto indifferent to alternatives $supp(p)$. This means that there exists a utility profile u such that welfare is maximized by p but not by b. Hence q does not SW-dominate p. □

Based on the lemmas proved above, we prove the main result.

Theorem 1. *A lottery is SW-efficient iff it is ex post efficient and uninteresting.*

Proof. By Lemma 4, an ex post efficiency and uninteresting lottery is SW-efficient.

We now prove that if lottery is not ex post efficient or uninteresting, it not SW-efficient. Due to Lemma 2, if a lottery is not ex post efficient, it is not SW-efficient. Similarly, by Lemma 3, if a lottery is interesting, it is not SW-efficient. □

Next we prove that if A contains no Pareto indifferent alternatives, then a lottery is SW-efficient iff it is ex post efficient and degenerate.

Lemma 5. *If A contains no Pareto indifferent alternatives, then if a lottery is uninteresting and not degenerate, then it is not ex post efficient.*

Proof. Assume that a lottery p is uninteresting and not degenerate. Since p is not degenerate, $|supp(p)| \geq 2$. Since p is uninteresting, there do not exist $a, b \in supp(p)$ such that there exist $i, j \in N$ such that $a \succ_i b$ and $b \succ_j a$. Thus either a Pareto dominates b, or b or Pareto dominates a or a and b are Pareto indifferent. The third case is not possible because we assumed that A does not contain Pareto indifferent alternatives. Since a Pareto dominates b or b or Pareto dominates a, $supp(p)$ contains a Pareto dominated alternative. Hence p is not ex post efficient. □

Theorem 2. *If A contains no Pareto indifferent alternatives, then a lottery is SW-efficient iff it is ex post efficient and degenerate.*

Proof. Assume that A contains no Pareto indifferent alternatives. If a lottery p is SW-efficient, then by Theorem 1, it is ex post efficient and uninteresting. By Lemma 5, since p is ex post efficient, it is degenerate.

Now assume that a lottery p is ex post efficient and degenerate. Since p is degenerate, it is uninteresting by definition. Since it is both ex post efficient and uninteresting, then by Theorem 1, it is SW-efficient. □

Theorem 2 gives us more insight into the results of Dogan et al. [9],

Lemma 6. *An assignment problem with strict preferences does not admit Pareto indifferent deterministic assignments.*

Proof. Consider two deterministic assignments M and M' such that all agents are indifferent among them. Then this means that each agent gets the same item in both M' and M'. But this implies that $M' = M$. □

Corollary 1 (Dogan et al. [9]). *If preferences are strict, the only undominated probabilistic assignments are the Pareto efficient deterministic assignments.*

Proof. By Lemma 6, no two deterministic assignments are completely indifferent for all agents. Hence, by Theorem 2, if a random assignment that is SW-efficient, then it is a deterministic Pareto optimal assignment. □

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