Economics Bulletin

Volume 37, Issue 3

Conditional Factor Demands and Positive Output Effects: A Necessary and Sufficient Condition

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Abstract

The duality between cost and production functions has been thoroughly studied and is well-known. A given set of assumptions on the technology implies a set of restrictions on the Jacobian of the cost function and on a subset of its Hessian matrix. The vector of second derivatives of the cost function with respect to the input prices and the output has not been fully characterized, however. In this note, we present a necessary and sufficient condition to ensure that the components of this vector are all strictly positive. That is, we specify the condition for all conditional demand functions to be simultaneously increasing in output. This condition is interpreted as a strengthening of the quasi-concavity of the production function.

Citation: Pierre Ouellette and Stéphane Vigeant, (2017) "Conditional Factor Demands and Positive Output Effects: A Necessary and Sufficient Condition", *Economics Bulletin*, Volume 37, Issue 3, pages 1549-1554
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Submitted: February 09, 2017. **Published:** July 08, 2017.

1. Introduction

The duality between production and cost functions is a fundamental result in economic theory. It implies that the technology has an equivalent representation in both, the quantity space and the price space. This form of duality proved to be extremely useful in applied economics as the estimation of cost functions became a very active field of application, for production economists in particular. The number of such implementations is too large to try to credit them all here. To serve such an important growing field of applications, economists have provided the applied researchers with a fairly complete theoretical characterization of the cost functions. This characterization is not complete however, and the note here provides another piece to this characterization effort. In particular, it focuses on determining the conditions under which the quantity effects are always positive. That is, we show a necessary and sufficient condition to ensure that all conditional factor demands increase when output increases. We use the rest of this introduction to set up the problem by quickly reviewing the cost minimization problem. This allows us to set up the notation used.

It is common practice in every microeconomic textbook to study very carefully the properties of the cost function. The argument usually follows a well-ordered path. One considers a production function, $y = f(x)$, where y is the output and x the n-vector of inputs. Then, the cost function is defined by the following problem:

$$C(w,y) = Min_x \{w'x : y = f(x)\}$$

where w is the n-vector of input prices and the prime superscript is the transposition operator.

Conditions imposed on the production function translate into properties of the cost function. Suppose that the production function, f, is twice continuously differentiable ($f \in \mathcal{C}^2$), increasing in all of its arguments ($f_x > 0$, where f_x is the $1 \times n$ vector of partial derivatives of f), and strongly quasi-concave (if $dx \in \Re^n \setminus \{0\}$ is such that $f_x dx = 0$ then $dx' f_{xx} dx < 0$). These conditions imply that the cost function C is twice continuously differentiable ($C \in \mathcal{C}^2$), homogeneous of degree one in w ($C(\lambda w, y) = \lambda C(w,y)$ for $\lambda > 0$), increasing in w and y ($C_w > 0$ and $C_y > 0$), and strongly concave in w ($dw' C_{ww}\, dw < 0$ for all $dw \neq \alpha w$ and $\alpha \in \Re \setminus \{0\}$). Finally, if we strengthen the condition on the curvature of the production function by supposing it is concave in x (implying decreasing returns to scale) then the cost function is convex in y.

From the preceding paragraph discussion, we observe that the Jacobian of the cost function is characterized in every dimension (w and y), but the Hessian matrix is only partially characterized. To see this, note that the Hessian matrix of the cost function can be written:

$$D^2 C = \begin{bmatrix} C_{ww} & C_{wy} \\ C_{yw} & C_{yy} \end{bmatrix}.$$

From the discussion above, one can see that the matrices on the diagonal (namely, C_{ww} and C_{yy}) are fully characterized, but the off-diagonal terms (C_{wy} and C_{yw}) are not.

From Young's theorem, we know that $C_{wy} = C_{yw}'$ so we might just focus on C_{wy}. From Shephard's Lemma ($C_w' = x$) we deduce that $C_{wy} = x_y$. That is, C_{wy} is the conditional factor demand output effect vector. Using the definition of the cost function and the envelope theorem we obtain $w'x_y = C_y = \lambda$. But since the Lagrange multiplier is non-negative and the vector of input prices is strictly positive, we know that the quantity effects ($x_{y\cdot}$) cannot be all simultaneously negative. We also know that homotheticity of the production function is sufficient to ensure that all x_{iy} are positive. For empirical work, this might often be an excessive assumption, as it imposes too much structure on the characteristics of the technology. For example, the Cobb-Douglas production function is homothetic but it comes at the price of imposing elasticity of substitution equal to one. This is, in general, overly demanding on the technology. We wish to be able to free us from this constraint, but at the same time we wish sometimes to avoid cases where some inputs might be used less intensively as production increase. That is, we wish that the technology be compatible with an increasing output expansion path for all inputs. Figure 1 illustrates the cases of homotheticity and quasi-concave production function with $x_{iy} > 0$ and $x_{jy} < 0$. Intuitively, in this paper a condition is given to avoid situation like the one represented in panel 2 of Figure 1, without imposing conditions so restrictive, such as homotheticiy, that we end up in situations like the one represented in panel 1.

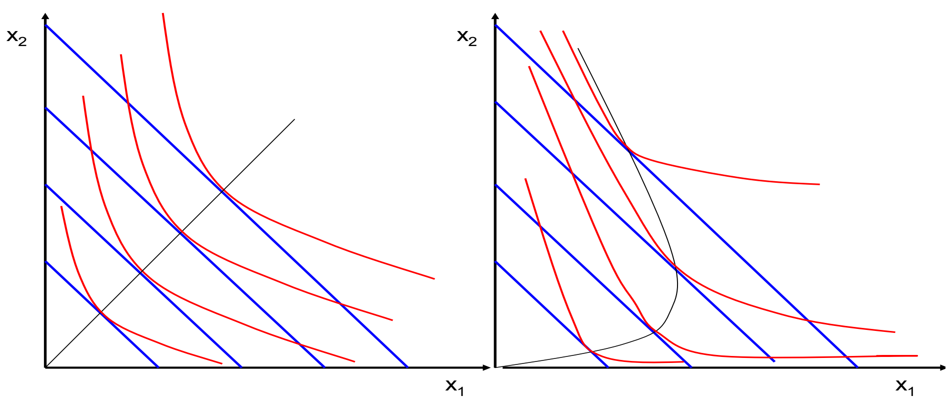

Figure 1: Output expansion paths with quasi-concave production functions – Homotheticity on the left panel, negative output effect on the right.

In this note, the characterization of the Hessian matrix of the cost function is extended by relating the positivity of the output effect vector to conditions on the production function, conditions that will be seen to be less demanding than homotheticity. More precisely, it will be shown that to have all terms of the vector x_y positive ($C_{wy} = x_y > 0$) is equivalent to a curvature condition on f.

The rest of the paper examines the relationship between x and y for a given input price vector, *i.e.* $w=\overline{w}$ and an interior solution to the cost minimization problem. This allows us to write the n-vector of conditional factor demands as $x = x(y)$. Given $w=\overline{w}$, we deduce from the solution to the cost minimization problem that the marginal rates of substitution are constant for all $y > 0$. That is, let $T_{ni} = -f_i / f_n$ for all $i = 1, \ldots, n$-1, where f_i is the partial derivative of f with respect to x_i and let $T : \Re^n \to \Re^{n-1}$ be the map from the space of the partial derivatives into the marginal rate of substitutions, $T = [T_{ni}]$. Then, we must have:

$$0 = \frac{dT(x_1(y), x_2(y), \ldots, x_n(y))}{dy} = \sum_{i=1}^{n} \frac{\partial T}{\partial x_i} \frac{dx_i}{dy}. \qquad (1)$$

This vector-equation condition gives us the key elements to characterize the curvature property of the production function in order to have a positive output effect, *i.e.* $x_{iy} > 0$ for all $i = 1, \ldots n$. We first show how the characterization works in the two input case and then generalize it to arbitrary n.

2. The two-input case

In the two-input case, it is immediate to show that:

$$\frac{dT}{dx_1} = -\frac{f_2 f_{11} - f_1 f_{12}}{f_2^2} \text{ and } \frac{dT}{dx_2} = -\frac{f_2 f_{12} - f_1 f_{22}}{f_2^2}. \qquad (2)$$

By substituting equation (2) into equation (1), we obtain:

$$\frac{C_{w_1 y}}{C_{w_2 y}} = \frac{dx_1/dy}{dx_2/dy} = \frac{f_1 f_{22} - f_2 f_{12}}{f_2 f_{11} - f_1 f_{12}}. \qquad (3)$$

For equation (3) to be positive, the sign of both the numerator and the denominator must be the same. But we have seen above that the dx_i / dy cannot be all simultaneously negative, so we conclude that $dx_i / dy > 0$ for all i if and only if $(f_1 f_{22} - f_2 f_{12}) / (f_2 f_{11} - f_1 f_{12}) > 0$. This result is not implied by quasi-concavity, and we now show that it is in fact a strengthening of the curvature condition.

Strong quasi-concavity of the production function implies that: [1]

$$f_2 (f_2 f_{11} - f_1 f_{12}) + f_1 (f_1 f_{22} - f_2 f_{12}) < 0. \qquad (4)$$

Because the production function is increasing, $f_x > 0$, it follows that $(f_2 f_{11} - f_1 f_{12})$ and $(f_1 f_{22} - f_2 f_{12})$ cannot be simultaneously positive under quasi-concavity. Strong quasi-concavity is only compatible with one of the following cases: either both terms are negative or they have opposite signs. It follows from equation (3) that when the sign of these terms is the same, the two output effects are positive. When the signs are different, one output effect is positive while the other is negative.

Table 1. Signs of output effects ($\partial x_1/\partial y$, $\partial x_2/\partial y$)

	$(f_2 f_{11} - f_1 f_{12}) > 0$	$(f_2 f_{11} - f_1 f_{12}) < 0$
$(f_1 f_{22} - f_2 f_{12}) > 0$	Not compatible with strong quasi-concavity of f	(+, -) or (-, +)
$(f_1 f_{22} - f_2 f_{12}) < 0$	(+, -) or (-, +)	(+, +)

The (+, +)-cell in Table 1 shows that for positive output effect, $C_{wy} = x_y > 0$, we need a strengthening of the strong quasi-concavity condition. That is, both $(f_2 f_{11} - f_1 f_{12})$ and $(f_1 f_{22} - f_2 f_{12})$ must be negative. This condition is not only sufficient, it is also necessary, as imposing that

[1] This is a rewriting of the determinant condition on the bordered Hessian matrix for quasi-concavity of the production function.

every output effect is positive implies a condition over each term, $(f_2 f_{11} - f_1 f_{12})$ and $(f_1 f_{22} - f_2 f_{12})$, separately.

Now, note that from quasi-concavity at least one of $(f_2 f_{11} - f_1 f_{12})$ and $(f_1 f_{22} - f_2 f_{12})$ is negative, so we may suppose that $(f_1 f_{22} - f_2 f_{12}) < 0$ without lost of generality. Using again the determinant condition on the bordered Hessian matrix of the production function, it follows from strong quasi-concavity that:

$$\frac{f_2 f_{11} - f_1 f_{12}}{f_1 f_{22} - f_2 f_{12}} > -\frac{f_1}{f_2}.$$

Quasi-concavity bounds the input effect to be larger than the marginal rate of substitution (MRS). Since the MRS (the right hand side of the equation above) is negative, it is clear that quasi-concavity is not enough to ensure positive output effect. Positive output effect is therefore more demanding on the curvature of the isoquant than what is necessary to ensure the existence of the conditional factor demands. Intuitively, the result is appealing. The curvature of the isoquant under strong quasi-concavity allows the conditional factor demands to have either positive or negative output effects. However, elimination of "backward bending" output expansion paths (as in Figure 1) requires that the optimal choice of inputs consistent with the (constant) MRS must be located in a smaller region than when it is not restricted. In other words, the curvature of the isoquant must be large enough. Of course, infinite curvature (Leontief technology) provides positive output effect. Nevertheless, the necessary and sufficient condition for positive output effects, as it was shown here, is weaker than homotheticity (the output path being linear). As it is also clear from the discussion above, homotheticity is also overly demanding to ensure positive output effects.

3. The *n*-inputs case

In this section, we generalize the results of the first section to the n-inputs case. Let $x_{-n} \equiv (x_1, \ldots, x_{n-1})$ be the input vector obtained by deleting the n^th^ component from x. Thus, the input vector is given by $x = \left(x_{-n}, x_n\right) \equiv (x_1, \ldots, x_{n-1}, x_n)$. Equation (1) can be rewritten as follows:

$$\sum_{j=1}^{n-1} \frac{\left(f_n f_{ij} - f_i f_{nj}\right)}{\left(f_i f_{nn} - f_n f_{in}\right)} \frac{dx_j}{dy} = \frac{dx_n}{dy}, \forall i = 1, \ldots, n-1. \qquad (5)$$

This equation can be expressed more compactly. Let A be a matrix with typical element $A_{ij} = \left(f_n f_{ij} - f_i f_{nj}\right), \forall (i,j) < n$ and let A_n be a $(n-1)\times(n-1)$ diagonal matrix with diagonal element $A_{in} = \left(f_i f_{nn} - f_n f_{in}\right), \forall i < n$, and zeros elsewhere. Then, equation (5) can be written as follows:

$$A_n^{-1} A \left[\frac{dx_{-n}}{dy}\right] = \left[\frac{dx_n}{dy}\right] \qquad (6)$$

where $\left[dx_n / dy\right]$ is a $(n-1)$ vector with dx_n / dy repeated in all $(n-1)$ entries of the vector. Now, note that $A_n^{-1} A$ is full ranked; It is obvious for the diagonal matrix A_n, and it follows from

Debreu (1952) Theorem 5 and quasi-concavity of the production function that A is non singular. Consequently, a sufficient condition for the output effects to be positive is:[2]

$$\frac{\left(f_n f_{ij} - f_i f_{nj}\right)}{\left(f_i f_{nn} - f_n f_{in}\right)} > 0, \forall (i,j) < n. \tag{7}$$

For n = 2, equation (7) is identical to equation (2) of the previous section, and both cases are equivalent. Consequently, the sufficient condition given by equation (6) reduces to the necessary and sufficient condition of the previous section.

When n>2, this condition is no longer necessary and the requirement for positive output effect can be weakened. Using the matrix notation above, equation (1) can be written as follows:

$$A\left[\frac{\partial x_{-n}}{\partial y}\right] = A_n\left[\frac{\partial x_n}{\partial y}\right].$$

Since A is full rank, we have:

$$\left[\frac{\partial x_{-n}}{\partial y}\right] = A^{-1}A_n\left[\frac{\partial x_n}{\partial y}\right] = \left[A_{ij}\right]^{-1}\left[A_{jn}\right]\left[\frac{\partial x_n}{\partial y}\right]. \tag{8}$$

Let A^{ij} denotes a typical element of A^{-1} and write Equation (8) as follows:

$$\left[\frac{\partial x_{-n}}{\partial y}\right] = \left[\sum_{j=1}^{n-1} A^{ij}A_{jn}\frac{\partial x_n}{\partial y}\right] = \frac{\partial x_n}{\partial y}\left[\sum_{j=1}^{n-1} A^{ij}A_{jn}\right]$$

Therefore, if the $\sum_{j=1}^{n-1} A^{ij}A_{jn}$ for $i=1,...,n-1$ are all positive, then the $\partial x_i / \partial y$ are of the same sign. But since at least one of these must be positive, they must be all positive.

The Proposition below summarizes this discussion.

Proposition: The necessary and sufficient condition for every output effects to be positive is:
$\left[\frac{\partial x_{-n}}{\partial y}\right] > 0$ and $\frac{\partial x_n}{\partial y} > 0$ if and only if $\sum_{j=1}^{n-1} A^{ij}A_{jn} > 0\ \forall i = 1,...,n-1.$[3]

It is possible to show that this condition is related to complementarity measures. It is also a strengthening of the quasi-concavity condition as we have stressed above. Finally, note that for n = 2, this condition reduces to the result obtained in the first section.

4. Translog technology

A Cobb-Douglas technology is homothetic, implying that the output-effects are necessarily positive. This is well known, so in order to show that the above condition is not trivial, we use a translog production function with two inputs and one output:

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[2] This condition is similar to Leroux (1987) sufficient condition for every consumption goods to be normal.

[3] This condition is similar to Alarie et al. (1990) necessary and sufficient condition for every consumption goods to be normal.

$$\ln y = a + b_1 \ln x_1 + b_2 \ln x_2 + 0,5 b_{11} \ln x_1 \ln x_1 + b_{12} \ln x_1 \ln x_2 + 0,5 b_{22} \ln x_2 \ln x_2 .$$

The first- and second-order partial derivatives are (for i and j = 1, 2):

$$f_i = \left(b_i + b_{i1} \ln x_1 + b_{i2} \ln x_2\right) \frac{y}{x_i} = \varepsilon_i \frac{y}{x_i},$$

$$f_{ij} = \frac{y}{x_i x_j} \left\{ b_{ij} + \varepsilon_i \varepsilon_j - \delta_{ij} \varepsilon_i \right\},$$

where $\varepsilon_i \triangleq \partial \ln y / \partial \ln x_i = \left(b_i + b_{i1} \ln x_1 + b_{i2} \ln x_2\right)$ is the elasticity of output with respect to input x_i and $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

Substituting these results in the determinant condition for quasi-concavity given by equation (4), we obtain:

$$\varepsilon_2 \left(b_{11} \varepsilon_2 - b_{12} \varepsilon_1\right) + \varepsilon_1 \left(b_{22} \varepsilon_1 - b_{12} \varepsilon_2\right) < \varepsilon_2 \varepsilon_1 \left(\varepsilon_1 + \varepsilon_2\right).$$

Since b_{ij} = 0 (the elasticities of substitution are constant) for a Cobb-Douglas technology, this equation reduces to $0 < \varepsilon_2 \varepsilon_1 \left(\varepsilon_1 + \varepsilon_2\right)$. Clearly, the Cobb-Douglas functional form always satisfies this condition.

Positive output-effects require that $\left(b_{11} \varepsilon_2 - b_{12} \varepsilon_1\right) < \varepsilon_2 \varepsilon_1$ and $\left(b_{22} \varepsilon_1 - b_{12} \varepsilon_2\right) < \varepsilon_1 \varepsilon_2$. Once again, in the Cobb-Douglas case (a homothetic technology), these conditions are trivially satisfied, and the conditions reduce to $0 < \varepsilon_2 \varepsilon_1$ and $0 < \varepsilon_1 \varepsilon_2$, since b_{ij} = 0.

This result can be more revealing if we analyze it in terms of the parameters alone. Suppose that we normalize the inputs, *i.e.* we set values $x_1 = x_2 = 1$. This implies that $\ln x_1 = \ln x_2 = 0$ which in turn implies that $\varepsilon_1 = b_1$ et $\varepsilon_2 = b_2$. Then, the positivity conditions become $\left(b_2 b_{11} - b_{12} b_1\right) < b_1 b_2$ and $\left(b_{22} b_1 - b_{12} b_2\right) < b_1 b_2$. It is now easy to see that this is a stringier requirement than quasi-concavity, as this curvature requirement is satisfied at this specific point if and only if $\frac{b_2}{\left(b_1 + b_2\right)} \left(b_{11} b_2 - b_{12} b_1\right) + \frac{b_1}{\left(b_1 + b_2\right)} \left(b_{22} b_1 - b_{12} b_2\right) < b_1 b_2$.

5. References

Alarie, Y., C. Bronsard and P. Ouellette (1990) "Preferences and Normal Goods: A Necessary and Sufficient Condition" *Journal of Economic Theory* **51**, 423-430.

Debreu, G. (1952) "Definite and Semidefinite Quadratic Forms" *Econometrica* **20**, 295-300.

Leroux, A. (1987) "Preferences and Normal Goods: A Sufficient Condition" *Journal of Economic Theory* **43**, 192-199.