Economics Bulletin

Volume 37, Issue 4

Stochastic optimization without Ito's lemma: applications to the portfolio model

Moawia Alghalith
UWI, St Augustine

Abstract

We show that the key results of the stochastic models (that use stochastic calculus) can be easily derived using classical calculus and without restrictive assumptions. We apply our method to two major areas in stochastic analysis: optimization and partial differential equations. For example, we apply the method to the portfolio model and the Black-Scholes partial differential equations.

Citation: Moawia Alghalith, (2017) "Stochastic optimization without Ito's lemma: applications to the portfolio model", *Economics Bulletin*, Volume 37, Issue 4, pages 2533-2536
Contact: Moawia Alghalith - malghalith@gmail.com.
Submitted: August 08, 2017. **Published:** November 19, 2017.

1. Introduction

In this paper, especially in mathematical finance, we show that the key results of the stochastic models (that use stochastic calculus) can be derived without Ito's lemma or Itos's calculus. That is, we derive these results using classical calculus and without the restrictive assumptions adopted by Ito (see Ito (1944)). Moreover, our approach is more precise than Ito's approach in the sense that we don't use the famous, controversial, imprecise rules, such as $(dt)^2 = 0$.

We apply our method to two major areas in stochastic analysis: optimization and partial differential equations PDEs. In particular, we apply our methods to the Black-Scholes PDE and the portfolio model.

2. Optimization

In this section, we apply our method to the area of continuous-time, stochastic optimization. As an example, we choose the portfolio model (a dominant model in mathematical finance; see, for example, Cvitanic and Zapatero (2004), Alghalith (2009) and Detemple (2014)). In this paper, we use standard technical assumptions.

Similar to previous literature, the price of the risky asset, at an arbitrary time u, is given by

$$S(u) = S(t)\, e^{\theta u + \sigma W(u)}; t \leq u,$$

where $W(u)$ is a standard Brownian motion, while θ and σ are constants. Clearly (using classical calculus)

$$dS(u) = S(u)\left(\theta du + \sigma dW(u)\right). \tag{1}$$

The wealth process $X(u)$ satisfies the following equation (a well-known result in the portfolio literature)

$$dX(u) = \left[rX(u) + \pi(u)(\theta - r)\right] du + \pi(u)\,\sigma dW(u), \tag{2}$$

where r is the risk-free rate of return, and $\pi(u)$ is the risky portfolio process. Thus,

$$\begin{aligned}(dX(u))^2 &= \left[rX(u) + \pi(u)(\theta - r)\right]^2 (du)^2 + \left(\pi(u)\,\sigma dW(u)\right)^2 + \\ &\quad 2\left[rX(u) + \pi(u)(\theta - r)\right]\pi(u)\,\sigma du dW(u). \end{aligned} \tag{3}$$

Proposition 1. The optimal portfolio is given by $\pi(t) = -\frac{(\theta - r)V_X(t,X(t))}{\sigma^2 V_{XX}(t,X(t))}$.

Proof. Consider the following exact Taylor's expansion of the value function around the point of expansion (t, x)

$$\begin{aligned} dV(u,X) &= V_u du + V_X dX + V_{uX} du dX \\ &\quad + \frac{1}{2}\left[V_{XX}(dX)^2 + V_u (du)^2\right] + R(u,X), \end{aligned} \tag{4}$$

where R is the remainder. Substituting (3) and (2) into (4) yields

$$\begin{aligned} dV(u,X) &= V_u du + V_X \left[rX(u) + \pi(u)(\theta - r)\right] du \\ &\quad + V_{uX}\left[rX(u) + \pi(u)(\theta - r)\right](du)^2 + \\ &\quad \frac{1}{2}\left\{ \begin{array}{c} V_{XX} \left[\begin{array}{c} \left[rX(u) + \pi(u)(\theta - r)\right]^2 (du)^2 + \\ \left(\pi(u)\sigma dW(u)\right)^2 + \\ 2\left[rX(u) + \pi(u)(\theta - r)\right]\pi(u)\sigma dW(u) du \end{array} \right] \\ + V_{uu}(du)^2 \end{array} \right\} \\ &\quad + R(u,X). \end{aligned}$$

Taking expectations on both sides, we obtain

$$\begin{aligned} E\left[dV(u,X)\right] &= E[V_u du + V_X \left(rX(u) + \pi(u)(\theta - r)\right) du \\ &+ V_{uX}\left[rX(u) + \pi(u)(\theta - r)\right](du)^2 \\ &+ \frac{1}{2}\left\{ \begin{array}{c} V_{XX}\left(\left[rX(u) + \pi(u)(\theta - r)\right]^2\right)(du)^2 \\ + V_{uu}(du)^2 \end{array} \right\} \\ &\quad + R(u,X)] + \frac{1}{2}\sigma^2 du E\pi^2(u) V_{XX}. \end{aligned} \tag{5}$$

Differentiating (5) w.r.t. $\pi(u)$ yields

$$E\left[\begin{array}{c} V_X(\theta - r) du + V_{uX}(\theta - r)(du)^2 + \\ V_{XX}\left[\left(rX(u) + \pi(u)(\theta - r)\right)(\theta - r)(du)^2 + \pi(u)\sigma^2 du\right] \end{array} \right] = 0.$$

Dividing by du yields

$$E\left[\begin{array}{c} V_X(\theta - r) + V_{uX}(\theta - r) du + \\ V_{XX}\left[\left(rX(u) + \pi(u)(\theta - r)\right)(\theta - r)) du + \pi(u)\sigma^2\right] \end{array} \right] = 0.$$

Evaluating at time t (the current time), we obtain

$$V_X(\theta - r) + V_{XX}\pi(t)\sigma^2 = 0,$$

since $du = 0$ at t. Therefore, the optimal portfolio is given by

$$\pi(t) = -\frac{(\theta - r) V_X(t, X(t))}{\sigma^2 V_{XX}(t, X(t))}.\square$$

Needless to say, this is the same formula obtained by stochastic calculus.

3. PDEs– the Black-Scholes equation

In this section, we apply the method to PDEs. As an example, we apply the method to the Black-Scholes PDEs (see Black and Scholes (1973)) .

Using (1) , we obtain

$$(dS)^2 = S^2 \left[\theta^2 (du)^2 + \sigma^2 (dW(u)^2 + 2\theta\sigma du dW(u)\right]. \quad (6)$$

Proposition 2. The price of the option satisfies this PDE

$$\begin{aligned} & C_t(t, S(t)) + r\left[C_S(t, S(t)) S(t) - C(t, S(t))\right] + \frac{1}{2}\sigma^2 C_{SS}(t, S(t)) S(t)^2 \\ = \ & 0. \end{aligned}$$

Proof. Consider the following exact Taylor's expansion of the price of the European call option $C(u, S)$ around (t, s)

$$\begin{aligned} dC(u, S) &= C_u du + C_S dS + C_{Su} du dS + \\ & \frac{1}{2}\left[C_{SS}(dS)^2 + C_{uu}(du)^2\right] + R(u, S). \end{aligned} \quad (7)$$

Substituting (1) and (6) into (7) , and taking expectations yields

$$\begin{aligned} EdC(u, S) &= E\left[\begin{array}{c} C_u du + C_S S\theta du + C_{Su} S\theta (du)^2 + \\ \frac{1}{2}\left[C_{SS} S^2\theta^2 (du)^2 + C_{uu}(du)^2 + R(u, S)\right] \end{array}\right] \\ & + \frac{1}{2}\sigma^2 du E C_{SS} S^2. \end{aligned}$$

Using the Black-Scholes assumptions (replicating portfolio and risk neutrality), we obtain

$$E\left[\begin{array}{c} C_u du + C_S S r du + C_{Su} S r \left(du\right)^2 + \\ \frac{1}{2}\left[C_{SS} S^2 r^2 \left(du\right)^2 + C_{uu} \left(du\right)^2 + R\left(u, S\right)\right] \end{array}\right] +$$

$$\frac{1}{2}\sigma^2 du E C_{SS} S^2 - r E C\left(u, S\right) = 0.$$

Dividing by du, then evaluating at t, we obtain the well-known Black-Scholes PDEs

$$\begin{aligned} & C_t\left(t, S\left(t\right)\right) + r\left[C_S\left(t, S\left(t\right)\right) S\left(t\right) - C\left(t, S\left(t\right)\right)\right] + \frac{1}{2}\sigma^2 C_{SS}\left(t, S\left(t\right)\right) S\left(t\right)^2 \\ = \ & 0.\square \end{aligned}$$

Needless to say, our approach can be applied to many other models in stochastic analysis.

References

[1] Alghalith, M. (2009) “A new stochastic factor model: general explicit solutions” *Applied Mathematics Letters* **22**, 1852-1854.

[2] Black, F and M. Scholes (1973) “The pricing of options and corporate liabilities” *Journal of Political Economy* **81**, 637–654.

[3] Cvitanic, J. and F. Zapatero (1994) *Introduction to the economics and Mathematics of financial markets*, MIT Press, Cambridge, MA.

[4] Detemple, J. (2014) Portfolio selection: a review *Journal of Optimization Theory and Applications* **161**, 1-121.

[5] Itô, K. (1944) ”Stochastic Integral” *Proc. Imperial Acad. Tokyo,* **20**, 519-524.