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The Golden Rule Without Marginal Productivities and Differential Rents: A Remark

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Abstract

The "Golden Rule of Accumulation" characterizes the savings rate that entails the highest possible rate of per capita consumption in the long term. Any permanent change in the savings rate, be it an increase or a reduction, would eventually bring about lower per capita consumption. In the models studied, such a maximum was characterized by the equality of the savings rate and the profit rate or, equivalently, by a rate of interest being equal to the growth rate. The present note shows that the rule applies also to the hybrid model of endogenous growth proposed by Schlicht (2016).

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1 Introduction

The "Golden Rule of Accumulation", sometimes also referred to as the "Neo-Classical Theorem", characterizes the savings rate that entails the highest possible rate of per capita consumption in the long term. If the economy grows along such a path, any permanent change in the savings rate, be it an increase or a reduction, would eventually bring about lower per capita consumption. In the models studied, such a maximum was characterized by the equality of the savings rate and the profit rate or, equivalently, by a rate of interest being equal to the growth rate.[1]

The Rule was derived in a variety of growth models such as the standard neoclassical growth model or vintage models with Leontief production functions. In models with endogenous growth rates, as initiated by Frankel (1962), Vogt (1966/69), and Romer (1986), Golden equilibria typically do not exist, because higher savings entail more investment and increased technical progress. Consumption achievable from a path obtained from a savings rate arbitrarily close to unity would eventually dominate consumption achievable on any path with a lesser rate of savings and, therefore, lower growth. Under such circumstances, no Golden savings rate can exist. Yet some other models of endogenous growth do permit Golden equilibria, such as von Weizsäcker (1966) or Phelps (1966). The present note shows that the model proposed by Schlicht (2016) can be added to this list.

Several writers have emphasized that the problem of the existence of a Golden saving rate is a purely theoretical question, with possibly little practical relevance (Phelps 1965, 813; Robinson 1962, 224). Yet the analysis may help to complete the theoretical picture and may be apposite in this sense. Further, the theory of interest proposed in the hybrid model relates to the rate of capital deepening and differs both from the standard neoclassical explanation in terms marginal productivity of capital as well as from the vintage models with Leontief technology that appeal to Ricardian differential rents accruing to newer and more productive machinery as compared to older production units.

The hybrid model appeals to the present writer for reasons mentioned in Schlicht (2016). In particular it avoids the central weakness of the prototype neoclassical approach that has been stated by Aghion and Howitt (2009, 47) as follows:

> people must be given an incentive to improve technology. But because the aggregate production function F exhibits constant returns in K [capital] and L [labor] alone, Euler's theorem tells us that it will take all of the economy's output to pay capital and labor their marginal products in producing final output, leaving nothing over to pay for the resources used in improving technology.

The hybrid model avoids also, to some extent, a central weakness of some AK-models of endogenous growth, where "cross-country variation in parameters … will result in permanent differences in rates of economic growth." (Aghion and Howitt, 2009, 55). The main advantage of the hybrid model is, in this writer's view, that it accounts more naturally for Harrod-neutral technological progress than other approaches do that I am aware of. Let me add that that I don't see the hybrid model as a "correct" model of economic growth. It offers just another macro perspective on processes of economic growth. Future applications may show how useful it may be for dealing with definite questions.

[1] The Golden Rule was, according to Allais (1962, 726), first stated by Jacques Desrousseaux in 1959 in an apparently unpublished paper, see also Desrousseaux (1961). The rule was also independently discovered by Edmund Phelps (1961), Joan Robinson (1962), Carl-Christian von Weizsäcker (1962), and Trevor Swan (1964).

The argument proceeds in two steps. In the following section, the Golden savings rate will be determined in a purely technical way from the slope of the technical progress function. In Section 3, the savings rate will be related to the rate of interest and the profit share. This gives rise to the Golden Rule in its two variants.

2 The Golden Savings Rate in the Hybrid Model

Consider an economy that produces an output Y with capital K and labor N. With capital productivity x and labor productivity y, production is given by the Leontief production function $Y = \min\{xK, yN\}$. The changes of capital productivity and labor productivity over time are determined by the firms' choices of capital deepening, induced by changes in wages and the rate of interest such that full employment of capital and labor is maintained. The full employment assumption appears appropriate in the present context where an optimal savings rate is to be determined, but may be highly problematic under other circumstances.

Labor grows with a growth rate v and labor productivity y grows over time due to technical progress and capital deepening. Denoting time derivatives by a dot and growth rates by a hat, the growth rate of labor productivity is written as $\hat{y} = \frac{\dot{y}}{y} = \frac{1}{y}\frac{dy}{dt}$. Capital deepening occurs if the capital-labor ratio $k = \frac{K}{N}$ increases over time. The rate of capital deepening is $\hat{k} = \left(\frac{\hat{K}}{N}\right) = \hat{K} - \hat{N} = \hat{K} - v$.

A fraction $s \in (0, 1)$ of output is saved. These savings are invested and enlarge the capital stock. Depreciation of the capital stock occurs at the rate δ. The change of the capital stock over time $\dot{K}$ is therefore given by the equation

$$\dot{K} = sY - \delta K. \tag{1}$$

Dividing by K gives

$$\hat{K} = sx - \delta$$

with $x = \frac{Y}{K}$ as capital productivity. This implies for the rate of capital deepening

$$\hat{k} = sx - (\delta + v) \tag{2}$$

The sum of depreciation and population growth $(\delta + v)$ is assumed to be non-negative. Equation (2) is the standard accumulation equation encountered in many growth models, sometimes referred to as the Solow equation.

In the model to be analyzed in the following, no distinction is made between investment in physical capital and investment in developing new technology; rather investment outlays are optimally divided between development and production of new machinery. Hence investment in new equipment and investment in technology occur jointly, as in AK models.

If the capital labor ratio k remains constant, there is pure capital widening and no capital deepening. Although no net investment per worker is realized, some investment occurs that just compensates depreciation and population growth, and this investment entails some improvements in technology. So we will have some growth in labor productivity for pure capital widening ($\hat{y} > 0$ for $\hat{k} = 0$).

With capital deepening ($\hat{k} > 0$), workers will be equipped with more and better machinery over time. Such entailed productivity increases will be more pronounced with more capital deepening, but the effect will be subject to decreasing returns such that very high rates of capital deepening will induce only very small increases in productivity growth. It will thus be postulated that the growth of labor productivity $\hat{y}$ is an increasing function of capital deepening $\hat{k}$:

$$\hat{y} = \varphi\left(\hat{k}\right), \ \varphi(0) > 0, \ \varphi' > 0, \ \varphi'' < 0, \ \lim_{\hat{k}\to\infty} \varphi'\left(\hat{k}\right) = 0. \tag{3}$$

This is the technical progress function proposed by Kaldor (1957). Because it is positive throughout but flattens out for large rates of capital deepening, there exists a root $\gamma > 0$ such that

$$\varphi\left(\gamma\right) = \gamma, \ \varphi'\left(\gamma\right) \in (0,1) \tag{4}$$

and

$$\varphi\left(\hat{k}\right) \gtreqless \hat{k} \ \Leftrightarrow \ \hat{k} \lesseqgtr \gamma. \tag{5}$$

As the output-capital ratio is $x = \frac{Y}{K}$, we have $x = \frac{y}{k}$ and its growth rate is $\hat{x} = \hat{y} - \hat{k}$ or

$$\hat{x} = \varphi\left(\hat{k}\right) - \hat{k}. \tag{6}$$

Together with (2) the growth rate of capital productivity is obtained as

$$\hat{x} = \varphi\left(sx - (\delta + \nu)\right) - \left(sx - (\delta + \nu)\right). \tag{7}$$

With (5) we find

$$\hat{x} \gtreqless 0 \ \Leftrightarrow \ x \lesseqgtr \frac{1}{s}\left(\gamma + \delta + \nu\right). \tag{8}$$

Hence

$$\bar{x} = \frac{1}{s}\left(\gamma + \delta + \nu\right) \tag{9}$$

is the globally stable equilibrium of the differential equation (7) that describes the movement of the output-capital ratio x over time. In this equilibrium, per capita income y grows with rate $\hat{y} = \varphi\left(s\bar{x} - (\delta + \nu)\right) = \varphi\left(\gamma\right) = \gamma$.

In the following we consider such a balanced growth path with

$$\bar{y}_t = y_0 e^{\gamma t} \tag{10}$$

for some initial labor productivity y_0 and and compare this with the growth path for per-capita income obtained from a system with a slightly different savings rate $s + \varepsilon$ with the same initial values for labor productivity and capital productivity:

$$\begin{aligned}
\hat{x} &= \varphi\left((s+\varepsilon)x - (\delta+\nu)\right) - \left((s+\varepsilon)x - (\delta+\nu)\right) && (11)\\
\hat{y} &= \varphi\left((s+\varepsilon)x - (\delta+\nu)\right) && (12)\\
x_0 &= \bar{x} = \frac{1}{s}\left(\gamma+\delta+\nu\right) && (13)\\
y_0 &= \bar{y}_0 && (14)
\end{aligned}$$

The disturbance ε in the savings rate is taken as arbitrarily small.

The comparison is done by looking at the ratio of the entailed output-labor ratios $\upsilon = \frac{y}{\bar{y}}$. For υ we obtain the differential equation

$$\hat{\upsilon} = \varphi\left((s+\varepsilon)x - (\delta+\nu)\right) - \gamma. \tag{15}$$

The analysis proceeds by analyzing the system (11),(15) with initial values $x_0 = \bar{x}$ and $z_0 = 1$ and by replacing φ around γ by its Taylor approximation

$$\varphi\left(\hat{k}\right) = \gamma + \alpha\left(\hat{k} - \gamma\right). \tag{16}$$

where

$$\alpha = \varphi'\left(\gamma\right) \in (0,1)$$

denotes the slope of the technical progress function at equilibrium. We thus obtain the system

$$\hat{x} = -(1-\alpha)\left((s+\varepsilon)x - \left(\delta+\nu+\gamma\right)\right) \tag{17}$$

$$\hat{v} = \alpha\left((s+\varepsilon)x - \left(\delta+\nu+\gamma\right)\right). \tag{18}$$

$$x_0 = \frac{1}{s}\left(\gamma+\delta+\nu\right) \tag{19}$$

$$v_0 = 1. \tag{20}$$

With the output-capital ratio at its initial value (19), equation (17) has the solution

$$x_t = \frac{\left(\gamma+\delta+\nu\right)}{s+\varepsilon\left(1-e^{-(1-\alpha)\left(\gamma+\delta+\nu\right)t}\right).} \tag{21}$$

Plugged into (18), we obtain a differential equation for v with the solution

$$v_t = \left(s+\varepsilon\left(1-e^{-(1-\alpha)\left(\gamma+\delta+\nu\right)t}\right)\right)^{\frac{\alpha}{1-\alpha}} s^{-\frac{\alpha}{1-\alpha}} \tag{22}$$

that has the limit

$$v_\infty = \left(1+\frac{\varepsilon}{s}\right)^{\frac{\alpha}{1-\alpha}}. \tag{23}$$

As a consequence, an increase in the savings rate by ε will increase the output-capital ratio permanently. This is similar to the neoclassical model.

The relevant comparison is not between between income levels, but rather between consumption levels. Per capita consumption on the equilibrium path is $\bar{c}_t = (1-s)\,\bar{y}_t$ whereas per capita consumption on the disturbed path is $c_t = (1-(s+\varepsilon))\,y_t$. So the ratio of per capita consumption on the disturbed path to per capita consumption on the equilibrium path is given by

$$q_t = \frac{c_t}{\bar{c}_t} = \frac{(1-(s+\varepsilon)}{(1-s)} v_t. \tag{24}$$

This expression – the ratio of per capita consumption on the two paths – has the limit

$$q_\infty = \left(1-\frac{\varepsilon}{1-s}\right)\left(1+\frac{\varepsilon}{s}\right)^{\frac{\alpha}{1-\alpha}}. \tag{25}$$

The derivative with respect to ε at $\varepsilon = 0$ is given by

$$\left.\frac{\partial q_\infty}{\partial \varepsilon}\right|_{\varepsilon=0} = \frac{\alpha - s}{s(1-s)(1-\alpha)}. \tag{26}$$

Starting from a balanced growth path where the slope of the technical progress function $\alpha = \varphi'(\gamma)$ exceeds the savings rate s, a slight increase in the savings rate will entail an increase in per capita consumption in the long term. Conversely, a reduction of the savings that initially exceeds the slope α will lead to higher per capita consumption in the entire future.

More formally, a necessary condition for an optimal savings rate is that derivative (26) is zero at $\varepsilon = 0$. This gives the Golden savings rate

$$s^* = \alpha. \tag{27}$$

The sufficient condition for a maximum q_∞ is that the second derivative at $\varepsilon = 0$ and $s = \alpha$ is negative. This is the case:

$$\left.\frac{\partial^2 q_\infty}{\partial \varepsilon^2}\right|_{\varepsilon=0} = \frac{-1}{(1-\alpha)^2\,\alpha} < 0. \tag{28}$$

Remark, following a suggestion by Peter Skott. Around the equilibrium path the linearized technical progress function (16) can be integrated into a Cobb-Douglas production function with capital elasticity α and labor elasticity $(1-\alpha)$ (Black, 1962). Hence the standard neoclassical Golden Rule derivation can be formally applied and gives the same result, *i. e.* (27).

3 The Rate of Interest, the Profit Share, and the Rate of Savings

In the hybrid model, income distribution is determined by the condition that the choice of the rate of capital deepening is cost minimizing. This implies that the slope of the technical progress function equals the profit share. Such a theory of distribution implies together with (27) the Golden Rule, as will be explained in the following.

With a rate of interest r and a real wage rate w, unit costs are

$$u = \frac{wN + (r+\delta)K}{Y} = \frac{w}{y} + \frac{(r+\delta)}{x}. \tag{29}$$

The change in unit costs over time is $\dot{u} = -\frac{r+\delta}{x}\hat{x} - \frac{w}{y}\hat{y} + \frac{\dot{w}}{y} + \frac{\dot{r}}{x}$. Using the technical progress function as in (3) and (6) this can be written as

$$\dot{u} = -u\varphi\left(\hat{k}\right) + \frac{(r+\delta)}{x}\hat{k} + \frac{\dot{w}}{y} + \frac{\dot{r}}{x}. \tag{30}$$

Hence the change in unit costs depend on the rate of capital deepening that can be determined by the firms, and on changes in the wage rate and the rate of interest that are determined by the market and are taken as given by the firms. The strongest decline in unit costs is obtained when the first order condition

$$\frac{\partial \dot{u}}{\partial \hat{k}} = -u\varphi' + \frac{(r+\delta)}{x} = 0 \tag{31}$$

is satisfied. (The second order condition $\frac{\partial^2 \dot{u}}{\partial \hat{k}^2} = -u\varphi'' > 0$ is satisfied because of (6). A linear technical progress function would rule out this mechanism.) In equilibrium pure profits are eliminated. Unit costs u are one and the slope of the technical progress function is α. As

$$\pi = \frac{(r+\delta)}{x} \tag{32}$$

gives the share of (gross) profits, condition (31) states that the strongest decline in unit costs is achieved if the profit share equals the slope of the technical progress function. As is has been established previously that the Golden savings rate equals the slope of the technical progress function in equilibrium (see (27)) we arrive at the Golden Rule in its first version, namely that maximum per capita consumption in the long term is achieved if the savings rate equals the share of profits:

$$s^* = \pi. \tag{33}$$

Together with the equilibrium condition (9) and the definition (32), the first version of the Golden Rule (33) implies the Golden Rule in its second version, namely that the rate of interest is to be equal to the growth rate as the sum of productivity growth and population growth:

$$r = \gamma + \nu. \tag{34}$$

The above derivation in terms of cost reduction, or "gradient cost minimization," is the simplest way to achieve the result. In Schlicht (2016, Appendix) it is shown that the equality of the slope of the technical progress function and the share of profits is also implied by discounted cost minimization in equilibrium. Hence both versions of the Golden Rule can be derived also in a more elaborate fashion.

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Proofs

A *Mathematica* notebook containing alI the proofs is available at http://www.semverteilung.vwl.uni-muenchen.de/mitarbeiter/es/software/Golden%20Rule%20Proofs.nb.

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Appendix: A Note on Kaldor's 1957 Model

The hybrid model underying this note is based on Kaldor's (1957) model, rather than on his later contributions on growth. Peter Skott, who has analyzed Kaldor's contributions to growth and distribution extensively and carefully, concluded that "the prototype model in Kaldor (1957) thus gives the best representation of Kaldor's theory." I fully agree, and I would like to add that this seems to correspond with Kaldor's own assessment at the beginning of the 1970ties where he remarked at a lecture at the Max-Planck Institute in Starnberg, Germany, that his newer models are not necessarily better than his older ones. Perhaps this is of interest to future historians of economic thought.