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Infinite Population and Positive Responsiveness: A Note

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Abstract

This paper shows an Arrovian impossibility result with acyclic social preferences. The set of individuals is assumed to be infinite. Positive responsiveness and a strong version of non-dictatorship are imposed.

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1 Introduction

Arrow's (1951) impossibility theorem does not hold if population is infinite (Fishburn, 1970).[1] Even positive responsiveness can be satisfied: there exists a social welfare function that satisfies Arrow's axioms and positive responsiveness (Cato, 2017a). This is a "possibility result." One possible criticism on this result is the definition of non-dictatorship, which requires that nobody can have the power to determine social rankings. It might be reasonable to require no "very small" coalition to be the decisive power in the infinite-population setting. This "coalitional" version of non-dictatorship is called *strong non-dictatorship*.

The size of coalitions is crucial here. Infinite sets of individuals are compared under a society with an infinite population. Kirman and Sondermann (1972) introduce an *atomless measure space* of agents to measure the size. The space of agents is well-structured, and the size of each coalition is well-defined. They show that there exist no social welfare function if we impose strong non-dictatorship instead of non-dictatorship.[2]

In this paper, we assume that the set of individuals is countably infinite and that social preferences are required only to be acyclic and complete. This population structure is particularly important when we consider the problem of intergenerational decision making. If we use the *natural density* to define the size of coalitions, there exist no social welfare function that satisfies weak Pareto, the independence of irrelevant alternatives, and strong non-dictatorship (Cato, 2017b). This impossibility result relies on the assumption that social preferences are orderings ("social welfare function"): if we weaken social rationality from transitivity to acyclicity, a possibility result arises.[3] If positive responsiveness is imposed, the impossibility theorem is recovered. Our theorem states that there is no social decision function that satisfies weak Pareto, the independence of irrelevant alternatives, strong dictatorship, and positive responsiveness.

A main result of Nagahisa (1993) is closely related to our theorem. Considering an atomless measure space of agents, he obtains the impossibility result under the same set of axioms, although definitions of the axioms are different from ours. There are several differences between his work and ours. First, Nagahisa (1993) consider a continuum of agents, while we consider the discrete set of agents. Second, Nagahisa's measure determines the size of any coalition, while the natural density, which we employ here, cannot measure all coalitions. The third point is related with a structural assumption on the population. If there is an atom, a possibility result arises in his framework. On the other hand, we do not have such an exception.

2 Setting

Let X be a set of outcomes. We assume that $\#X \geq 3$. The set of orderings (transitive and complete binary relations) is denoted by $\wp$.[4] The set of individuals N is the set $\mathbb{N}$ of natural numbers. Each individual has a preference ordering $\succsim_i \in \wp$ on X. Then, a preference profile is a list of individual preferences $(\succsim_i)_{i\in\mathbb{N}} \in \wp^{\mathbb{N}}$.

[1]Several authors address social choice with an infinite population: Lauwers and Van Liedekerke (1995), Cato (2013), and Takayama and Yokotani (2017).

[2]See Proposition 5 of Kirman and Sondermann (1972).

[3]We can apply the Pareto extension rule, which is proposed by Sen (1969, 1970). See also Hansson (1976) and Iritani, Kamo, and Nagahisa (2013).

[4]Cato (2016) provides a basic analysis on properties, such as transitivity, completeness, and acyclicity.

In this paper, social preferences $\succsim$ are assumed to be acyclic and complete. Acyclicity requires that for all $K \in \mathbb{N}$ and $x^0, x^1, \dots, x^K \in X$,

$$\left[x^{k-1} \succ x^k \text{ for all } k \in \{1, 2, \dots, K\}\right] \Rightarrow x^K \not\succ x^0.$$

A *social decision function* f is a mapping from the set of individual profiles to the set of acyclic and complete binary relations.[5] As usual, $\succsim$ and $\succsim'$ are social preferences under $(\succsim_i)_{i\in\mathbb{N}}$ and $(\succsim'_i)_{i\in\mathbb{N}}$, respectively. Note that we implicitly impose the unrestricted domain.

A coalition $A \subseteq \mathbb{N}$ is *decisive* if, for all $x, y \in X$ with $x \neq y$,

$$[x \succ_i y \text{ for all } i \in A] \Rightarrow x \succ y.$$

Given f, let $\mathcal{D}_f$ denote the family of decisive coalitions.

We next introduce Arrow's axioms on f.

Weak Pareto: $\mathbb{N}$ is decisive.

Independence of Irrelevant Alternatives: For all $(\succsim_i)_{i\in\mathbb{N}}, (\succsim'_i)_{i\in\mathbb{N}} \in \wp^{\mathbb{N}}$, and for all $x, y \in X$, if $\succsim_i$ and $\succsim'_i$ agree on $\{x, y\}$ for all $i \in \mathbb{N}$, then $\succsim$ and $\succsim'$ agree on $\{x, y\}$.

f is called *Arrovian* if it satisfies weak Pareto and independence of irrelevant alternatives.

Positive responsiveness is formulated as follows.

Positive Responsiveness: For all $(\succsim_i)_{i\in\mathbb{N}}, (\succsim'_i)_{i\in\mathbb{N}} \in \wp^{\mathbb{N}}$, and for all $x, y \in X$, if $\forall i \in \mathbb{N} : \big[[x \succ_i y \Rightarrow x \succ'_i y] \wedge [x \sim_i y \Rightarrow x \succsim'_i y]\big]$ and $\exists k \in \mathbb{N} : \big[[x \sim_k y \wedge x \succ'_k y] \vee [y \succ_k x \wedge x \succsim'_k y]\big]$, then $[x \succsim y \Rightarrow x \succ' y]$.

This axiom requires that social preferences monotonically changes for changes in individual preferences. May (1952) introduces this axiom as the core of the simple majority rule in the case of two alternatives. Mas-Colell and Sonnenschein (1972) employ it in the Arrovian framework with $\#X \geq 3$.

3 Results

For each $A \subseteq \mathbb{N}$, define

$$\alpha^*(A; n) = \frac{\#A \cap \{1, \dots, n\}}{n}.$$

The natural density of A is given by

$$\alpha(A) = \lim_{n\to\infty} \alpha^*(A; n),$$

if the limit exists. For example, $\alpha(\{2n : n \in \mathbb{N}\}) = 1/2$ and $\alpha(\{n^2 : n \in \mathbb{N}\}) = 0$. Note that $\alpha(A)$ is not always defined.

The coalitional version of non-dictatorship in the current framework is defined as follows.

Strong Non-Dictatorship: For some $\varepsilon > 0$, there exists no decisive coalition $A \subseteq \mathbb{N}$ such that $\alpha(A) < \varepsilon$.

For $A \subseteq N$ and $a \in N$, let A_{+a} be $A \cup \{a\}$.

[5] Arrow's social welfare function is a social decision function whose range is restricted to the set of orderings.

Theorem 1. *There exists no Arrovian social decision function that satisfies positive responsiveness and strong non-dictatorship.*

The following result is proved by Cato (2017a, Theorem 4).

Lemma 1. If an Arrovian social decision function f satisfies positive responsiveness, then $\mathcal{D}_f$ satisfies the following conditions:
(i) if $A^0, A^1, \ldots, A^K \in \mathcal{D}_f$, then $B_{+a} \in \mathcal{D}_f$ for all $a \notin B$, where $B = \bigcap_{k\in\{1,\ldots,K\}} A^k$.
(ii) if $\#A \geq 3$ and $A \notin \mathcal{D}_f$, then $N \setminus A_{-a} \in \mathcal{D}_f$ for all $a \in A$.

Proof of Theorem 1. By strong non-dictatorship, there exists $\varepsilon > 0$ such that $A \notin \mathcal{D}_f$ for all A such that $\alpha(A) < \varepsilon$. Choose $K \in \mathbb{N}$ such that $1/K < \varepsilon$. We can consider K coalitions, $A_1, \ldots, A_K$, as follows:

$$A_k = \{i \in \mathbb{N} : i = k + K(n-1) \text{ for some } n \in \mathbb{N}\}. \quad (k = 1, \ldots, K).$$

That is,

$$\begin{aligned} A_1 &= \{1, K+1, 2K+1, 3K+1, \ldots\}, \\ A_2 &= \{2, K+2, 2K+2, 3K+2, \ldots\}, \\ &\vdots \\ A_K &= \{K, 2K, 3K, 4K, \ldots\}. \end{aligned}$$

Note that $\alpha(A_k) = 1/K$. Thus, $A_k \notin \mathcal{D}_f$. Define

$$C_k = (N \setminus A_k) \cup \{k\}.$$

Lemma 1(ii) implies that $C_k \in \mathcal{D}_f$ for all $k \in \{1, \ldots, K\}$. Note that

$$\bigcap_{k\in\{1,\ldots,K\}} C_k = \{1, \ldots, K\}.$$

Lemma 1(i) implies that $\{1, \ldots, K, K+1\} \in \mathcal{D}_f$. Note that for a number n which is larger than K,

$$\alpha^*(\{1, \ldots, K, K+1\}, n) = \frac{K+1}{n}.$$

Thus, $\alpha(\{1, \ldots, K, K+1\}) = \lim_{n\to\infty} \alpha^*(\{1, \ldots, K, K+1\}, n) = 0$. This contradicts strong non-dictatorship. ■

This result strengthens a classical result by Mas-Colell and Sonnenschein (1972), which states that there exists a vetoer for an Arrovian social decision function satisfying positive responsiveness under a finite population.

Nagahisa (1993) shows a related result. He considers a measure space $(N, \mathcal{N}, \mu)$ of individuals, where $\mathcal{N}$ is a σ-algebra and μ is a measure on N. He imposes the following axiom: there is $\varepsilon > 0$ such that any coalition A is not decisive if $\mu(A) < \varepsilon$. He shows that there is no social decision function that satisfies this axiom, weak Pareto, independence irrelevant alternatives, and positive responsiveness. Clearly, our result is an extension of his result.

An implication of Nagahisa's axiom is dependent on the structure of the measure. For example, let us assume that there is an atom $\{a\}$ in $(N, \mathcal{N}, \mu)$ such that $\mu(\{a\}) = 1/10$. Define

$$f(\mathbf{R}) = R_a.$$

That is, a social preference is identical with individual a's preference. This rule is a dictatorial social decision function. However, this satisfies Nagahisa's strong non-dictatorship because there is no decisive coalition that is smaller than 1/10. Since it is also Arrovian and satisfies positive responsiveness, all axioms in Theorem 1 are compatible in this framework. In this sense, Nagahisa's strong non-dictatorship can be independent of non-dictatorship. On the other hand, our strong non-dictatorship always implies non-dictatorship.

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