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Cournot tatonnement and Nash equilibrium in binary status games

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Abstract

We study a rather simplified game model of competition for status. Each player chooses a scalar variable (say, the level of conspicuous consumption), and then those who chose the highest level obtain the "high" status, while everybody else remains with the "low" status. Each player strictly prefers the high status, but they also have intrinsic preferences over their choices. The set of all feasible choices may be continuous or discrete, whereas the strategy sets of different players can only differ in their upper and lower bounds. The resulting strategic game with discontinuous utilities does not satisfy the assumptions of any general theorem known as of today. Nonetheless, the existence of a (pure strategy) Nash equilibrium, as well as the "finite best response improvement property," are established.

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1 Introduction

There exists a considerable literature on the modeling of social status, which started with Veblen (1899) and has grown steadily, especially during the last four decades. An important step was made by Akerlof (1997). Meanwhile, some empirical literature is also appearing, see, e.g., Immorlica et al. (2017) and references therein. Status relates to the observation that people care about their relative standing in society. It is not so clear how "status" should be defined, let alone how to measure it. Various discussions about the nature of status are going on. In this article we shall not interfere in this debate. Our goal is more technical: we address the problem of the existence of pure Nash equilibrium in the game model of Haagsma and von Mouche (2010), further referred to as the *HvM game*. To achieve a positive result, we have to simplify the notion of the status of an agent in the model; hence the "binary status games." Simultaneously, we generalize the model in another aspect.

The model of Haagsma and von Mouche (2010) is a strategic, non-cooperative game. It is abstract in the sense that it does not rely on any specific interpretation of the actions of the players. On the other hand, it can be linked to the intertemporal model of social status recently developed by Haagsma (2018). That latter model can provide formal explanations of the Easterlin Paradox (Easterlin, 1974) and the Kuznets (1942) consumption puzzle.

In contrast to the main bulk of the previous literature, the HvM game assumes a finite set of players, rather than a continuum, sharing this feature with Immorlica et al. (2017). Obviously, this assumption cannot be called unrealistic. While creating some technical problems absent in the continual models, it agrees with the standard approach of the theory of strategic games.

Another specific feature of the HvM game is its ordinal approach: The status of a player is determined by the *comparisons* with other players' actions rather than *differences* between them. This feature makes inevitable "bad" discontinuities of the utility functions, i.e., discontinuities in the choices of other players.

There are various Nash equilibrium existence results for strategic games with discontinuous utility functions, the most well-known being that of Reny (1999). However, this result is not applicable to the HvM game since it presupposes that each player's utility function is quasi-concave in own strategy. The quasi-concavity assumption was relaxed by McLennan et al. (2011), and even further by Reny (2016). It remains unclear whether their theorems could be applied to this game; anyway, nobody has shown such possibility so far. The strategy sets in the HvM game are linearly ordered, but there is neither strategic complements, nor strategic substitutes. Thus, the most recent results of Prokopovych and Yannelis (2017) or Kukushkin (2018) also cannot help.

In the present paper, we take another route, somewhat related to the notion of potential games (Monderer and Shapley, 1996). We start with simplifying the model: the status of a player may be either "high" or "low," without any subtler gradations. On the other hand, we generalize the model, allowing the set of all feasible choices to be discrete; after that modification, the theorem of Reny (2016), which presupposes convex strategy sets, becomes clearly inapplicable.

We achieve our goal by studying the behavior of the "Cournot tâtonnement paths," i.e., results of consecutive unilateral best response improvements by the players. It turns out that every binary status game admits a "Cournot potential" (Kukushkin, 2004, 2015). Moreover, every Cournot path, regardless of where it was started and in what order the players made their improvements, inevitably reaches a Nash equilibrium after a finite number of steps. This property of a strategic game was observed (in an absolutely unrelated context) and named the "finite best response improvement property" by Milchtaich (1996). From a purely technical viewpoint, the fact that such a property is discovered in a class of *infinite* games is very interesting.

As far as we know, the present article is the first in the status literature where potentials are used. The prospects for extending our theorem to the HvM game in general remain unclear at the moment.

Section 2 contains basic definitions and notations associated with a strategic game. Section 3 provides a formal description of our binary status model as well as a comparison with the HvM game. Section 4 contains definitions related to the Cournot tâtonnement process and the formulation of our main theorem. Its proof is in Section 5.

2 Strategic games

We start with the standard model of a *strategic game*. It is defined by a finite set of players N, and strategy sets X_i and utility functions $u_i\colon X_N \to \mathbb{R}$, where $X_N = \prod_{i\in N} X_i$, for all $i \in N$. We assume that $\#N \geq 2$ and denote $X_{-i} = \prod_{j\in N\setminus\{i\}} X_j$ for each $i \in N$. Given a strategy profile $x_N \in X_N$ and $i \in N$, we denote x_i and x_{-i} its projections to X_i and X_{-i}, respectively; a pair (x_i, x_{-i}) uniquely determines x_N.

We define the *best response correspondence* $\mathcal{R}_i\colon X_{-i} \to 2^{X_i}$ for each $i \in N$ in the usual way,

$$\mathcal{R}_i(x_{-i}) := \operatorname*{Argmax}_{x_i\in X_i} u_i(x_i, x_{-i}).$$

In the games we consider here, $\mathcal{R}_i(x_{-i}) \neq \emptyset$ for every $x_{-i} \in X_{-i}$ although models where the best responses may fail to exist are not unusual.

A strategy profile $x^0_N \in X^0_N$ is a (pure strategy) *Nash equilibrium* if the inequality $u_i(x^0_i, x^0_{-i}) \geq u_i(x_i, x^0_{-i})$ holds for all $i \in N$ and $x_i \in X_i$. An equivalent definition of a Nash equilibrium is

$$x^0_i \in \mathcal{R}_i(x^0_{-i})$$

for all $i \in N$.

3 Binary status games

A *binary status game* is a strategic game satisfying these additional requirements: (i) There is a closed subset $X \subseteq \mathbb{R}$ ("conceivable strategies") and there are $a_i \leq b_i$ in $\mathbb{R}$ for each $i \in N$ such that $X_i = [a_i, b_i] \cap X$; hence each X_i is compact. (ii) Each player's utility depends on her strategy $x_i \in X_i$ and her *status* $s_i \in S := \{0,1\}$, which, in turn, is determined by a mapping $p_i\colon X_N \to S$:

$$p_i(x_N) := \begin{cases} 1, & x_i = \max_{j\in N} x_j; \\ 0, & x_i < \max_{j\in N} x_j. \end{cases}$$

To be more precise, there is a function $U_i\colon X_i \times S \to \mathbb{R}$ such that $u_i(x_N) = U_i(x_i, p_i(x_N))$ for all $i \in N$ and $x_N \in X_N$. (iii) Each function $U_i(x_i, s)$ is strictly increasing in s, and upper semicontinuous in x_i; moreover, there are $\hat{x}_i^s \in X_i$ for all $i \in N$ and $s \in S$ such that $U_i(x_i, s)$ strictly increases when $x_i \le \hat{x}_i^s$ and strictly decreases when $x_i \ge \hat{x}_i^s$.

A comparison with the HvM game is in order. First, the rank of a player in that model is determined by the number of players with greater strategies (the fewer, the better). Thus, there are $\#N$ potential ranks; the players with $x_i = \max_{j\in N} x_j$ are still of the highest rank, but there may be distinctions between the other players. Note that, when there are just two players, both ways to define their ranks coincide. Second, the strategy sets in the HvM game are closed intervals, while here we allow essentially arbitrary subsets of $\mathbb{R}$. It is not that such arbitrariness is needed in economic or social models, but the possibility to restrict the possible strategies to, say, integers, may come handy in some contexts.

For each $i \in N$, we denote $\bar{x}_i^1 := \sup\{x_i \in X_i \mid U_i(x_i, 1) > U_i(\hat{x}_i^0, 0)\}$ and $X_i^1 := \{x_i \in X_i \mid \hat{x}_i^1 \le x_i \le \bar{x}_i^1\}$; the upper semicontinuity implies that $U_i(\bar{x}_i^1, 1) \ge U_i(\hat{x}_i^0, 0)$. The best response correspondence $\mathcal{R}_i$ ($i \in N$) is essentially the same as in the two-person case of the HvM game: If $\max_{j\ne i} x_j \le \hat{x}_i^1$, then $\mathcal{R}_i(x_{-i}) = \{\hat{x}_i^1\}$; if $\hat{x}_i^1 \le \max_{j\ne i} x_j < \bar{x}_i^1$, then $\mathcal{R}_i(x_{-i}) = \{\max_{j\ne i} x_j\}$; if $\max_{j\ne i} x_j = \bar{x}_i^1$, then either $\mathcal{R}_i(x_{-i}) = \{\bar{x}_i^1\}$ or $\mathcal{R}_i(x_{-i}) = \{\bar{x}_i^1, \hat{x}_i^0\}$; if $\max_{j\ne i} x_j > \bar{x}_i^1$, then $\mathcal{R}_i(x_{-i}) = \{\hat{x}_i^0\}$.

Remark. Unlike Haagsma and von Mouche (2010), we do not assume that $U_i(\max X_i, 1) < U_i(\hat{x}_i^0, 0)$ for each i. In particular, it is possible that $X_i^1 = X_i$ for some $i \in N$. The situation of $\mathcal{R}_i(x_{-i}) = \{\bar{x}_i^1\}$ when $\max_{j\ne i} x_j = \bar{x}_i^1$ was impossible in that paper because functions $U_i(x_i, s)$ were *continuous* in x_i while strategy sets were intervals.

4 Cournot paths and potentials

A *best response improvement path*, or, for brevity, a *Cournot path*, in a strategic game is a finite or infinite sequence of strategy profiles $\langle x_N^k \rangle_{k=0,1,\dots}$ such that x_N^k and x_N^{k+1} only differ in the strategy of one player, $i(k) \in N$, and that player chooses a best response to the strategies of others, $x_{i(k)}^{k+1} \in \mathcal{R}_i(x_{-i(k)}^k) = \mathcal{R}_i(x_{-i(k)}^{k+1})$, whereas her previous choice was not optimal, i.e., $x_{i(k)}^k \notin \mathcal{R}_i(x_{-i(k)}^k)$. A game has the *finite best response improvement property* (*FBRP*) if it admits no infinite Cournot path. Then every Cournot path, if extended whenever possible, ends at a Nash equilibrium.

The FBRP was introduced in Milchtaich (1996) although only for finite games. The property is stronger than the mere existence of a Nash equilibrium, but weaker than the *finite improvement property* (*FIP*) of Monderer and Shapley (1996).

Remark. The FBRP implies the existence of a Nash equilibrium only when the best responses exist everywhere (as is the case here). We do not discuss the question of how the FBRP "should" be defined when $\mathcal{R}_i(x_{-i})$ may be empty for some $i \in N$, $x_{-i} \in X_{-i}$.

Now we are ready to formulate our main result.

Theorem. *Every binary status game has the FBRP, and hence possesses a Nash equilibrium.*

The proof is deferred to Section 5.

It is technically convenient to reformulate the definition of Cournot paths with the help of these binary relations on X_N ($i \in N$, $y_N, x_N \in X_N$):

$$y_N \rhd_i^{\mathrm{BR}} x_N \rightleftharpoons [y_{-i} = x_{-i} \ \&\ x_i \notin \mathcal{R}_i(x_{-i}) \ni y_i];$$
$$y_N \rhd^{\mathrm{BR}} x_N \rightleftharpoons \exists i \in N\, [y_N \rhd_i^{\mathrm{BR}} x_N].$$

Now the defining property of a Cournot path is $x_N^{k+1} \rhd_{i(k)}^{\mathrm{BR}} x_N^k$ whenever $k \geq 0$ and x_N^{k+1} is defined.

A *Cournot potential* (Kukushkin, 2004, 2015) is an irreflexive and transitive binary relation $\succ$ on X_N such that

$$\forall x_N, y_N \in X_N \left[y_N \rhd^{\mathrm{BR}} x_N \Rightarrow y_N \succ x_N\right]. \tag{1}$$

The existence of a Cournot potential means that X_N can be partially ordered in such a way that every Cournot path goes upwards. This property is equivalent to the absence of *Cournot cycles*, i.e., Cournot paths $\langle x_N^0, x_N^1, \ldots, x_N^m\rangle$ such that $m > 0$ and $x_N^0 = x_N^m$. For a finite game, it implies, actually, is equivalent to, the FBRP. Example 1 of Kukushkin (2011) shows that a compact-continuous game may admit a Cournot potential and still possess no Nash equilibrium, to say nothing of the FBRP. Nonetheless, an important part of the proof of our theorem consists in constructing a Cournot potential for an arbitrary binary status game.

5 Proof

Given a strategy profile $x_N \in X_N$, we denote $m(x_N) := \max_{i \in N} x_i$ and $M(x_N) := \operatorname{Argmax}_{i \in N} x_i \subseteq N$. Then we define a number of auxiliary constructions (sets, functions, binary relations): $M^1(x_N) := \{i \in M(x_N) \mid x_i \in X_i^1\}$; $M^+(x_N) := \{i \in M(x_N) \mid x_i = \hat{x}_i^1\}$; $M^0(x_N) := \{i \in N \mid x_i = \hat{x}_i^0\}$;

$$\eta(x_N) := \begin{cases} 1, & M^+(x_N) \neq \emptyset \text{ or } \#M^1(x_N) > 1; \\ 0, & \text{otherwise;} \end{cases}$$

$$y_N \succeq^{\mathrm{M}} x_N \rightleftharpoons \left[M^1(y_N) \supset M^1(x_N) \text{ or } \left(M^1(y_N) = M^1(x_N) \ \&\ M^0(y_N) \supseteq M^0(x_N)\right)\right].$$

Clearly, $\succeq^{\mathrm{M}}$ is a *preorder*, i.e., a reflexive and transitive binary relation on X_N; its asymmetric and symmetric components are, respectively,

$$y_N \succ^{\mathrm{M}} x_N \rightleftharpoons \bigl[M^1(y_N) \supset M^1(x_N) \text{ or } \bigl(M^1(y_N) = M^1(x_N) \ \& \ M^0(y_N) \supset M^0(x_N)\bigr)\bigr]$$

and

$$y_N \sim^{\mathrm{M}} x_N \rightleftharpoons \bigl[M^1(y_N) = M^1(x_N) \ \& \ M^0(y_N) = M^0(x_N)\bigr].$$

Finally, we define our presumptive Cournot potential:

$$y_N \succ x_N \rightleftharpoons \Bigl[\eta(y_N) > \eta(x_N) \text{ or } \bigl[\eta(y_N) = \eta(x_N) = 1 \ \& \ \bigl(m(y_N) > m(x_N) \text{ or } [m(y_N) = m(x_N) \ \& \ y_N \succ^{\mathrm{M}} x_N]\bigr)\bigr] \text{ or } \bigl[\eta(y_N) = \eta(x_N) = 0 \ \& \ \bigl(y_N \succ^{\mathrm{M}} x_N \text{ or } [y_N \sim^{\mathrm{M}} x_N \ \& \ m(y_N) < m(x_N)]\bigr)\bigr]\Bigr]. \quad (2)$$

$\succ$ is obviously irreflexive and transitive.

Claim 5.1. *The relation $\succ$ defined by (2) is a Cournot potential of the game.*

An informal explanation of these constructions may be helpful in following the proof below. If $i \in M^+(x_N)$, then player i enjoys the absolute maximum of her utility and hence is not interested in any changes. If $i, j \in M^1(x_N)$ and $i \neq j$, then players i and j may be better off with lesser choices, but making such a choice *unilaterally*, either of them would only lose the "high" status. Thus, $\eta(x_N) = 1$ means that $m(x_N)$ cannot go down. Moreover, any increase in $m(x_N)$ can only happen when someone chooses $y_i = \hat{x}_i^1$; so once η has reached the level 1, it will remain at that level forever. The meaning of $\succeq^{\mathrm{M}}$ is clear by itself.

Proof. We have to check (1). Let $y_N, x_N \in X_N$, $i \in N$, and $y_N \rhd_i^{\mathrm{BR}} x_N$; we have to show $y_N \succ x_N$. Let us consider several alternatives.

A. Let $p_i(y_N) = 0$. Then $i \notin M^1(x_N)$ and $y_i = \hat{x}_i^0$; hence $M^0(y_N) \supset M^0(x_N)$. We consider two alternatives.

A.1. Let $M(x_N) = \{i\}$. Then $m(y_N) < m(x_N)$ and $M^1(x_N) = \emptyset$; hence $\eta(x_N) = 0$. Now if $\eta(y_N) = 1$, then $y_N \succ x_N$ by the first disjunctive term in (2). If $\eta(y_N) = 0$, then $y_N \succeq^{\mathrm{M}} x_N$, and hence $y_N \succ x_N$ by the last disjunctive term in (2).

A.2. Let $M(x_N) \neq \{i\}$. Then $m(y_N) = m(x_N)$, $M^1(y_N) = M^1(x_N)$, and $M^0(y_N) \supset M^0(x_N)$; hence $\eta(y_N) = \eta(x_N)$ and $y_N \succ^{\mathrm{M}} x_N$. Now we have $y_N \succ x_N$ by either the middle or the last disjunctive term in (2).

B. Let $p_i(y_N) = 1$; then $y_i \in X_i^1$ and hence $i \in M^1(y_N)$. We consider two alternatives.

B.1. Let $y_i < x_i$. Then $m(y_N) < m(x_N)$ and $M(x_N) = \{i\}$; hence $\eta(x_N) = 0$. Further, we have $M^1(y_N) \supseteq M^1(x_N)$ and $M^0(y_N) \supseteq M^0(x_N)$. Thus, $y_N \succ x_N$ for exactly the same reason as in the case **A.1**.

B.2. Let $y_i > x_i$. Then $m(y_N) \geq m(x_N)$ and we again consider two alternatives. If $m(y_N) > m(x_N)$ then $M(y_N) = \{i\} = M^+(y_N)$; hence $\eta(y_N) = 1 \geq \eta(x_N)$. Now $y_N \succ x_N$ by the first or the middle disjunctive term in (2). Finally, if $m(y_N) = m(x_N)$, then we have $\eta(y_N) \geq \eta(x_N)$ and $M^1(y_N) \supset M^1(x_N)$; hence $y_N \succ^{\mathrm{M}} x_N$ and we have $y_N \succ x_N$ again. $\square$

Claim 5.2. *There is no infinite Cournot path in the game.*

Proof. Obviously, the function $\eta(x_N)$ can only increase once. Trying to imagine an infinite Cournot path $\langle x_N^k \rangle_{k=0,1,\ldots}$, therefore, we may impose an assumption that either $\eta(x_N^k) = 1$ for all $k \in \mathbb{N}$, or $\eta(x_N^k) = 0$ for all $k \in \mathbb{N}$. The relation $\succ^{\mathrm{M}}$, being defined in terms of subsets of a finite set, also admits only a finite number of consecutive improvements. Therefore, there must hold $m(x_N^{k+1}) > m(x_N^k)$ for an infinite number of $k \in \mathbb{N}$ in the first case, or $m(x_N^{k+1}) < m(x_N^k)$ for an infinite number of $k \in \mathbb{N}$ in the second case.

Let us start with the case of $\eta(x_N^k) = 1$ for all $k \in \mathbb{N}$. The condition $m(x_N^{k+1}) > m(x_N^k)$ implies that $m(x_N^{k+1}) = x_{i(k)}^{k+1} = \hat{x}_{i(k)}^1$; hence each player i can play such a role only once. Since the set of players is finite, the sequence $\langle x_N^k \rangle_{k=0,1,\ldots}$ cannot be infinite.

The analysis of the case of $\eta(x_N^k) = 0$ for all $k \in \mathbb{N}$ is a bit more complicated. Let $\langle x_N^k \rangle_{k=0,1,\ldots}$ be an infinite Cournot path such that $\eta(x_N^k) = 0$ for each k. As noted in the preceding paragraph, the inequality $m(x_N^{k+1}) > m(x_N^k)$ would immediately imply $x_{i(k)}^{k+1} = \hat{x}_{i(k)}^1$, and hence $\eta(x_N^{k+1}) = 1$; therefore, $m(x_N^{k+1}) \leq m(x_N^k)$ for all k, while $m(x_N^{k+1}) < m(x_N^k)$ for an infinite number of them. Furthermore, on each step $k \in \mathbb{N}$, we must have one of these two cases: either $p_{i(k)}(x_N^{k+1}) = 0$ and $x_{i(k)}^{k+1} = \hat{x}_{i(k)}^0$, or $p_{i(k)}(x_N^{k+1}) = 1$ and $x_{i(k)}^{k+1} \in X_{i(k)}^1 \setminus \{\hat{x}_{i(k)}^1\}$; in the latter case, $x_{i(k)}^{k+1} = m(x_N^{k+1}) = x_j^k$ for some $j \in N$. Denoting $\mu := \inf_k m(x_N^k)$ [$= \lim_{k\to\infty} m(x_N^k)$] and $Y := [\mu, m(x_N^0)] \cap \bigcup_{i\in N} \{x_i^0, \hat{x}_i^0\}$, we immediately see that $m(x_N^k) \in Y$ for all $k \in \mathbb{N}$. Since Y is finite, we must have an infinite monotone sequence in a finite set, which is clearly impossible. □

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