Economics Bulletin

Volume 39, Issue 3

On the existence of undominated alternatives in convex sets

Nikolai S. Kukushkin
Dorodnicyn Computing Centre, FRC CSC RAS; Moscow Institute of Physics and Technology, Russia

Abstract

Necessary and sufficient conditions are found for an interval order to admit a maximizer in every convex, or convex and compact, subset of its domain. The conditions are formulated in terms of "improvement paths" and are somewhat similar to conditions characterizing interval orders admitting a maximizer in every compact subset. For a stronger, "von Neumann-Morgenstern," property - every dominated alternative is dominated by an undominated one - only a sufficient condition is obtained.

Financial support from the Russian Foundation for Basic Research (project 17-07-00507) is acknowledged. I thank two anonymous referees for their helpful suggestions.
Citation: Nikolai S. Kukushkin, (2019) "On the existence of undominated alternatives in convex sets", *Economics Bulletin*, Volume 39, Issue 3, pages 2129-2136
Contact: Nikolai S. Kukushkin - ququns@inbox.ru.
Submitted: June 07, 2019. **Published:** September 30, 2019.

1 Introduction

An important tool in the mathematical theory of decision making (Fishburn, 1973; Sen, 1984; Kreps, 1988; Aizerman and Aleskerov, 1995) is a *choice function*, i.e., a correspondence singling out, in every potential feasible set, the subset of acceptable choices (from the viewpoint of the individual, or team, or committee, etc. under consideration). Understandably, the non-emptiness of all its values is a desirable property of such a function. When the choice is determined by a binary relation, this justifies the search for conditions ensuring the existence of undominated alternatives in all potential feasible sets.

When attention is restricted to finite sets, the existence question is easily resolved; moreover, the interrelations between various properties of choice functions have been unraveled. Going beyond finite sets, the landscape becomes more entangled.

Nonetheless, there is quite a number of sufficient conditions for a binary relation defined on a topological space to admit undominated alternatives in every nonempty compact subset (Gillies, 1959; Bergstrom, 1975; Kalai and Schmeidler, 1977; Mukherji, 1977; Walker, 1977; Campbell and Walker, 1990). Smith (1974) found that a very weak version of upper semicontinuity is *necessary* and sufficient for the property, provided the relation in question is an ordering (weak order); Kukushkin (2008a) obtained a similar characterization result for interval orders.

An attempt to move beyond interval orders meets a peculiar obstacle: Kukushkin (2008b) showed the impossibility of a "simple" (in an exact sense) condition that would be *necessary* and sufficient for the existence of undominated alternatives in all nonempty compact subsets. As a way around that obstacle, Kukushkin (2008a) showed that Smith's condition characterizes binary relations with a stronger property (the "NM property": every dominated alternative is dominated by an undominated one) on every compact subset.

A clarification is in order. There is a considerable literature developing conditions ensuring the existence of undominated alternatives in a single set, see, e.g., Bosi and Zuanon (2017) and the references therein. Quite a number of such conditions are *necessary* and sufficient, in which case, however, they cannot be inherited by the restriction of the same relation to subsets; therefore, they have no relevance to choice functions.

In economic models, sets of feasible alternatives are often convex (e.g., budget sets). The existence problem in this framework has been studied intensely and quite a number of sufficient conditions have been obtained (Kiruta et al., 1980; Yannelis and Prabhakar, 1983; Danilov and Sotskov, 1985). However, there is no characterization result in that literature.

This paper strives to fill the gap. We define "convex" analogs of the conditions from Kukushkin (2008a), and obtain a characterization result for "reasonable" interval orders. Thus, the situation proves not so bleak as was suggested in Kukushkin (2008a, Subsection 6.3). On the other hand, only a sufficient condition is obtained for the NM property.

In Section 2, basic definitions and some well-known results are reproduced. Section 3 contains specific conditions needed for our results and auxiliary statements about them. The main theorems are in Section 4. Section 5 contains a few examples showing the impossibility of easy extensions.

2 Basic notions and known characterization results

The set of all nonempty subsets of a set A is denoted $\mathfrak{B}_A$. Given a binary relation $\succ$ on A and $X \in \mathfrak{B}_A$, we denote

$$M(X, \succ) := \{x \in X \mid \nexists y \in X\,[y \succ x]\},$$

the set of *maximizers* of $\succ$ on X. It is often convenient to use an auxiliary relation $y \succeq x \rightleftharpoons x \not\succ y$; then $M(X, \succ) = \{x \in X \mid \forall y \in X\,[x \succeq y]\}$. We say that $\succ$ has the *NM property* on $X \in \mathfrak{B}_A$ if, for every $x \in X \setminus M(X, \succ)$, there is $y \in M(X, \succ)$ such that $y \succ x$. The property means that $M(X, \succ)$ is a von Neumann–Morgenstern solution on X; it implies that $M(X, \succ) \neq \emptyset$, but is stronger than that.

Quite a few useful conditions are naturally formulated with the help of "improvement paths." Given a binary relation $\succ$, an *improvement path* is a (finite or infinite) sequence $\langle x^k \rangle_{k=0,1,\ldots}$ such that $x^{k+1} \succ x^k$ whenever both sides are defined. A relation $\succ$ is *acyclic* if it admits no *finite improvement cycle*, i.e., no improvement path such that $x^m = x^0$ for an $m > 0$. A relation is *strictly acyclic* if it admits no infinite improvement path.

It seems impossible to ascribe any authorship to the following well-known statements.

Proposition A. *A binary relation $\succ$ on A has the property that $M(X, \succ) \neq \emptyset$ for every $X \in \mathfrak{B}_A$ if and only if it is strictly acyclic.*

Proposition B. *A binary relation $\succ$ on A has the property that $M(X, \succ) \neq \emptyset$ for every finite $X \in \mathfrak{B}_A$ if and only if it is acyclic.*

Proposition C. *A binary relation $\succ$ on A has the NM property on every $X \in \mathfrak{B}_A$ if and only if it is strictly acyclic and transitive.*

Proposition D. *A binary relation $\succ$ on A has the NM property on every finite $X \in \mathfrak{B}_A$ if and only if it is irreflexive and transitive.*

Similar characterization results concerning *compact* subsets of metric spaces are less straightforward. Some standard conditions expressing the "degree of rationality" of the underlying preferences are needed. An *interval order* is an irreflexive binary relation $\succ$ such that $[y \succ x \;\&\; a \succ b] \Rightarrow [y \succ b \;\text{ or }\; a \succ x]$ for all $x, y, a, b \in A$; every interval order is transitive. An interval order is a *semiorder* if $z \succ y \succ x \Rightarrow \forall a \in A\,[z \succ a \;\text{ or }\; a \succ x]$ for all $x, y, z, a \in A$. A binary relation $\succ$ is a (*strict*) *ordering* if it is asymmetric, i.e., $y \succ x \Rightarrow y \succeq x$, and *negatively transitive*, i.e., $z \succeq y \succeq x \Rightarrow z \succeq x$; every strict ordering is a semiorder.

Alternative, equivalent, definitions are also available: $\succ$ is an ordering if there is a chain (i.e., a linearly ordered set) C and a mapping $u\colon A \to C$ such that $y \succ x \iff u(y) > u(x)$ for all $x, y \in A$; $\succ$ is an interval order if there is a chain C and two mappings $u^-, u^+\colon A \to C$ such that $u^+(x) \geq u^-(x)$ and $y \succ x \iff u^-(y) > u^+(x)$ for all $x, y \in A$.

We call a binary relation $\succ$ on a metric space *ω-transitive* if it is transitive and, whenever $\langle x^k \rangle_{k \in \mathbb{N}}$ is an infinite improvement path and $x^k \to x^\omega$, there holds $x^\omega \succ x^0$; we call $\succ$ *ω-acyclic* if, whenever $\langle x^k \rangle_{k \in \mathbb{N}}$ is an infinite improvement path and $x^k \to x^\omega$, there holds $x^\omega \neq x^0$ (the prohibition of such cycles was introduced by Mukherji (1977) as "Condition (A5)").

Remark. If $\succ$ is ω-transitive, respectively ω-acyclic, then $x^\omega \succ x^k$, respectively $x^\omega \succeq x^k$, in the above situation for all $k \in \mathbb{N}$.

Theorem E (Smith, 1974, Theorem 1)**.** *An ordering $\succ$ on a metric space A has the property that $M(X, \succ) \neq \emptyset$ for every compact $X \in \mathfrak{B}_A$ if and only if it is ω-transitive.*

Remark. The sufficiency part was first proved by Gillies (1959).

Theorem F (Kukushkin, 2008a, Theorem 3)**.** *An interval order $\succ$ on a metric space A has the property that $M(X, \succ) \neq \emptyset$ for every compact $X \in \mathfrak{B}_A$ if and only if it is ω-acyclic.*

Remark. ω-transitivity and ω-acyclicity are equivalent for semiorders, but not for interval orders.

Theorem G (Kukushkin, 2008a, Theorem 1)**.** *A binary relation $\succ$ on a metric space A has the NM property on every compact $X \in \mathfrak{B}_A$ if and only if it is irreflexive and ω-transitive.*

Throughout the rest of the paper, we consider "*preference relations*," i.e., irreflexive and transitive binary relations on a convex subset A of a real vector space, often, a convex subset of a locally convex topological vector space. The set of all nonempty convex subsets of A is denoted $\mathfrak{C}_A^{\text{conv}} \subset \mathfrak{B}_A$; the set of all nonempty compact and convex subsets, $\mathfrak{C}_A^{\text{mpx}} \subset \mathfrak{C}_A^{\text{conv}} \subset \mathfrak{B}_A$. The convex hull of $X \subseteq A$ is denoted $\operatorname{co} X$; the topological closure of X, $\operatorname{cl} X$.

3 Key assumptions

We start with "convex modifications" of the conditions used in Proposition A and Theorems E and F.

A preference relation $\succ$ on a convex subset of a vector space is called *strictly C-transitive* if, whenever $\langle x^k \rangle_{k\in\mathbb{N}}$ is an infinite improvement path, there is y in $\operatorname{co}\{x^k\}_{k\in\mathbb{N}}$ (convex hull) such that $y \succ x^k$ for each $k \in \mathbb{N}$, see (1) below; $\succ$ on a convex subset of a locally convex topological vector space is called *ω-C-transitive* if, whenever $\langle x^k \rangle_{k\in\mathbb{N}}$ is an infinite improvement path and $x^k \to x^\omega$, there is y in $\operatorname{cl}\operatorname{co}\{x^k\}_{k\in\mathbb{N}}$ (topological closure) such that $y \succ x^k$ for each $k \in \mathbb{N}$ (2).

A preference relation $\succ$ on a convex subset of a vector space is called *strictly C-acyclic* if, whenever $\langle x^k \rangle_{k\in\mathbb{N}}$ is an infinite improvement path, there is $y \in \operatorname{co}\{x^k\}_{k\in\mathbb{N}}$ such that $y \succeq x^k$ for each $k \in \mathbb{N}$ (3); $\succ$ on a convex subset of a locally convex topological vector space is called *ω-C-acyclic* if, whenever $\langle x^k \rangle_{k\in\mathbb{N}}$ is an infinite improvement path and $x^k \to x^\omega$, there is $y \in \operatorname{cl}\operatorname{co}\{x^k\}_{k\in\mathbb{N}}$ such that $y \succeq x^k$ for each $k \in \mathbb{N}$ (4).

$$\forall k \in \mathbb{N}\, [x^{k+1} \succ x^k] \Rightarrow \exists y \in \operatorname{co}\{x^k\}_{k\in\mathbb{N}}\, \forall k \in \mathbb{N}\, [y \succ x^k]. \tag{1}$$

$$\big[x^k \to x^\omega \;\&\; \forall k \in \mathbb{N}\, [x^{k+1} \succ x^k]\big] \Rightarrow \exists y \in \operatorname{cl}\operatorname{co}\{x^k\}_{k\in\mathbb{N}}\, \forall k \in \mathbb{N}\, [y \succ x^k]. \tag{2}$$

$$\forall k \in \mathbb{N}\, [x^{k+1} \succ x^k] \Rightarrow \exists y \in \operatorname{co}\{x^k\}_{k\in\mathbb{N}}\, \forall k \in \mathbb{N}\, [y \succeq x^k]. \tag{3}$$

$$\big[x^k \to x^\omega \;\&\; \forall k \in \mathbb{N}\, [x^{k+1} \succ x^k]\big] \Rightarrow \exists y \in \operatorname{cl}\operatorname{co}\{x^k\}_{k\in\mathbb{N}}\, \forall k \in \mathbb{N}\, [y \succeq x^k]. \tag{4}$$

If any one of those conditions holds on A, then it obviously holds on every convex subset of A. Both conditions (2) and (4) hold if $\succ$ is *upper semicontinuous*, i.e., has open lower contour sets; the converse implications are generally wrong (e.g., the lexicographic order on $\mathbb{R}^m$). The following implications are straightforward:

$$\begin{array}{ccccc} \text{Strict acyclicity} & & & & \text{Strict acyclicity} \\ \Downarrow & & & & \Downarrow \\ \text{Strict C-transitivity (1)} & \Rightarrow & \omega\text{-C-transitivity (2)} & \Leftarrow & \omega\text{-transitivity} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{Strict C-acyclicity (3)} & \Rightarrow & \omega\text{-C-acyclicity (4)} & \Leftarrow & \omega\text{-acyclicity.} \end{array}$$

Proposition 1. *A semiorder on a convex subset A of a vector space is strictly C-transitive if and only if it is strictly C-acyclic. A semiorder on a convex subset A of a locally convex topological vector space is ω-C-transitive if and only if it is ω-C-acyclic.*

Proof. Let $\succ$ be a strictly C-acyclic semiorder and $\langle x^k \rangle_{k\in\mathbb{N}}$ be an infinite improvement path. By (3), there is $y \in \mathrm{co}\{x^k\}_{k\in\mathbb{N}}$ such that $y \succeq x^k$ for each $k \in \mathbb{N}$. On the other hand, for each $k \in \mathbb{N}$, we have $x^{k+2} \succ x^{k+1} \succ x^k$. Since $\succ$ is a semiorder and $y \succeq x^{k+2}$, we have $y \succ x^k$, hence (1) holds. The second statement is proven in virtually the same way. □

Following Kukushkin (2012), we call an infinite improvement path $\langle x^k \rangle_{k=0,1,\ldots}$ *maximizing* in $X \in \mathfrak{B}_A$ if, for every $x \in X$, there is $k \in \mathbb{N}$ such that $x^k \succeq x$. We call a preference relation *reasonable*, just for want of a better term, if, for every $X \in \mathfrak{B}_A$, either $M(X, \succ) \neq \emptyset$, or there exists a maximizing improvement path in X (those alternatives need not be mutually exclusive). We call a preference relation *C-weakly reasonable* if the same condition holds for every $X \in \mathfrak{C}_A^{\mathrm{conv}}$. The weaker property is sufficient for our main results.

Proposition 2. *Every interval order defined by two functions $u^-, u^+ \colon X \to \mathbb{R}$ is reasonable.*

Proof. Let $X \in \mathfrak{B}_A$ and $M(X, \succ) = \emptyset$. We denote $v^\infty := \sup_{x\in X} u^-(x)$; note that $u^-(x) < v^\infty$ for all $x \in X$. Then we pick a sequence $v^k \in \mathbb{R}$ such that $v^{k+1} > v^k$ and $v^k \to v^\infty$ (in this casuistic way we cover both cases $v^\infty < +\infty$ and $v^\infty = +\infty$). Picking $x^0 \in X$ arbitrarily, we recursively construct a sequence of $x^k \in X$ such that $x^{k+1} \succ x^k$ and $u^-(x^{k+1}) > v^{k+1}$ for all $k \in \mathbb{N}$. Obviously, $\langle x^k \rangle_{k\in\mathbb{N}}$ is a maximizing improvement path in X. □

Proposition 2 covers the case of an ordering defined by a utility function $u \colon X \to \mathbb{R}$, and, a bit less obviously, a Pareto dominance order defined by a finite family of functions $u^\alpha \colon X \to \mathbb{R}$. Actually, $\mathbb{R}$ in each case can be replaced with $\mathbb{R}^m$ lexicographically ordered.

A binary relation $\succ$ is *quasiconcave* if $\{y \in A \mid y \succ x\}$ is convex for every $x \in A$.

Proposition 3. *Every quasiconcave preference relation on a convex subset A of a finite-dimensional vector space is C-weakly reasonable.*

Proof. The proof of this technical statement heavily relies on the Axiom of Choice. Let $\succ$ be a quasiconcave preference relation on a convex $A \subseteq \mathbb{R}^n$, and let $X \in \mathfrak{C}_A^{\mathrm{conv}}$. First, a straightforward application of Zorn's Lemma gives us the existence of a "maximal chain" in X, i.e., a subset $C \subseteq X$ such that (i) $y \succ x$ or $x \succ y$ whenever $x, y \in C$ and $x \neq y$, and (ii) for every $y \in X \setminus C$, there is $x \in C$ for which $y \succeq x$ and $x \succeq y$. If there exists a maximum of C, i.e. $x \in C$ such that $x \succ y$ whenever $y \in C$ and $y \neq x$, then $x \in M(X, \succ)$; let there be no maximum in C.

For every $x \in C$, we denote $G(x) := \{y \in X \mid y \succ x\}$; note that $G(x) \neq \emptyset$ and hence $G(x) \in \mathfrak{C}_A^{\text{conv}}$. Since C is maximal, we must have $\bigcap_{x \in C} G(x) = \emptyset$. The key step in the proof is producing an infinite improvement path $\langle x^k \rangle_{k \in \mathbb{N}}$ in C such that $\bigcap_{k \in \mathbb{N}} G(x^k) = \emptyset$. Then $\langle x^k \rangle_{k \in \mathbb{N}}$ will obviously be maximizing.

Whenever $x, y \in C$ and $y \succ x$, we have $G(y) \subset G(x)$; hence the dimension of $G(x)$ is non-increasing. Since it is integer, it must stabilize at some stage, i.e., there is $x^* \in C$ such that all $G(x)$ for $x \succ x^*$ are of the same dimension, say, $m \leq n$. Denoting $C^* := C \cap G(x^*)$, we have $G(x) \subseteq \mathbb{R}^m$ for all $x \in C^*$, and every $G(x)$ has a non-empty interior in $\mathbb{R}^m$. Clearly, $\bigcap_{x \in C^*} G(x) = \emptyset$ as well.

Now, every open subset of $\mathbb{R}^m$ is the union of some open balls with rational centers and rational radii; the set $\mathfrak{O}$ of all such balls is countable, so we may denote $\mathfrak{O} = \{O_h\}_{h \in \mathbb{N}}$. Since $\bigcap_{x \in C^*} G(x) = \emptyset$, for every $h \in \mathbb{N}$, there is $x_h \in C^*$ such that $G(x_h)$ does not contain O_h. We denote $L := \{x_h\}_{h \in \mathbb{N}} \subseteq C^*$. Being a countable chain (w.r.t. $\succ$), L can be embedded into the chain of rational numbers $\mathbb{Q}$, and hence contains a cofinal subset isomorphic to the chain of natural numbers. In other words, there is an infinite improvement path $\langle x^k \rangle_{k \in \mathbb{N}} \subseteq L$ such that, for every $h \in \mathbb{N}$, there is $k \in \mathbb{N}$ for which $x^k \succ x_h$. It follows immediately that the interior of $\bigcap_{k \in \mathbb{N}} G(x^k)$ is empty, even though we cannot yet assert the emptiness of the intersection itself.

Suppose, for a moment, that there is $y \in C$ such that $y \succ x^k$ for every $k \in \mathbb{N}$; then $y \in \bigcap_{k \in \mathbb{N}} G(x^k) \supset G(y)$. Thus, $G(y)$ is a convex subset of $\mathbb{R}^m$ with an empty interior, but of dimension m: a contradiction. Suppose, now, that there is $y \in X$ such that $y \succ x^k$ for every $k \in \mathbb{N}$; then $y \in X \setminus C$ by the preceding argument and $C \cup \{y\}$ is a chain (w.r.t. $\succ$), contradicting the maximality of C. In other words, we have proved that $\bigcap_{k \in \mathbb{N}} G(x^k) = \emptyset$, which means that $\langle x^k \rangle_{k \in \mathbb{N}}$ is indeed maximizing. □

Remark. Without the restriction to finite-dimensional spaces, the notion of the dimension of a convex set would be meaningless. It remains unclear what could be done in a more general case.

4 Main results

Theorem 4. *Let $\succ$ be a C-weakly reasonable interval order on a convex subset A of a vector space. Then $\succ$ has the property that $M(X, \succ) \neq \emptyset$ for every $X \in \mathfrak{C}_A^{\text{conv}}$ if and only if $\succ$ is strictly C-acyclic.*

Proof. Necessity: If $\langle x^k \rangle_{k \in \mathbb{N}}$ is an infinite improvement path and $y \in M(\text{co}\{x^k\}_{k \in \mathbb{N}}, \succ) \neq \emptyset$, then $y \succeq x^k$ for each $k \in \mathbb{N}$, i.e., (3) holds. Note that the reasonableness of $\succ$ is not needed.

Sufficiency: Suppose, to the contrary, that $M(X, \succ) = \emptyset$ for an $X \in \mathfrak{C}_A^{\text{conv}}$. Since $\succ$ is C-weakly reasonable, there is a maximizing improvement path $\langle x^k \rangle_{k \in \mathbb{N}}$ in X. By (3), there is $y \in \text{co}\{x^k\}_{k \in \mathbb{N}} \subseteq X$ such that $y \succeq x^k$ for all $k \in \mathbb{N}$. Since we assumed that $M(X, \succ) = \emptyset$, there is $z \in X$ such that $z \succ y$. Since the improvement path $\langle x^k \rangle_{k \in \mathbb{N}}$ is maximizing, we have $x^k \succeq z$ for some k. Since $x^{k+1} \succ x^k$ and $\succ$ is an interval order, we have $x^{k+1} \succ y$: a contradiction. □

Theorem 5. *Let $\succ$ be a C-weakly reasonable interval order on a convex subset A of a locally convex topological vector space. Then $\succ$ has the property that $M(X, \succ) \neq \emptyset$ for every $X \in \mathfrak{C}_A^{\text{mpx}}$ if and only if $\succ$ is ω-C-acyclic.*

Proof. The argument goes along the same lines as in the proof of Theorem 4. If $\langle x^k \rangle_{k \in \mathbb{N}}$ is an infinite improvement path converging to $x^\omega \in A$, then $\text{cl}\,\text{co}\{x^k\}_{k \in \mathbb{N}}$ is compact; picking $y \in M(\text{co}\{x^k\}_{k \in \mathbb{N}}, \succ) \neq \emptyset$, we have $y \succeq x^k$ for each $k \in \mathbb{N}$, i.e., (4) holds. The reasonableness of $\succ$ is also not needed.

An assumption that $M(X, \succ) = \emptyset$ for an $X \in \mathfrak{C}_A^{\text{mpx}}$ would imply the existence of a maximizing improvement path $\langle x^k \rangle_{k \in \mathbb{N}}$ in X. Since X is compact, without restricting generality, $x^k \to x^\omega$. By (4), there is $y \in \text{cl}\,\text{co}\{x^k\}_{k \in \mathbb{N}} \subseteq X$ such that $y \succeq x^k$ for all $k \in \mathbb{N}$. Now the relation $z \succ y$ for $z \in X$ would lead to the same contradiction as in the proof of Theorem 4. □

Proposition 6. *Let $\succ$ be a reasonable preference relation on a convex subset A of a vector space. Then $\succ$ has the NM property on every $X \in \mathfrak{C}_A^{\text{conv}}$ if it is strictly C-transitive.*

Proof. Let $x \in X \in \mathfrak{C}_A^{\text{conv}}$. Denoting $Y := \{y \in X \mid y \succ x\}$, we have to prove that $Y \cap M(X, \succ) \neq \emptyset$. Since $\succ$ is reasonable, either $M(Y, \succ) \neq \emptyset$ or there is a maximizing improvement path $\langle x^k \rangle_{k \in \mathbb{N}}$ in Y. In the first case, we are home immediately since $M(Y, \succ) \subseteq M(X, \succ)$: if $y \in M(Y, \succ)$, then $z \succ y \succ x$ would imply $z \in Y$ as well, i.e., a contradiction. In the second case, we invoke (1), obtaining $y \in X$ such that $y \succ x^k$ for all $k \in \mathbb{N}$. Now, $y \in Y$ for the same reason as $z \in Y$ in the previous case; therefore, $x^k \succeq y$ for some k: the final contradiction. □

Proposition 7. *Let $\succ$ be a reasonable preference relation on a convex subset A of a locally convex topological vector space. Then $\succ$ has the NM property on every $X \in \mathfrak{C}_A^{\text{mpx}}$ if it is ω-C-transitive.*

The proof is very close to that of Proposition 6 and is omitted.

Remark. If an assumption that the preference relation $\succ$ is quasiconcave is added to the conditions of Proposition 6 or 7, then the requirement on $\succ$ can be weakened to only C-weakly reasonable.

5 "Counterexamples"

If $\succ$ is *not* an interval order, the sufficiency parts of both Theorems 4 and 5 become wrong.

Example 8. On $A := [0,1] \subset \mathbb{R}$, we define an equivalence relation $y \sim x \rightleftharpoons y - x \in \mathbb{Q}$. Then A is partitioned into equivalence classes; we denote $E := A/_\sim$ the set of those equivalence classes, and, for every $x \in A$, $e(x) \in E$ the class where x belongs. Now we define a partial order $\succ$ on A by

$$y \succ x \rightleftharpoons \big[(x = 1 \;\&\; 0 \le y < 1) \;\text{ or }\; (x, y \in [0,1[\;\&\; e(y) = e(x) \;\&\; y > x) \big].$$

Clearly, $M(A, \succ) = \emptyset$ even though A is compact and convex. On the other hand, $\succ$ is strictly C-acyclic, i.e., satisfies (3): Whenever $\langle x^k \rangle_{k \in \mathbb{N}}$ is an infinite improvement path in A, we have $e(x^k) = e(x^h)$ for all $k, h \in \mathbb{N}$ (with a possible exception of x^0), but $\mathrm{co}\{x^k\}_{k \in \mathbb{N}}$ inevitably contains y for which $e(y)$ is different. It is also easy to check that $\succ$ is reasonable.

Without the reasonableness assumption, the sufficiency parts of both Theorems 4 and 5 are also wrong.

Example 9. We define $A \subset \mathbb{R}$, $\sim$ on A, E, and the mapping $e \colon A \to E$ in exactly the same way as in Example 8. Let $\gg$ be a well-order on E (existing by the Axiom of Choice); we define a linear order $\succ$ on A by

$$y \succ x \rightleftharpoons \big[e(y) \gg e(x) \;\text{ or }\; [e(y) = e(x) \;\&\; y > x] \big].$$

Without restricting generality, we may assume that $M(E, \gg) = \emptyset$; hence $M(A, \succ) = \emptyset$ too, even though A is compact and convex.

Meanwhile, $\succ$ is even strictly C-transitive, i.e., satisfies (1). Let $\langle x^k \rangle_{k \in \mathbb{N}}$ be an infinite improvement path in A. The set $E^* := \{e(x^k)\}_{k \in \mathbb{N}} \subset E$ is countable; therefore, there is $\varepsilon := \sup E^* \in E$. Now the set $e^{-1}(\varepsilon)$ is dense in A; therefore, there is $y \in \mathrm{co}\{x^k\}_{k \in \mathbb{N}}$ such that $e(y) = \varepsilon$, and hence $y \succ x^k$ for each $k \in \mathbb{N}$.

In contrast to Theorem G, the converse to either Proposition 6 or 7 is wrong.

Example 10. We define $A := [0,1] \subset \mathbb{R}$, $\sim$ on A, and E in exactly the same way as in Example 8. Clearly, $A \cap \mathbb{Q}$ is one of the equivalence classes in E; let C be another class from E. Now we define an interval order $\succ$ on A by these functions $u^-, u^+ \colon A \to \mathbb{R}$:

$$u^-(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q} \cup C; \\ 0, & \text{if } x \in A \setminus (\mathbb{Q} \cup C); \end{cases} \qquad u^+(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}; \\ x, & \text{if } x \in C; \\ 0, & \text{if } x \in A \setminus (\mathbb{Q} \cup C). \end{cases}$$

Clearly, $M(A, \succ) = A \cap \mathbb{Q}$. Let $x \in X \in \mathfrak{C}_A^{\mathrm{conv}}$ and $\#X > 1$; then $M(X, \succ) \supseteq X \cap \mathbb{Q} \neq \emptyset$. Let $x \notin \mathbb{Q}$. If $x \notin C$, then $y \succ x$ for every $y \in X \cap \mathbb{Q} \setminus \{0\}$. If $x < \sup X$, then there is $y \in X \cap \mathbb{Q}$ such that $y > x$; hence $y \succ x$ again. Finally, if $x = \max X \in C$, then $x \in M(X, \succ)$. Thus, $\succ$ has the NM property on every convex subset of A.

On the other hand, even (2) does not hold. We pick $x^\omega \in A \setminus (\mathbb{Q} \cup C)$ and $x^0 \in C$ such that $x^0 < x^\omega$. Let $\langle q^k \rangle_{k \in \mathbb{N}}$ be an infinite sequence of rational numbers such that $q^0 = 0$, $q^{k+1} > q^k$ for all k, and $q^k \to x^\omega - x^0$. Defining $x^k := x^0 + q^k$, we have $x^k \to x^\omega$ while $x^k \in C$ for all k, and hence $x^{k+1} \succ x^k$ for all k. Now, $\mathrm{cl}\,\mathrm{co}\{x^k\}_{k \in \mathbb{N}} = [x^0, x^\omega]$ and $M([x^0, x^\omega], \succ) = [x^0, x^\omega] \cap \mathbb{Q}$ since $x^\omega \in A \setminus (\mathbb{Q} \cup C)$. For every $y \in [x^0, x^\omega] \cap \mathbb{Q}$, there is $k \in \mathbb{N}$ for which $x^k > y$ and hence $x^k \succeq y$, i.e., (2) does indeed not hold.

References

Aizerman, M.A. and F.T. Aleskerov (1995) *Theory of Choice*, North-Holland: Amsterdam.

Bergstrom, T.C. (1975) "Maximal elements of acyclic relations on compact sets" *Journal of Economic Theory* **10**, 403–404.

Bosi, G. and M.E. Zuanon (2017) "Maximal elements of quasi upper semicontinuous preorders on compact spaces" *Economic Theory Bulletin* **5**(1), 109–117.

Campbell, D.E. and M. Walker (1990) "Maximal elements of weakly continuous relations" *Journal of Economic Theory* **50**, 459–464.

Danilov, V.I. and A.I. Sotskov (1985) "Rational choice and convex preferences" *Izvestiya Akademii Nauk SSSR, Tekhnicheskaya Kibernetika* 2, 14–23. (in Russian)

Fishburn, P.C. (1973) *The Theory of Social Choice*, Princeton University Press: Princeton.

Gillies, D.B. (1959) "Solutions to general non-zero-sum games" in *Contributions to the Theory of Games 4* (*Annals of Mathematical Studies* **40**) by A.W. Tucker and R.D. Luce, Eds., Princeton University Press: Princeton, 47–85.

Kalai, E. and D. Schmeidler (1977) "An admissible set occurring in various bargaining situations" *Journal of Economic Theory* **14**, 402–411.

Kiruta, A.Ya., A.M. Rubinov and E.B. Yanovskaya (1980) *Optimal Choice of Allocations in Complex Social-Economic Problems*, Nauka: Leningrad. (in Russian)

Kreps, D. (1988) *Notes on the Theory of Choice*, Routledge: New York.

Kukushkin, N.S. (2008a) "Maximizing an interval order on compact subsets of its domain" *Mathematical Social Sciences* **56**, 195–206.

Kukushkin, N.S. (2008b) "On the existence of maximal elements: An impossibility theorem" *Discrete Applied Mathematics* **156**, 1988–1993.

Kukushkin, N.S. (2012) "On the existence of optima in complete chains and lattices" *Journal of Optimization Theory and Applications* **154**, 759–767.

Mukherji, A. (1977) "The existence of choice functions" *Econometrica* **45**, 889–894.

Sen, A.K. (1984) *Collective Choice and Social Welfare*, North-Holland: Amsterdam.

Smith, T.E. (1974) "On the existence of most-preferred alternatives" *International Economic Review* **15**, 184–194.

Walker, M. (1977) "On the existence of maximal elements" *Journal of Economic Theory* **16**, 470–474.

Yannelis, N.C. and N.D. Prabhakar (1983) "Existence of maximal elements and equilibria in linear topological spaces" *Journal of Mathematical Economics* **12**, 233–245.