Economics Bulletin

Volume 39, Issue 4

Unnested Aversion to s-th Degree Inequality

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Abstract

The paper characterizes the necessary and sufficient condition under which additively separable social welfare functions are averse to s-th degree inequality. For s>1, the result states that aversion to (s+1)-th degree inequality is neither a weaker nor a stronger attitude than aversion to s-th degree inequality, hence it is not an extension of the Pigou-Dalton condition. This statement makes room to several attitudes to inequality such as aversion to upside inequality and elitism, among others.

I would thank the two reviewers for very helpful remarks and suggestions. The usual disclaimer applies.
Citation: Marc Dubois, (2019) "Unnested Aversion to s-th Degree Inequality", *Economics Bulletin*, Volume 39, Issue 4, pages 2374-2380
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Submitted: November 27, 2018.   **Published:** October 13, 2019.

1 Introduction

Several tools are candidates to measure economic inequality, poverty, and social welfare. So, how to choose the right one? It may depend on the particular aversion to inequality each tool supports. In axiomatic approaches, aversion to inequality is characterized by principles of transfer. The most popular principle is the Pigou-Dalton condition which requires – in welfare terms – that social welfare does not decrease as the result of a transfer of positive amount of income from one individual to a poorer one. Such a transfer (hereafter a rich-to-poor transfer) preserves the first moment of the income distribution but it alters the second moment. Hence, Gayant and Le Pape (2017) state that the transfer increases second-degree inequality. The Pigou-Dalton condition characterizes aversion to (an incease in) second-degree inequality.[1]

Atkinson and Kolm inequality indices, as well as a limited class of Generalized Entropy indices rely on additively separable social welfare functions (SWFs) being *averse to downside inequality*. The attitude requires a given rich-to-poor transfer to be increasingly valuable as the recipient is poorer. It is characterized by both the Pigou-Dalton condition and the *principle of transfer sensitivity* (Shorrocks and Foster, 1987), which together require social welfare not to decrease as the result of a rich-to-poor transfer and/or a *FAvourable Composite Transfer* (FACT). A FACT is a combination of a rich-to-poor transfer and a poor-to-rich transfer at higher income levels, provided that the income difference between affected individuals is the same in both transfers. Since aversion to downside inequality implies aversion to second degree inequality, the set of SWFs that exhibit the former attitude is a subfamily of the set of SWFs that exhibit the latter attitude.

To be precise, the principle of transfer sensitivity solely requires that social welfare does not decrease as the result of a FACT, which leaves the mean and the variance unchanged but it alters the third moment of the income distribution. Hence, the principle of transfer sensitivity defines aversion to third degree inequality. In its content, nothing demands that a poor-to-rich transfer should decrease social welfare. Chiu (2007) points out that inequality indices satisfying the principle may exhibit "inequality tolerance". Indeed, the author analyses that such inequality indices may report a decrease in inequality as the result of an increase in inequality according to the Pigou-Dalton condition *and* a decrease in inequality following the principle of transfer sensitivity. This is formally demonstrated by Chateauneuf et al. (2002). While the concavity of the utility function is necessary (and sufficient) for additively separable SWFs to be averse to second degree inequality, they show that no condition on the concavity is needed for SWFs to exhibit aversion to third degree inequality. From their result, aversion to downside inequality implies aversion to third degree inequality but the converse is not true. Therefore, the set of SWFs that are averse to downside inequality is a subfamily of the set of SWFs that are averse to third degree inequality.

Exhaustively, Atkinson and Kolm inequality indices and the SWFs they rely on exhibit aversion to *any* degree inequality (Gayant and Le Pape, 2017). Aversion to higher than third degree inequality is defined by considering both FACTs and UNFAvourable Composite

[1] Aversion to first degree inequality should be defined by means of transfers that alter the first moment of the income distribution. The attitude would deal with the trade-off between efficiency and inequality, which is beyond the scope of this paper. For more details, see Fleurbaey and Michel (2001), Aboudi and Thon (2003), and Dubois (2016).

Transfers (UNFACTs).[2] Indeed, they show that Transfers of order s (T^s) *à la* Fishburn and Willig (1984), which are combinations of FACTs and UNFACTs when $s > 3$, generate decreases in sth degree inequality. Aversion to sth degree inequality is characterized by the Principle of Transfer of order s (PoT^s), which requires that social welfare does not decrease as the result of a T^s. Following this notation, aversion to second and third degrees inequality respectively are characterized by PoT^2 and PoT^3.

The objective of this paper is to characterize the condition under which the additively separable SWFs are averse to *any* degree inequality, and so to exhibit the weakness of the proper normative content of such an attitude. While Fishburn and Willig (1984) strengthen aversion to downside inequality taking recourse to income transfers T^s, this paper makes room to more redistributive judgments thanks to the very same transfers. Technically, the result shows that, for $s \geq 2$, PoT^{s+1} does not imply PoT^s. To shed light on the economic relevance of the result, consider the normative interpretation of aversion to fourth degree inequality. The attitude is characterized by PoT^4 and the result shows that SWFs satisfying the principle do not necessarily satisfy PoT^3 and/or PoT^2. Consider SWFs that satisfy PoT^2, do not satisfy PoT^3 and satisfy PoT^4. Such SWFs are averse to second degree inequality and some of them place more emphasis on inequality among richer individuals rather than on inequality among poorer individuals. Moreover, they report that social welfare does not decrease as the result of T^4. Such a transfer is a combination of a FACT and an UNFACT at higher income levels. Precisely, it involves two rich-to-poor transfers, one at the lowest affected income levels, and one at the highest affected income levels. Since the SWFs are averse to fourth degree inequality, they may place a great emphasis on inequality among the very rich. The normative statement would be consistent with social decision makers who aim at reducing the gap between the richest individuals and the remainder part of the population. Such an attitude is called aversion to upside inequality by analogy with upside positional inequality aversion, which is a similar attitude for SWFs that are not additively separable (Aaberge, 2009).

In practice, numerous redistribution policies involve complex sequences of income transfers that are relevant to principles of transfer of higher orders than 3. For instance, consider two generations x and y whose income distributions are respectively $(20, 20, 40, 60)$ and $(10, 30, 50, 50)$. Every individual of generation x gives 1 unit of income to one individual of generation y. Before transfers, the aggregate distribution is $(10, 20, 20, 30, 40, 50, 50, 60)$ and after transfers it becomes $(11, 19, 19, 31, 39, 51, 51, 59)$. Whenever a SWF satisfies PoT^4, it states that the income redistribution does not decrease social welfare. The result of the paper shows that while the income redistribution is deemed (weakly) welfare-improving, it does imply that such a judgment is due to the fact that there is less inequality between the two poorest individuals in the distribution after transfers. Indeed, the redistribution also reduces income inequality between the two richest individuals. For this reason, a SWF that satisfies PoT^2 and PoT^4 and does not satisfy PoT^3 may place more emphasis on the inequality between the richer individuals so that it judges the redistribution to be welfare-improving as well.

Section 2 introduces the framework and useful notations. Section 3 presents the result

[2] An UNFACT is a combination of a poor-to-rich transfer and a rich-to-poor transfer at higher income levels, provided that the income difference between involved individuals is the same in both transfers.

and Section 4 concludes.

2 Setup

The setup exploited in this paper is the one defined by Fishburn and Willig (1984). The density function $f(x)$ is the proportion of the population with income $x \in \mathbb{R}_+$, with $\mathbb{R}_+$ being the non-negative part of the real line. Let Ω be the set of all random variables representing income distributions f. The cumulative distribution function of x is $F(x)$, then $F(x) = \int_0^x f(y)dy$. The utility derived from x is denoted $u(x)$ and social welfare is measured by an additively separable SWF:

$$W(F) = \int_{\mathbb{R}_+} u(x)dF(x). \tag{1}$$

Let $\mathscr{C}^s$ be the set of s-time differentiable functions, and $u^{(s)}$ the s-order derivative of u such that s belongs to the set of positive integers $\mathbb{N} := \{1, 2, 3, \ldots\}$. Throughout the note, $u(x) \in \mathbb{R}_+$ and $u^{(1)}(x) > 0$ for all $x \in \mathbb{R}_+$.

Let $T^1(\alpha, x, \delta)$ denote the shift of proportion α of the population from income level x to $x + \delta$, with $\delta > 0$ and $\alpha(x + \delta) \leqslant \mu$ (where μ is the mean income before T^1). $T^1(\alpha, x, \delta)$ is equal to $-\alpha$ at x, α at $x + \delta$, and 0 elsewhere. Let $T^2(\alpha, x, \delta)$ be the transfer which is obtained by pairing $T^1(\alpha, x, \delta)$ with a similar shift in the opposite direction at uniformly higher income level $(x + \delta)$:[3]

$$T^2(\alpha, x, \delta) = T^1(\alpha, x, \delta) - T^1(\alpha, x + \delta, \delta).$$

$T^2(\alpha, x, \delta)$ equals $-\alpha$ at x, 2α at $x+\delta$ and $-\alpha$ at $x+2\delta$. Proceeding recursively, for $s+1 \geqslant 2$ let[4]

$$T^{s+1}(\alpha, x, \delta) = T^s(\alpha, x, \delta) - T^s(\alpha, x + \delta, \delta). \tag{2}$$

The transfer of order $s+1$ changes the proportion of the population at income level $x+i\delta$ by $(-1)^{i+1}\binom{s+1}{i}\alpha$ for $i = \{0, 1, \ldots, s+1\}$. The change in social welfare as the result of $T^{s+1}(\alpha, x, \delta)$ is denoted $\Delta W(T^{s+1}(\alpha, x, \delta))$:

$$\begin{aligned}\Delta W(T^{s+1}(\alpha, x, \delta)) &= \int_0^\infty u(y)f(y) + T^{s+1}(\alpha, x, \delta)dy - \int_0^\infty u(y)f(y)dy \\ &= \sum_{i=0}^{s+1} (-1)^{i+1} \binom{s+1}{i} \alpha u\left(x + i\delta\right). \end{aligned} \tag{3}$$

Following Fishburn and Willig (1984), a principle of transfer of order s states that social welfare increases as the result of any transfer of order at most as high as s. In this paper, a principle of transfer *solely* judges the impact on social welfare of the transfer of corresponding order. The PoT^{s+1} requires that social welfare does not decrease as the result of T^{s+1}. Hence,

[3] T^2 can be applied when $0 < 2\alpha \leqslant 1$, $2\alpha(x + \delta) \leqslant \mu$.

[4] T^{s+1} can be applied when $0 < 2^s\alpha \leqslant 1$, $\alpha(2^s x + \beta_s\delta) \leqslant \mu$ where $\beta_1 = 2$ and $\beta_{s+1} = 2\beta_s + 2^s$.

principles are defined on transfers *à la* Fishburn and Willig (1984) but they are weaker than the principles introduced by the authors.[5]

Definition 2.1. Principle of Transfer of order $\mathbf{s+1}$. *For $s \in \mathbb{N}$,*

$$\Delta W(T^{s+1}(\alpha, x, \delta)) \geqslant 0. \tag{PoT^{s+1}}$$

As pointed out by Fishburn and Willig (1984, Theorem 1), the alternation in sign of the successive derivatives of u *up to* the order $s+1$ is necessary and sufficient for (1) to satisfy *all* principles of transfer up to order $s+1$.

3 The result

To determine the conditions under which (1) satisfies *any* principle of transfer up to order $(s+1)$, let $u(x) \in \mathscr{C}^{s+1}$, for all $s \in \mathbb{N}$. Consider the first-order forward difference of the utility function:

$$\Delta_1(x, a_1) = u(x + a_1) - u(x), \ \forall a_1 > 0.$$

Recursively, the second-order forward difference is:

$$\Delta_2(x, a_1, a_2) = \Delta_1(x + a_2, a_1) - \Delta_1(x, a_1), \ \forall a_1, a_2 > 0.$$

More generally, for $i \in \{1, \ldots, s+1\}$, the $(s+1)$th-order forward difference is:

$$\Delta_{s+1}(x, a_1, \ldots, a_{s+1}) = \Delta_s(x + a_{s+1}, a_1, \ldots, a_s) - \Delta_s(x, a_1, \ldots, a_s), \ \forall a_i > 0.$$

Consider a particular case of the $(s+1)$th-order forward difference of the u function by setting $a_1 = \ldots = a_{s+1} = \delta$. It follows that for any $s \in \mathbb{N}$ and $\delta > 0$:

$$(-1)^{s+1} u^{(s+1)}(x) = (-1)^{s+1} \lim_{\delta \to 0} \left[\frac{\Delta_{s+1}(x, \delta, \ldots, \delta)}{\delta^{s+1}} \right], \tag{4}$$

where

$$\Delta_{s+1}(x, \delta, \ldots, \delta) = \sum_{i=0}^{s+1} (-1)^i \binom{s+1}{i} u\left(x + (s+1-i)\delta\right).$$

The scaled $(s+1)$th-order forward difference $(-1)^s \alpha \Delta_{s+1}$ characterizes the change in social welfare as the result of a transfer $T^{s+1}(\alpha, x, \delta)$.

Lemma 3.1. *The following expression is true for any $s \in \mathbb{N}$:*
$[\mathbf{H}^{s+1}]: \ \Delta W(T^{s+1}(\alpha, x, \delta)) = (-1)^s \alpha \Delta_{s+1}(x, \delta, \ldots, \delta)$.

Proof. See the Appendix. □

From Lemma 3.1 and (4), if the $(s+1)$th-order derivative of the u function is non-negative [non-positive] whenever s is odd [even], then social welfare does not decrease as the result of T^{s+1}. On these grounds, the sign of the $(s+1)$th order derivative of the u function yields a necessary and sufficient condition for (1) to satisfy PoT^{s+1}.

[5]Precisely, the principle of transfer of order 2 is the same as that due to Fishburn and Willig (1984). Otherwise, all principles of transfer of higher order than 2 are weaker than those introduced by the authors.

Theorem 3.1. *For any given $s \in \mathbb{N}$, the two following statements are equivalent:*
(i) *SWFs (1) satisfy* PoT^{s+1}.
(ii) $(-1)^{s+1}u^{(s+1)}(x) \leqslant 0$.

Proof. See the Appendix. □

For $s > 1$, Condition (ii) of Theorem 3.1 neither implies nor is implied by $(-1)^s u^{(s)}(x) \leqslant 0$. Therefore, PoT^{s+1} is neither weaker nor stronger than PoT^s. In words, aversion to $(s+1)$th degree inequality is neither a weaker nor a stronger attitude than aversion to sth degree inequality. This result generalizes the result of Chateauneuf et al. (2002) who state the equivalence between conditions (i) and (ii) only for $s = 2$. It also generalizes the result of Fishburn and Willig (1984).

It is possible to characterize other attitudes to inequality than aversion to downside inequality (*i.e.* PoT^2 *and* PoT^3). As outlined in Introduction, aversion to upside inequality might be defined. Moreover, Theorem 3.1 makes room to a redistributive judgment, which may fit with elitism. Consider the possible case where the utility function of (1) is increasing, convex, and its third-order derivative is positive so that it satisfies PoT^3 but not PoT^2. Social welfare does not decrease as the result of T^3 (equivalently, a FACT) and/or a poor-to-rich transfer. Such a SWF states that a given poor-to-rich transfer is increasingly valuable insofar as the recipient is richer. The normative content is consistent with social decision makers who aim at increasing the gap between the richest individuals and the remainder part of the population.

4 Conclusion

The principle of transfer of order s, whose definition depends only on the transfer of order s, does not imply lower-order principles of transfer. Therefore, it is not an extension of the Pigou-Dalton condition. On these grounds, for $s > 1$, the paper shows that aversion to $(s+1)$th degree inequality is neither a weaker nor a stronger attitude than aversion to sth degree inequality. This statement is a first step to characterize rigorously several attitudes to inequality such as aversion to upside inequality and elitism, among others.

Gayant and Le Pape (2017) designate sth degree inequality by analogy with the concept of sth degree risk presented by Ekern (1980). As Crainich et al. (2013) disentangle prudence from risk aversion, the result could help to disentangle temperance from both risk aversion and prudence. More generally, it could be helpful to disentangle aversion to sth degree risk from aversion to lower degree risk.

Appendix

Proof of Lemma 3.1.

Proof. We proceed by mathematical induction, *i.e.* we first prove that $\mathbf{H}^{s+1}$ is true for $s = 1$ and then we prove that if $\mathbf{H}^{s+1}$ is assumed to be true for any positive integer s, then so is $\mathbf{H}^{s+2}$.

For $s = 1$, with $a \to 0$,

$$\begin{aligned}
&\Delta W(T^2(\alpha, x, \delta)) = \int_0^{x-a} u(y)f(y)dy + u(x)[f(x) - \alpha] \\
&+ \int_{x+a}^{x+\delta-a} u(y)f(y)dy + u(x+\delta)[f(x+\delta) + 2\alpha] + \int_{x+\delta+a}^{x+2\delta-a} u(y)f(y)dy \\
&+ u(x+2\delta)[f(x+2\delta) - \alpha] + \int_{x+2\delta+a}^{\infty} u(y)f(y)dy - \int_0^{\infty} u(y)f(y)dy \\
&= (-\alpha)\ u(x) + 2\alpha\ u(x+\delta) - \alpha\ u(x+2\delta) \\
&= (-\alpha)\,[\Delta_1(x+\delta, \delta) - \Delta_1(x, \delta)] = (-1)^{2-1}\alpha\ \Delta_2(x, \delta, \delta).
\end{aligned}$$

That is, $\mathbf{H}^2$ is true. Now we want to prove the relation $\mathbf{H}^{s+2}$:

$$\Delta W(T^{s+2}(\alpha, x, \delta)) = \int_0^{\infty} u(y)T^{s+2}(\alpha, x, \delta)dy. \tag{5}$$

From (2), the right-hand part of (5) can be rewritten as:

$$\int_0^{\infty} u(y)[T^{s+1}(\alpha, x, \delta)dy - \int_0^{\infty} u(y)T^{s+1}(\alpha, x+\delta, \delta)]dy. \tag{6}$$

Since $\mathbf{H}^{s+1}$ is assumed to be true, (6) becomes:

$$\begin{aligned}
&(-1)^s\alpha\Delta_{s+1}(x, \delta, \dots, \delta) - (-1)^s\alpha\Delta_{s+1}(x+\delta, \delta, \dots, \delta) \\
&= (-1)^{s+1}\alpha\Delta_{s+2}(x, \delta, \dots, \delta),
\end{aligned}$$

which concludes the proof. □

Proof of Theorem 3.1.

Proof. $(i) \implies (ii)$ Assume that $\Delta W(T^{s+1}(\alpha, x, \delta)) \geqslant 0$. From Lemma 3.1, we have:

$$(-1)^s\alpha\Delta_{s+1}(x, \delta, \dots, \delta) \geqslant 0.$$

Since $\alpha > 0$, it turns out that:

$$(-1)^{s+1}\Delta_{s+1}(x, \delta, \dots, \delta) \leqslant 0.$$

From (4) and dividing by δ with $\delta \to 0$, we obtain:

$$\begin{aligned}
&(-1)^{s+1}\lim_{\delta\to 0}\left[\frac{\Delta_{s+1}(x, \delta, \dots, \delta)}{\delta^{s+1}}\right] \leqslant 0 \\
&\iff (-1)^{s+1}u^{(s+1)}(x) \leqslant 0.
\end{aligned}$$

$(ii) \implies (i)$ Assume that $(-1)^{s+1}u^{(s+1)}(x) \leqslant 0$. From (4), it turns out that:

$$(-1)^{s+1}\Delta_{s+1}(x, \delta, \dots, \delta) \leqslant 0 \text{ with } \delta > 0.$$

From Lemma 3.1,

$$\Delta W(T^{s+1}(\alpha, x, \delta)) \geqslant 0.$$

□

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