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An asset market with backwards price comparative statics

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Abstract

A simple asset market is developed in which an increase in supply can increase the equilibrium price, and an increase in demand can decrease the price. The key economic feature of the market is risk aversion. Furthermore, the equilibrium with the backward comparative statics is learnable by the market participants, even if they don't start out fully-informed rational.

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1. Introduction

This paper develops a simple model of an asset market in which changes in supply and demand can have the opposite of the usual effects on the equilibrium price. Specifically, an increase in supply (in the sense of an increased sensitivity of quantity supplied to the price) can drive the equilibrium price up, instead of down as is normally the case. And a symmetric result holds for demand. These "backward" comparative statics do not occur throughout the parameter space, but they do occur in a positive-measure region, which requires (among other things) that the shocks are bounded.

Prices and quantities, the two most studied variables in economics, are of course determined by the interaction of supply and demand. As is shown below, the standard comparative statics are not robust to the incorporation of price risk—modeled here as the price's variance—on demand. Aside from the inherent interest to economists of price behavior, the standard price comparative statics form the basis for the analysis of e.g. tax and subsidy policies. The Appendix provides a policy application with a sales tax and shows the opposite of the standard result in that setting.

The results are shown here in the canonical linear supply-demand model with uncertainty. They can also be shown to hold in the classic stochastic cobweb model of e.g. Muth (1961). For the demand coefficient, the analogous result for the cobweb model is exact. For the supply coefficient a difference is that in the cobweb model the production lag forces suppliers to choose output based on their forecasted future price, not the current price. But *mutatis mutandis*, the result can be shown to hold for that coefficient too.

Economically, the model feature that leads to the result is that demanders dislike variation in the price of the asset after they purchase it. They dislike variance of the price because they are risk-averse and are concerned with the effects of the asset value on their net wealth or may want to sell the asset in the future. The way that the price variability affects asset demand is similar in spirit to, but rather more general than, mean-variance preferences.

The model can also be compared with models where the asset does not yield any direct utility but is valued only for its monetary return. For example, the Capital Asset Pricing Model from finance (e.g. Nicholson (2002), p. 221-4) considers purely financial assets—they are not consumer durables (e.g. cars), which is the main interpretation here—which leads to a different analysis. Notably, the CAPM assumes that asset variances are exogenous, while in the present model the variance of the price is endogenous, and the endogeneity, aside from probably being more realistic, drives the result.

After the result is shown the model is generalized with respect to information structure. Instead of having rational expectations, market participants are endowed with a standard learning algorithm and forced to learn about the price and the price's variance over time. The equilibrium with the backward comparative statics is shown to be learnable.

2. The Model

2.1. The Basic Setup and Main Result

First I develop a standard reference case in which demanders are not concerned about the price variance, and verify the standard comparative statics. Demand and supply for an asset at time t are

$$D_t = D_1 + D_2 p_t + u_t \tag{1}$$

$$S_t = S_1 + S_2 p_t + v_t \tag{2}$$

with $D_1 > 0$, $D_2 < 0$, $S_1 \leq 0$, $S_2 > 0$. It is assumed that the random shocks u and v are uncorrelated at all leads and lags and $E(u_t) = E(v_t) = 0$.[1] Equilibrium has

$$p_t = \frac{D_1 - S_1 + u_t - v_t}{S_2 - D_2}. \tag{3}$$

Using $E(u_t) = E(v_t) = 0$, the derivative of the price with respect to S_2 is negative in expectation. That is, as quantity supplied becomes more sensitive to the price, the equilibrium price is driven down.

Now add a concern for the variance to demand,

$$D_t = D_1 + D_2 p_t + u_t - f(\text{var}(p_{t+1})) \tag{4}$$

where f is any strictly increasing differentiable function with $f(0) = 0$. Now the equilibrium price is

$$p_t = \frac{D_1 - S_1 + u_t - v_t - f(\text{var}(p_{t+1}))}{S_2 - D_2}. \tag{5}$$

Restricting attention to equilibria with time-invariant first and second moments, we can focus on the unconditional variance of this, i.e. with $\text{var}(p_t) = \text{var}(p)$ for all t. That implies, with $\sigma_u^2, \sigma_v^2 > 0$ being the variances of u and v,

$$\text{var}(p) = \frac{\sigma_u^2 + \sigma_v^2}{(S_2 - D_2)^2}, \tag{6}$$

using the facts that the variance is a constant, and that u and v are uncorrelated, so $\text{var}(u - v) = \sigma_u^2 + \sigma_v^2$. Differentiating this with respect to S_2, we have

$$\frac{\partial \, \text{var}(p)}{\partial S_2} = \frac{-2(\sigma_u^2 + \sigma_v^2)}{(S_2 - D_2)^3} < 0. \tag{7}$$

The comparative statics for the price in S_2 are:

$$\frac{dp_t}{dS_2} = \frac{\frac{2(\sigma_u^2+\sigma_v^2)}{(S_2-D_2)^2} f'(\text{var}(p)) - D_1 + S_1 - u_t + v_t + f(\text{var}(p))}{(S_2 - D_2)^2}. \tag{8}$$

In expectation this is

$$E\left(\frac{dp_t}{dS_2}\right) = \frac{\frac{2(\sigma_u^2+\sigma_v^2)}{(S_2-D_2)^2} f'\left(\frac{\sigma_u^2+\sigma_v^2}{(S_2-D_2)^2}\right) - D_1 + S_1 + f\left(\frac{\sigma_u^2+\sigma_v^2}{(S_2-D_2)^2}\right)}{(S_2 - D_2)^2} \tag{9}$$

which generically can't be signed because $D_1 > 0$ and $S_1 \leq 0$. But it is positive, giving the backwards result, for a positive-measure subset of the parameter space. One necessary

[1] If we interpret the specification as in logarithms instead of levels, then D_2 and S_2 are the demand and supply elasticities instead of slopes.

condition for the result to hold, if the price and quantity are guaranteed to be positive, is bounded shocks.[2] Note it helps the expression to be positive if $f(\frac{\sigma_u^2+\sigma_v^2}{(S_2-D_2)^2})$ is larger (provided that price and quantity are positive). Since $f(\cdot)$ is demanders' dislike of variance, the intuition for large $f(\frac{\sigma_u^2+\sigma_v^2}{(S_2-D_2)^2})$ is that if the role of risk on the demand side is large, the change in the supply curve affects the price in the opposite of the standard way. Intuitively, a rise in the supply coefficient S_2 drives the variance down (see (7)). But since demanders are averse to the variance, the lower variance increases demand, and this effect can be strong enough to counteract the effect of the increased supply on the price. Overall, in such cases, the price rises and we have the "wrong" effect on the price.

Using a similar derivation one can show an analogous result for D_2, the price coefficient in the demand equation; specifically $E\left(\frac{dp_t}{dD_2}\right) = -E\left(\frac{dp_t}{dS_2}\right)$. Obviously, the conditions that make $E\left(\frac{dp_t}{dS_2}\right)$ positive make $E\left(\frac{dp_t}{dD_2}\right)$ negative. In particular, while the benchmark case has $dp_t/dD_2 > 0$ (i.e. as D_2 rises toward zero the equilibrium price rises), the derivative is negative for parameter values that give risk a large role in the market. Intuitively, a smaller concern for the price (D_2 closer to zero) raises the price variance (it can be shown that $\frac{\partial \operatorname{var}(p)}{\partial D_2} > 0$). The increased price variance is something demanders dislike, though, so it shifts the demand curve down. If this last effect is strong enough, the equilibrium price falls when demanders become less concerned about the price.

2.2. Example Parameters

If the model is interpreted as being in levels, the equilibrium price and quantity must be guaranteed to be positive—which requires that the shocks u_t and v_t have bounded support—and the basic result requires $E\left(\frac{dp_t}{dS_2}\right) > 0$. Therefore what is required is a set of parameters for which these three inequalities hold:

(1) The main result for the mean comparative statics:

$$E\left(\frac{dp_t}{dS_2}\right) = \frac{\frac{2(\sigma_u^2+\sigma_v^2)}{(S_2-D_2)^2} f'(\frac{\sigma_u^2+\sigma_v^2}{(S_2-D_2)^2}) - D_1 + S_1 + f(\frac{\sigma_u^2+\sigma_v^2}{(S_2-D_2)^2})}{(S_2-D_2)^2} > 0. \tag{10}$$

(2) The equilibrium price should be positive:

$$p_t = \frac{D_1 - S_1 + u_t - v_t - f(\frac{\sigma_u^2+\sigma_v^2}{(S_2-D_2)^2})}{S_2 - D_2} > 0. \tag{11}$$

(3) The equilibrium quantity (found by equating (2) and (4)) should be positive:

$$q_t = \frac{S_2[D_1 + u_t - f(\frac{\sigma_u^2+\sigma_v^2}{(S_2-D_2)^2})] - D_2(S_1 + v_t)}{S_2 - D_2} > 0. \tag{12}$$

Parameters which meet all these desiderata are as follows: Give u_t and v_t bounded support; specifically, let u_t and v_t be uniformly distributed on $[-\overline{u}, \overline{u}]$ and $[-\overline{v}, \overline{v}]$

[2] If the model is interpreted as being in logarithms, so that the sign of price and quantity do not matter, it is even easier to get the result.

respectively, so $\sigma_u^2 = \overline{u}^2/3$ and $\sigma_v^2 = \overline{v}^2/3$, due to the symmetry about zero. Set $\overline{v} = \overline{u} = 1$. Let $f(\text{var}(p)) = K\,\text{var}(p)$ (so $f'(\cdot) = K$) with $K = 10$, and $D_1 = 7$, $S_1 = -0.75$, $D_2 = -0.25$, $S_2 = 1$. To make sure that quantity and price are positive for all admissible shock values, I test the inequalities using the worst-case scenarios for the shocks. I.e. in each inequality in which u_t and v_t appear, I set each shock to either its lower or upper bound, whichever makes it most difficult for the inequality to be satisfied. It can be verified that even in the worst-case scenario all three inequalities hold, with

$$E\left(\frac{dp_t}{dS_2}\right) \approx 3.23 \tag{13}$$

$$p_t \approx 1.18 \tag{14}$$

$$q_t \approx 1.04. \tag{15}$$

That the backwards result holds in a positive-measure region of the parameter space follows from the continuity of the expressions in the three inequalities above, when evaluated at the indicated parameters.

3. Learning

Sometimes equilibria with exotic features are fragile, e.g., in that market participants would not converge to them if the economy begins away from them. In this section I show that is not the case for this equilibrium, using a standard model of learning by the demand side. Demanders' quantity demanded is a function of the current price, which they observe contemporaneously (*a la* a Walrasian auctioneer), but they must learn about the price variance. Of course, they must form a forecast of the price next period in order to form a belief about the variance. I use the simplest possibility, that learning employs the sample mean as the estimator.

Unless they have learned their way to a rational expectations equilibrium (REE), demanders' expectations are not necessarily rational. Let p_{t+1}^e denote their subjective expectation of p_{t+1} and let s_{t+1}^e denote their subjective variance of p_{t+1}. To avoid complications in the market determination of p_t due to simultaneity with demanders' beliefs, I have agents form start-of-date-t expectations with only the history through $t-1$ in their information sets.[3] Then demanders' forecast of p_{t+1} as they enter date t is

$$p_{t+1}^e = \frac{1}{t-1} \sum_{i=1}^{i=t-1} p_i. \tag{16}$$

This can be re-expressed as a recursion:

$$p_{t+1}^e = p_t^e + \frac{1}{t-1}\left(p_{t-1} - p_t^e\right). \tag{17}$$

Learners update their estimate s_t^e via

$$s_{t+1}^e = \frac{1}{t-1} \sum_{i=1}^{i=t-1} (p_i - p_i^e)^2. \tag{18}$$

[3] This isn't necessary; it just simplifies the learning. See Evans and Honkapohja (2001), Chapter 8.

We can write (18) recursively as

$$s^e_{t+1} = s^e_t + \frac{1}{t-1}\left((p_{t-1} - p^e_{t-1})^2 - s^e_t\right). \tag{19}$$

From (5), the actual temporary equilibrium price when learners form expectations this way is

$$\begin{aligned} p_t &= \frac{D_1 - S_1 + u_t - v_t - f(s^e_{t+1})}{S_2 - D_2} && (20)\\ &\equiv A + x_t - Bf(s^e_{t+1}) && (21) \end{aligned}$$

with $A = \frac{D_1 - S_1}{S_2 - D_2}$, $x_t = \frac{u_t - v_t}{S_2 - D_2}$ and $B = \frac{1}{S_2 - D_2}$.

Using this, (17) and (19) become

$$p^e_{t+1} = p^e_t + \frac{1}{t-1}\left(A + x_{t-1} - Bf(s^e_t) - p^e_t\right) \tag{22}$$

$$s^e_{t+1} = s^e_t + \frac{1}{t-1}\left([A + x_{t-1} - Bf(s^e_t) - p^e_{t-1}]^2 - s^e_t\right). \tag{23}$$

Writing this as a vector equation in $\theta_t = [\ p^e_t \quad s^e_t\ ]'$ we have:

$$\theta_{t+1} = \theta_t + \frac{1}{t-1}\begin{pmatrix} A + x_{t-1} - Bf(s^e_t) - p^e_t \\ [A + x_{t-1} - Bf(s^e_t) - p^e_{t-1}]^2 - s^e_t \end{pmatrix}. \tag{24}$$

This vector difference equation is a stochastic recursive algorithm, which can be approximated by an associated ordinary differential equation (ODE), formed by taking the infinite-horizon expectation of the term in parentheses, with the subjective parameters s^e_t and p^e_t frozen:[4]

$$\frac{d\theta}{d\tau} = \lim_{t\to\infty} E\begin{pmatrix} A + x_{t-1} - Bf(s^e) - p^e \\ [A + x_{t-1} - Bf(s^e) - p^e]^2 - s^e \end{pmatrix}. \tag{25}$$

Here τ is a notional time index. The foregoing can be expanded as

$$\frac{d\theta}{d\tau} = \begin{pmatrix} A - Bf(s^e) - p^e \\ A^2 + \sigma^2_x - 2Ap^e - 2ABf(s^e) - s^e + B^2[f(s^e)]^2 + 2Bf(s^e)p^e + (p^e)^2 \end{pmatrix} \tag{26}$$

where σ^2_x is the variance of x_t. At the REE given by (5) and (6),

$$p^e_t = \frac{D_1 - S_1 - f(s^e_t)}{S_2 - D_2} \tag{27}$$

$$s^e_t = \frac{\sigma^2_u + \sigma^2_v}{(S_2 - D_2)^2} \tag{28}$$

it can be shown that $\frac{d\theta}{d\tau} = \begin{pmatrix} 0 & 0 \end{pmatrix}'$. That is, the ODE describing the learners' belief dynamics has a rest point at the REE: Once they've learned an REE, their beliefs stop changing (on average).

[4] This is a standard technique in the literature on stochastic recursive algorithms. For an excellent exposition pertaining to economics, see Evans and Honkapohja (2001).

To assess the stability under learning of the REE, we assess the stability of this ODE at the REE. The Jacobian matrix of the right side of the ODE is

$$J = \begin{pmatrix} -1 & -Bf'(s^e) \\ 2(Bf(s^e) + p^e - A) & 2B^2 f(s^e) f'(s^e) - (2ABf'(s^e) + 1) + 2Bp^e f'(s^e) \end{pmatrix} \tag{29}$$

so the eigenvalues are

$$\frac{1}{2}\left(d - 1 \pm \sqrt{(d+1)^2 - 8Bf'(s^e)(Bf(s^e) + p^e - A)}\right) \tag{30}$$

where $d = 2B^2 f(s^e) f'(s^e) - (2ABf'(s^e) + 1) + 2Bp^e f'(s^e)$. If both eigenvalues are negative, or have negative real parts, at an REE, the REE is locally stable under the learning algorithm. While the eigenvalues may be complex in general (see e.g. Simon and Blume, Theorem 25.4) it can be verified that at the REE, $d = -1$, and this in turn implies the eigenvalues at the REE are both equal to -1. Therefore the REE is locally stable under sample-mean learning.

4. Conclusion

In summary, a plausible modification to demand, risk aversion, can induce "backward" price comparative statics. If the role of risk in the market is strong enough—essentially, if demanders' risk-aversion is sufficiently strong—then the price coefficients in the demand and supply equations affect the price in the opposite of the standard ways. That is, an increase in the price sensitivity of supply can drive the price up, instead of down, and symmetrically for demand. Furthermore, this is robust to a generalization of the information structure: when REE is not assumed and the demanders must learn about the price moments over time, the equilibrium is stable under the belief dynamics arising from a standard learning algorithm.

5. Appendix

Percentage sales taxes are a common type of tax. Consider a percentage tax $\tau \in \mathbb{R}_+$ and let p denote the price before the tax is applied. Then demanders pay $P \equiv (1+\tau)p$ for one unit of the good. For each unit sold the supplier remits τp to the government and receives p. With quantity q, the supplier receives total revenue pq. Total revenue expressed in terms of P is $Pq/(1+\tau)$. Assume the firm's profit maximization problem is

$$\max_{q_t} \Pi_t = \frac{P_t}{1+\tau} q_t + s_1 q_t - \frac{1}{2} q_t^2 \tag{31}$$

where $s_1 < 0$ is part of the cost function. The F.O.C. is

$$q_t = \frac{1}{1+\tau} P_t + s_1. \tag{32}$$

Adding a random shock ξ_t to the firm's quantity produced yields

$$q_t = s_1 + s_2 P_t + \xi_t \tag{33}$$

where $s_2 = 1/(1+\tau)$. To obtain the market supply curve, sum this over the number of firms in the market, n:

$$S_t = S_1 + S_2 P_t + v_t \tag{34}$$

where $S_t = nq_t$, $S_1 = ns_1$, $S_2 = ns_2$, and $v_t = \Sigma\xi_t$ where the sum is over firms.

Since the foregoing is expressed in terms of the price the demanders pay per unit, the associated demand curve is still

$$D_t = D_1 + D_2 P_t + u_t. \tag{35}$$

We can now carry out the comparative statics as in Section 2, since nothing has changed but the replacement of a lower-case p with an upper-case P. The following can be shown: In the standard case of no concern for the variance, a lower percentage tax raises S_2 and this has the effect of lowering the price buyers pay. And of course, the results in the main text immediately imply that when risk has a significant role in the market, a lower percentage tax can drive the price buyers pay up, the opposite of the standard result.

6. References

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