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General analysis of dynamic oligopoly with sticky price

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Abstract

We present a differential game analysis of an oligopoly with sticky price under general demand and cost functions. We show that the output of each firm at the steady state of the open-loop dynamic oligopoly is generally larger than the output of each firm at the equilibrium of the static oligopoly, and the output of each firm at the steady state of the memoryless closed-loop dynamic oligopoly is generally larger than that at the steady state of the open-loop dynamic oligopoly. We also present an analysis of the feedback dynamic oligopoly.

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1 Introduction

There are many studies of dynamic oligopoly by differential game approach, for example, Simaan and Takayama (1978), Fershtman and Kamien (1987), Cellini and Lambertini (2004) and Cellini and Lambertini (2007) about behaviors of firms and market structures with dynamics of sticky prices in an oligopoly with a homogeneous good or differentiated goods, Cellini and Lambertini (2003a) and Cellini and Lambertini (2003b) about advertising investment with dynamics of accumulated advertising effects in an oligopoly with a homogeneous good or differentiated goods, Cellini and Lambertini (2005) and Cellini and Lambertini (2011) about R&D investment with dynamics of accumulated cost reducing effects in an oligopoly with a homogeneous good or differentiated goods, Fujiwara (2006) about a Stackelberg duopoly, Fujiwara (2008) about competitiveness of markets in an oligopoly with renewable resource. For a comprehensive survey see Dockner et al. (2000) and Lambertini (2018).

However, most of these studies used a model of linear demand functions and quadratic cost functions. We extend an analysis of Cellini and Lambertini (2004) about an oligopoly with sticky price to a case of general demand and cost functions. We investigate steady states of an open-loop dynamic oligopoly, a memoryless closed-loop dynamic oligopoly and a feedback dynamic oligopoly. We show that the output of each firm at the steady state of the open-loop dynamic oligopoly is generally larger than the output of each firm at the equilibrium of the static oligopoly, and the output of each firm at the steady state of the memoryless closed-loop dynamic oligopoly is generally larger than that at the the steady state of the open-loop dynamic oligopoly. We also show that if the output of each firm is increasing with respect to the price, the output of each firm at the steady state of the feedback dynamic oligopoly is larger than the output of each firm at the steady state of the open-loop dynamic oligopoly, and show that an example of linear demand and quadratic cost functions presented by Cellini and Lambertini (2004) and Lambertini (2018) satisfies this condition.

2 Open-loop analysis of dynamic oligopoly

Consider a symmetric oligopoly where, at any $t \in [0, \infty)$, n firms, Firms $1, 2, \dots, n$ produce a homogeneous good. The firms intertemporally maximize their profits. Let $x_i(t)$, $i \in \{1, 2, \dots, n\}$, be the outputs of the firms, $p(t)$ be the price of the good.

The inverse demand function is written as

$$\hat{p}(x_1(t) + x_2(t) + \cdots + x_n(t)).$$

It is decreasing and twice differentiable. The cost function of Firm i, $i \in \{1, 2, \dots, n\}$, is

$$C(x_i(t)).$$

It is increasing and twice differentiable. All firms have the same cost function. The instantaneous profit of Firm i, $i \in \{1, 2, \dots, n\}$, is

$$\pi_i(t) = x_i(t)p(t) - C(x_i(t)).$$

The problem of Firm i, $i \in \{1, 2, \dots, n\}$, is written as

$$\max_{x_i(t)} \int_0^\infty e^{-\rho t}[x_i(t)p(t) - C(x_i(t))]dt,\ \rho > 0.$$

ρ is the discount rate. The moving of the price is governed by

$$\frac{dp(t)}{dt} = s\left[\hat{p}(x_1(t) + x_2(t) + \cdots + x_n(t)) - p(t)\right],\ s > 0,\ p(0) > 0.$$

The present value Hamiltonian function of Firm i, $i \in \{1, 2, \dots, n\}$, is

$$\mathcal{H}_i(t) = e^{-\rho t}\{x_i(t)p(t) - C(x_i(t)) + \lambda_i(t)s\left[\hat{p}(x_1(t) + x_2(t) + \cdots + x_n(t)) - p(t)\right]\}.$$

The current value Hamiltonian function of Firm i, $i \in \{1, 2, \dots, n\}$, is

$$\begin{aligned}\hat{\mathcal{H}}_i(t) =& e^{\rho t}\mathcal{H}_1(t)\\ =& x_i(t)p(t) - C(x_i(t)) + \lambda_i(t)s\left[\hat{p}(x_1(t) + x_2(t) + \cdots + x_n(t)) - p(t)\right]\}.\end{aligned}$$

Let

$$\mu_i(t) = e^{-\rho t}\lambda_i(t),\ i \in \{1, 2, \dots, n\}.$$

It is the costate variable. The first order condition for Firm i is

$$\frac{\partial \mathcal{H}_i(t)}{\partial x_i(t)} = 0.$$

Using the current value Hamiltonian function, it is written as

$$\frac{\partial \hat{\mathcal{H}}_i(t)}{\partial x_i(t)} = p(t) + \hat{p}'(x_1(t) + x_2(t) + \cdots + x_n(t))\lambda_i(t)s - C'(x_i(t)) = 0. \tag{1}$$

The second order condition is

$$\frac{\partial^2 \hat{\mathcal{H}}_i(t)}{\partial x_i(t)^2} = \hat{p}''(x_1(t) + x_2(t) + \cdots + x_n(t))\lambda_i(t)s - C''(x_i(t)) < 0. \tag{2}$$

The adjoint condition is

$$-\frac{\partial \mathcal{H}_i(t)}{\partial p(t)} = \frac{\partial \mu_i(t)}{\partial t} = e^{-\rho t}\left[\frac{\partial \lambda_i(t)}{\partial t} - \rho\lambda_i(t)\right],\ i \in \{1, 2, \dots, n\}.$$

Using the current value Hamiltonian function, it is written as

$$-\frac{\partial \hat{\mathcal{H}}_i(t)}{\partial p(t)} = -x_i(t) + \lambda_i(t)s = \frac{\partial \lambda_i(t)}{\partial t} - \rho\lambda_i(t),\ i \in \{1, 2, \dots, n\}.$$

This means

$$\frac{\partial \lambda_i(t)}{\partial t} = (\rho + s)\lambda_i(t) - x_i(t),\ i \in \{1, 2, \dots, n\}.$$

Differentiating (1) with respect to time, we get

$$[C''(x_i(t)) - \hat{p}''\lambda_i(t)s]\,\frac{dx_i(t)}{dt} = \frac{dp(t)}{dt} + \hat{p}'\frac{\partial \lambda_i(t)}{\partial t}s \tag{3}$$
$$= \frac{dp(t)}{dt} + \hat{p}'s[(\rho+s)\lambda_i(t) - x_i(t)],\ i \in \{1, 2, \dots, n\},$$

We denote $\hat{p}'(x_1(t) + x_2(t) + \cdots + x_n(t))$ by $\hat{p}'$, and $\hat{p}''(x_1(t) + x_2(t) + \cdots + x_n(t))$ by $\hat{p}''$.

At the steady sate $\frac{dp(t)}{dt} = 0$, $\frac{dx_i(t)}{dt} = 0$ and $\frac{\partial \lambda_i(t)}{\partial t} = 0$ for $i \in \{1, 2, \dots, n\}$. By symmetry of the oligopoly all x_i's are equal. Denote $x_i(t)$, $p(t)$ and $\lambda_i(t)$ at the steady state by x^*, p^* and λ^*. Then, from (3)

$$(\rho + s)\lambda^* = x^*.$$

Substituting this into (1) yields

$$(\rho + s)\frac{\partial \hat{\mathcal{H}}_i(t)}{\partial x_i(t)} = (\rho + s)[p^* - C'(x^*)] + \hat{p}'sx^* = 0. \tag{4}$$

Since $\hat{p}' < 0$, we have

$$p^* - C'(x^*) > 0. \tag{5}$$

The profit of Firm i in the static oligopoly is

$$\pi_i(t) = \hat{p}(x_1(t) + x_2(t) + \cdots + x_n(t))x_i(t) - C_i(x_i(t)).$$

The first order condition for profit maximization is

$$\hat{p}(x_1(t) + x_2(t) + \cdots + x_n(t)) + \hat{p}'(x_1(t) + x_2(t) + \cdots + x_n(t))x_i(t) - C_i'(x_i(t)) = 0. \tag{6}$$

Denote the output of each firm and the price of the good at the equilibrium of the static oligopoly by $\tilde{x}$ and $\tilde{p}$. Then,

$$\tilde{p} + \hat{p}'\tilde{x} - C'(\tilde{x}) = 0. \tag{7}$$

Suppose that $x_i(t) = \tilde{x}$ for all $i \in \{1, 2, \dots, n\}$. From (7)

$$\tilde{p} - C'(\tilde{x}) = -\hat{p}'\tilde{x}.$$

Substituting this into the left-hand side of (4) assuming $x^* = \tilde{x}$, we have

$$(\rho + s)\frac{\partial \hat{\mathcal{H}}_i(t)}{\partial x_i(t)} = -(\rho + s)\hat{p}'\tilde{x} + \hat{p}'s\tilde{x} = -\rho\hat{p}'s\tilde{x}.$$

Since $\hat{p}' < 0$, this is positive. Thus, by the second order condition,

$$x^* > \tilde{x}.$$

We have shown the following result.

Proposition 1. *The output of each firm at the steady state of the open-loop dynamic oligopoly is larger than the output of each firm at the static equilibrium.*

3 Closed-loop analysis of dynamic oligopoly

In this section, according to the analyses by Cellini and Lambertini (2003a), Cellini and Lambertini (2004), Cellini and Lambertini (2007) and Lambertini (2018)(p.65), we consider a memoryless closed-loop approach to a dynamic oligopoly. The current value Hamiltonian function and the first order condition for Firm i, $i \in \{1, 2, \dots, n\}$, are the same as those in the open-loop case as follows.

$$\hat{\mathscr{H}}_i(t) = x_i(t)p(t) - C(x_i(t)) + \lambda_i(t)s\,[\hat{p}(x_1(t) + x_2(t) + \cdots + x_n(t)) - p(t)]\}.$$

and

$$\frac{\partial \hat{\mathscr{H}}_i(t)}{\partial x_i(t)} = p(t) + \hat{p}'(x_1(t) + x_2(t) + \cdots + x_n(t))\lambda_i(t)s - C'(x_i(t)) = 0. \tag{1}$$

This condition for Firm j is written as

$$\frac{\partial \hat{\mathscr{H}}_j(t)}{\partial x_j(t)} = p(t) + \hat{p}'(x_1(t) + x_2(t) + \cdots + x_n(t))\lambda_j(t)s - C'(x_j(t)) = 0. \tag{8}$$

The second order condition for Firm i is the same as (2) as follows.

$$\frac{\partial^2 \hat{\mathscr{H}}_i(t)}{\partial x_i(t)^2} = \hat{p}''(x_1(t) + x_2(t) + \cdots + x_n(t))\lambda_i(t)s - C''(x_j(t)) < 0. \tag{2}$$

The adjoint condition for Firm $i \in \{1, 2, \dots, n\}$ is different from that in the open-loop case. In the closed-loop case it is written as

$$-\frac{\partial \hat{\mathscr{H}}_i(t)}{\partial p(t)} - \sum_{j \neq i} \frac{\partial \hat{\mathscr{H}}_i(t)}{\partial x_j(t)} \frac{\partial x_j(t)}{\partial p(t)} = \frac{\partial \lambda_i(t)}{\partial t} - \rho\lambda_i(t). \tag{9}$$

The term in (9)

$$-\sum_{j \neq i} \frac{\partial \hat{\mathscr{H}}_i(t)}{\partial x_j(t)} \frac{\partial x_j(t)}{\partial p(t)}$$

takes into account the interaction between the control variables of the firms other than Firm i and the current level of the state variable. We have

$$\frac{\partial \hat{\mathscr{H}}_i(t)}{\partial x_j(t)} = \hat{p}'\lambda_i(t)s.$$

About $\frac{\partial x_j(t)}{\partial p(t)}$ from (8),

$$\frac{\partial x_j(t)}{\partial p(t)} = \frac{1}{C''(x_j(t)) - \hat{p}''\lambda_j(t)s}.$$

Therefore, (9) is rewritten as

$$-\frac{\partial \hat{\mathscr{H}}}{\partial p(t)}-\sum_{j\neq i}\frac{\partial \hat{\mathscr{H}}_i(t)}{\partial x_j(t)}\frac{\partial x_j(t)}{\partial p(t)}$$
$$=-x_i(t)+\lambda_i(t)s-\hat{p}'\lambda_i(t)s\sum_{j\neq i}\frac{1}{C''(x_j(t))-\hat{p}''\lambda_j(t)s}=\frac{\partial \lambda_i(t)}{\partial t}-\rho\lambda_i(t).$$

This means

$$\frac{\partial \lambda_i(t)}{\partial t}=(\rho+s)\lambda_i(t)-x_i(t)-\hat{p}'\lambda_i(t)s\sum_{j\neq i}\frac{1}{C''(x_j(t))-\hat{p}''\lambda_j(t)s}.$$

Differentiating (1) with respect to time, we get

$$[C''(x_i(t))-\hat{p}''\lambda_i(t)s]\frac{dx_i(t)}{dt}=\frac{dp(t)}{dt}+\hat{p}'\frac{\partial \lambda_i(t)}{\partial t}s=\frac{dp(t)}{dt}+\hat{p}'s[(\rho+s)\lambda_i(t)-x_i(t)]$$
$$-(\hat{p}'s)^2\lambda_i(t)\sum_{j\neq i}\frac{1}{C''(x_j(t))-\hat{p}''\lambda_j(t)s},\ i\in\{1,2,\dots,n\},$$

At the steady sate $\frac{dp(t)}{dt}=0$, $\frac{dx_i(t)}{dt}=0$ and $\frac{\partial \lambda_i}{\partial t}=0$ for $i\in\{1,2,\dots,n\}$. By symmetry of the oligopoly all x_i's are equal and all $\lambda_i(t)$'s are equal. Denote $x_i(t)$, $p(t)$ and $\lambda_i(t)$ at the steady state by x^{**}, p^{**} and λ^{**}. Then,

$$\left[(\rho+s)-\hat{p}'s(n-1)\frac{1}{C''(x^{**})-\hat{p}''\lambda^{**}s}\right]\lambda^{**}=x^{**}.$$

Substituting this into (1), we obtain

$$\left[(\rho+s)-\hat{p}'s(n-1)\frac{1}{C''(x^{**})-\hat{p}''\lambda^{**}s}\right]\frac{\partial \hat{\mathscr{H}}_i(t)}{\partial x_i(t)} \tag{10}$$
$$=\left[(\rho+s)-\hat{p}'s(n-1)\frac{1}{C''(x^{**})-\hat{p}''\lambda^{**}s}\right][p^{**}-C'(x^{**})]+\hat{p}'sx^{**}=0.$$

Suppose $x_i(t)=x^*$, which is the output of each firm in the steady state of the open-loop dynamic oligopoly, for all i. From (4) in the open-loop case

$$p^*-C'(x^*)=-\frac{\hat{p}'s}{\rho+s}x^*.$$

Substituting this into the left-hand side of (10) assuming $x^{**}=x^*$, we have

$$\left[(\rho+s)-\hat{p}'s(n-1)\frac{1}{C''(x^*)-\hat{p}''\lambda^*s}\right]\frac{\partial \hat{\mathscr{H}}_i(t)}{\partial x_i(t)} \tag{11}$$
$$=-\hat{p}'s(n-1)\frac{1}{C''(x^*)-\hat{p}''\lambda^*s}[p^*-C'(x^*)].$$

From (5),

$$p^* - C'(x^*) > 0.$$

It means that the price is larger than the marginal cost. Note that from the first order condition (1), if $x^{**} = x^*$, we have

$$\lambda^{**} = \lambda^* = -\frac{1}{\hat{p}'s}[p^{**} - C'(x^{**})] > 0.$$

Since $\hat{p}' < 0$ and by the second order condition (2) in the open-loop case,

$$C''(x^*) - \hat{p}''\lambda^* s > 0,$$

(11) is positive. Therefore, we get

$$x^{**} > x^*.$$

We have shown the following result.

Proposition 2. *The output of each firm at the steady state of the closed-loop dynamic oligopoly is larger than the output of each firm at the steady state of the open-loop dynamic oligopoly.*

4 Feedback analysis of dynamic oligopoly

Finally we consider the feedback solution. Let $V_i(p(t))$ be the value function of Firm i, $i \in \{1, 2, \dots, n\}$. The Hamilton-Jacobi-Bellman equation for Firm i is written as

$$\rho V_i(p(t)) = \max_{x_i(t)}\{x_i(t)p(t) - C(x_i(t)) + \frac{\partial V_i(p(t))}{\partial p(t)} s[\hat{p}(x_1(t) + x_2(t) + \dots x_n(t)) - p(t)]\}.$$

The first order condition is

$$p(t) - C'(x_i(t)) + \frac{\partial V_i(p(t))}{\partial p(t)} s\hat{p}' = 0. \tag{12}$$

The second order condition is

$$-C''(x_i(t)) + \frac{\partial V_i(p(t))}{\partial p(t)} s\hat{p}'' < 0.$$

From (12) we get

$$\frac{dx_i(t)}{dp(t)} = \frac{1 + \frac{\partial^2 V_i(p(t))}{\partial p(t)^2} s\hat{p}'}{C''(x_i(t)) - \frac{\partial V_i(p(t))}{\partial p(t)} s\hat{p}''}, \tag{13}$$

and

$$\frac{\partial V_i(p(t))}{\partial p(t)} = -\frac{[p(t) - C'(x_i(t))]}{s\hat{p}'}. \tag{14}$$

Substituting this into the Hamilton-Jacobi-Bellman equation yields

$$\rho V_i(p(t)) = x_i(t)p(t) - C(x_i(t)) - \frac{p(t) - C'(x_i(t))}{\hat{p}'}[\hat{p}(x_1(t) + x_2(t) + \dots x_n(t)) - p(t)].$$

This is an identity. Differentiating this with respect to $p(t)$ taking (13) into account, given symmetry of the oligopoly we obtain

$$\rho\frac{\partial V_i(p(t))}{\partial p(t)} = x_i(t) + \frac{p(t) - C'(x_i(t))}{\hat{p}'} - \frac{\hat{p}(x_1(t) + x_2(t) + \dots x_n(t)) - p(t)}{\hat{p}'} \tag{15}$$
$$+\left\{p(t) - C'(x_i(t)) + \frac{n\hat{p}''[p(t) - C'(x_i(t))][\hat{p}(x_1(t) + x_2(t) + \dots x_n(t)) - p(t)]}{(\hat{p}')^2}\right.$$
$$\left.-n[p(t) - C'(x_i(t))] + \frac{C''(x_1(t))[\hat{p}(x_1(t) + x_2(t) + \dots x_n(t)) - p(t)]}{\hat{p}'}\right\}\frac{dx_i(t)}{dp(t)}.$$

At the steady state we have $\hat{p}(x_1(t) + x_2(t) + \dots x_n(t)) - p(t) = 0$. Denote the steady state values of $x_i(t)$ and $p(t)$ by x^F and p^F, we get

$$\rho\frac{\partial V_i(p(t))}{\partial p(t)} = x^F + \frac{p^F - C'(x^F)}{\hat{p}'} - (n-1)[p^F - C'(x^F)]\frac{dx_i(t)}{dp(t)}$$

Combining this with (14) yields

$$(\rho + s)[p^F - C'(x^F)] + \hat{p}'sx^F - (n-1)\hat{p}'s[p^F - C'(x^F)]\frac{dx_i(t)}{dp(t)} = 0.$$

If $\frac{dx_i(t)}{dp(t)} > 0$, that is, the output of each firm is increasing with respect to the price, we obtain $x^F > x^*$. Thus, we have shown the following result.

Proposition 3. *If the output of each firm is increasing with respect to the price, the output of each firm at the steady state of the feedback dynamic oligopoly is larger than the output of each firm at the steady state of the open-loop dynamic oligopoly.*

Linear and quadratic example

According to Cellini and Lambertini (2004), we assume that the inverse demand function is

$$\hat{p}(t) = a - \sum_{i=1}^{n} x_i(t).$$

a is a positive constant, the cost function of Firm i, $i \in \{1, 2, \dots, n\}$, is

$$C(x_i(t)) = cx_i(t) + \frac{1}{2}x_i(t)^2,$$

and $V_i(p(t))$ is

$$V_i(p(t)) = \frac{k_i p(t)^2}{2} + h_i p(t) + g_i.$$

Thus,

$$\frac{\partial V_i(p(t))}{\partial p(t)} = k_i p(t) + h_i,\ \frac{\partial^2 V_i(p(t))}{\partial p(t)^2} = k_i,\ \frac{dx_i(t)}{dp(t)} = 1 - k_i s.$$

k_i, h_i and g_i are constants. From (14),

$$x_i(t) = (1 - k_i s)p(t) - c - h_i s. \tag{16}$$

From (15) with symmetry of the oligopoly,

$$\begin{aligned}\rho(k_i p(t) + h_i) =& x_i - [p(t) - c - x_i(t)] + [a - nx_i - p(t)] \\ &+ \{-(n-1)[p(t) - c - x_i(t)] - [a - nx_i - p(t)]\}(1 - k_i s).\end{aligned} \tag{17}$$

From (16) and (17), we get

$$k_i = \frac{2ns + 2s + \rho - \sqrt{(4n^2+8)s^2 + (4n+4)\rho s + \rho^2}}{2(2n-1)s^2},$$

and

$$h_i = \frac{c - (cn + a)sk_i}{(2n-1)k_i s^2 - (n+1)s - r}.$$

Also we have

$$\frac{dx_i(t)}{dp(t)} = 1 - k_i s = \frac{\sqrt{(4n^2+8)s^2 + (4n+4)\rho s + \rho^2} + 2ns - 4s - \rho}{2(2n-1)s} > 0,$$

because

$$(4n^2+8)s^2 + (4n+4)\rho s + \rho^2 - (2ns - 4s - \rho)^2 = 4(2n-1)s(2s + \rho) > 0.$$

Therefore, this example satisfies the condition for Proposition 3. The steady state output of each firm is

$$\begin{aligned}x^F =& \frac{(a-c)n\sqrt{(4n^2+8)s^2 + (4n+4)rs + r^2} - \sqrt{(4n^2+8)s^2 + (4n+4)rs + r^2}}{n(n\sqrt{(4n^2+8)s^2 + (4n+4)rs + r^2} - 2\sqrt{(4n^2+8)s^2 + (4n+4)rs + r^2} + 2n^2s + 4s + 3nr)} \\ &+ \frac{(a-c)(2n^2s - 2ns + 2s + 3nr - r)}{n(n\sqrt{(4n^2+8)s^2 + (4n+4)rs + r^2} - 2\sqrt{(4n^2+8)s^2 + (4n+4)rs + r^2} + 2n^2s + 4s + 3nr)}.\end{aligned}$$

About details please see Cellini and Lambertini (2004) and Lambertini (2018)(pp.59-61).

5 Concluding Remark

Assumptions of linear demand and quadratic cost functions are very limited. Analyses of dynamic oligopoly with general demand and cost functions may be applicable to some situations, in particular, various comparative statics.

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