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A note on implied correlation for bivariate contracts

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Abstract

In this paper we develop a framework in which implied correlation can be rigorously defined for a class of derivative contracts written on two assets. Within this class, we show that implied correlation exists and is unique provided that the observed two-asset contract price is free of arbitrage. We also obtain an analytic result to compute the sensitivity to implied correlation of a contract's price. We then provide a numerical illustration of these results applied to spread options.

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1. Introduction

In spite of all of its flaws, volatility is a key financial indicator, used both by investors for portfolio allocation purposes and by market operators for managing derivative contracts. There are mainly two approaches for the estimation of volatility. The first one, historical volatility, is backward-looking and computed from the standard deviation of past returns. The second one, implied volatility, is forward-looking because it is based on the prices of options which mature some time in the future. More precisely, the implied volatility is equal to the volatility parameter in the Black-Scholes-Merton (BSM) pricing formula (Black and Scholes (1973) and Merton (1973)) that matches the market price of the option.

When dealing with two or more assets, the correlations between assets, in addition to the volatilities, play a crucial role. While historical correlations are easy to compute, there is no obvious way to proceed with *implied* correlation. This is a strong limitation for agents who are active in the markets for multi-asset derivative contracts because the value of their trading book is highly sensitive to these correlations.

Campa and Chang (1998) is an early reference where implied correlation in foreign exchange (FX) options markets is defined as a ratio of implied volatilities and used to forecast realized correlation. Burtschell et al. (2009) study the notion of implied correlation smile for Collateralized Debt Obligations and define it as the correlation parameters to be used in a one-factor Gaussian copula model to reproduce observed prices. Linders and Schoutens (2014) develop an implied correlation measure for equity index options based on the Gaussian copula. Markopoulou et al. (2016) define implied correlation for SP100 index options as a ratio involving implied volatilities and index weights.

For two-asset options, implied correlation has frequently been defined as the correlation parameter to be used in a bivariate Black-Scholes-Merton model (henceforth 2DBSM model) to reproduce an observed price. Following this approach, Carmona and Durrleman (2003) as well as Alexander and Scourse (2004) deal with spread options. Limitations of this definition are commented in the introduction of Alexander and Venkatramanan (2011). Da Fonseca et al. (2007) work with Wishart processes to build a model that is applied to the pricing of best-of options and define implied correlation as the parameter to be used in a 2DBSM model with volatility levels set at their implied values. Marabel Romo (2012) works with correlation model applied to the pricing of worst-of options and compute implied correlation from the implied covariance matrix to be used in a 2DBSM model.

In this paper we consider implied correlation for two-asset options as the correlation parameter to be used in a model where the marginal distributions are recovered from market prices of call and put options on each asset and the dependence is a bivariate Gaussian copula. Hence, the reference model we use to define implied correlations departs from a 2DBSM model because marginal distributions are not Gaussian and are consistent with the full observed volatility smile for each asset. The marginal distributions are recovered using the Breeden-Litzenberger formula (Breeden and Litzenberger (1978)).

Our contribution is to rigorously formalize a class a bivariate payoffs corresponding to two-asset derivative contracts for which implied correlation exists and is unique as soon as the observed price is free of arbitrage. Popular contracts such as basket, spread and min-max options, as well as double binary options, belong to this class. This formalization works with any form, parametric or non-parametric, chosen for the marginal distributions of the underlying assets. For this class of payoffs we obtain an analytic form for the sensitivity to the implied correlation parameter. We then work with an asymmetric extension of the Student t copula in order to provide a numerical illustration of our results applied to spread options. The asymmetric extension we consider is based on power functions and is described in Liebscher (2008, 2011) and Andersen and Piterbarg (2010).

The rest of the paper is structured as follows. In Section 2 we review the theoretical framework in which our analysis is developed. In Section 3 we formalize the class of bivariate contracts and the notion of implied correlation. Section 4 is dedicated to numerical illustrations. Finally, Section 5 concludes.

2. Financial framework

We consider a financial market where two risky assets are traded. These risky assets have initial prices S_0^1 and S_0^2. We also consider a finite time horizon T. The final prices of the risky assets are positive random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ denoted by S_T^1 and S_T^2, with $\mathbb{P}$ the physical measure. The risk-free interest rate is assumed to be constant and denoted by r. The discount factor corresponding to the final time horizon is computed as e^{-rT}.

We assume that standard call options (vanilla options) with maturity T and positive strikes are available for the two risky assets of our market. This assumption is equivalent to assuming that the corresponding volatility smiles are known. For $i = 1, 2$, we denote by $C^i(K_i, T)$ the call option written on S^i with maturity T and struck at $K_i \in [0, +\infty[$, with the special case $C^i(K = 0, T) = S^i$, that is the call with a zero strike is the asset itself.

In this market and in addition to these standard options, we deal with two-asset derivative contracts that are written on the two risky assets. A two-asset derivative contract Z has a final payoff defined as a positive random variable $Z_T = z(S_T^1, S_T^2)$ for a positive payoff function z on $[0, +\infty[^2$.

We assume that our market with vanilla options and two-asset contracts is free of arbitrage so that there exists at least one risk-neutral probability $\mathbb{Q}$. This probability measure $\mathbb{Q}$ is used in the sequel for pricing purposes. We refer to Tavin (2015) for further comments on this aspect.

At $t = 0$, the current price of a two-asset derivative contract Z is computed as an expectation under a risk-neutral probability measure $\mathbb{Q}$ as

$$Z_0 = e^{-rT} \mathbb{E}^{\mathbb{Q}} \left[z \left(S_T^1, S_T^2 \right) \right]. \tag{1}$$

Let H be the joint cumulative distribution function under $\mathbb{Q}$ of (S_T^1, S_T^2). As is customary, we decompose this bivariate distribution as, first, the univariate cumulative distribution functions F_1 and F_2 and, second, the associated bivariate copula C. The latter characterizes the chosen dependence between S_T^1 and S_T^2. The relationships between H, F_1, F_2 and C are given by Sklar's Theorem as

$$H(s_1, s_2) = C\left(F_1(s_1), F_2(s_2)\right),$$
$$C(u_1, u_2) = H\left(F_1^{-1}(u_1), F_2^{-1}(u_2)\right).$$

For a statement of this fundamental result and details about copula functions, see Nelsen (2006). Because we assume that (single-asset) call option prices, or implied volatilities, are available for all positive strikes it is possible to uniquely recover the marginal distributions of S_T^1 and S_T^2. It means that in the joint distribution H, the marginals F_1 and F_2 are consistent with the full volatility smile of each asset. In this framework, F_1 and F_2 are computed from call option prices using the Breeden-Litzenberger formula, see Breeden and Litzenberger (1978). For $i = 1, 2$ and $x \geq 0$

$$F_i(x) = 1 + \frac{1}{e^{-rT}} \frac{\partial C_0^i}{\partial K_i}(x). \tag{2}$$

We assume marginal distribution functions $F_i : [0, +\infty[\longrightarrow [0, 1], i = 1, 2$ to be continuous. This assumption is essentially technical and poses no restriction to practical applications of the results.

Additionally, the expectation under $\mathbb{Q}$ in (1) can be written as a double integral

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[Z_T] &= \iint_{[0,+\infty[^2} z(s_1, s_2) dC(F_1(s_1), F_2(s_2)) \\ &= \iint_{[0,1]^2} z(F_1^{-1}(u_1), F_2^{-1}(u_2)) dC(u_1, u_2).\end{aligned}$$

If C is absolutely continuous and has density $c = \frac{\partial^2 C}{\partial u_1 \partial u_2}$, the double integral becomes

$$\mathbb{E}^{\mathbb{Q}_T}[Z_T] = \iint_{[0,1]^2} z(F_1^{-1}(u_1), F_2^{-1}(u_2)) c(u_1, u_2) du_1 du_2. \tag{3}$$

3. Implied correlation

3.1. A class of bivariate payoffs

We introduce a class of bivariate payoff functions, the $\mathcal{I}$-payoffs, corresponding to a family of two-asset derivative contracts for which the computation of the double integral in (1) can be reduced to a single integral involving the joint distribution function H.

Definition 1. *A payoff function z is said to be of class $\mathcal{I}$ if it has one of the following forms.*

Integrated Indicator Function:

$$\forall (s_1, s_2) \in \mathbb{R}_+^2 \quad z(s_1, s_2) = \int_L^U \mathbb{1}_{\{\pm s_1 \leq g_1(x), \pm s_2 \leq g_2(x)\}} dx, \tag{4}$$

for two continuous functions g_1 and g_2 well defined on $\mathbb{R}_+$, $L \in \mathbb{R}_+$ and $U \in \mathbb{R}_+ \cup \{+\infty\}$, $L < U$. And with g_1, g_2 different from the constants 0 and $+\infty$.

Double Indicator Function:

$$\forall (s_1, s_2) \in \mathbb{R}_+^2 \quad z(s_1, s_2) = \mathbb{1}_{\{\pm s_1 \leq K_1, \pm s_2 \leq K_2\}}, \tag{5}$$

for $K_1, K_2 \in]0, +\infty[$.

The most widespread bivariate payoff functions fall under the umbrella of the $\mathcal{I}$-payoffs. They correspond to the first form in the definition above. Other obvious members of the $\mathcal{I}$-class, corresponding to the second form, are double binary options (also named double digital options), where K_1 and K_2 are the two strikes of the option.

Contracts corresponding to the first form can be seen as continuous portfolios of double binary options. The latter can be seen as the elementary building blocks of the two-asset payoff universe falling under the $\mathcal{I}$-class.

Table 1 summarizes the payoff functions of common two-asset derivative contracts as well as their $\mathcal{I}$-forms. Without any loss of generality, we work with either call or put options in order to simplify the expressions. The expression for the other contract can be obtained via put-call parity identities. We show in the next section in what way such a structure in the payoff function can be exploited to uniquely define an implied correlation.

Table 1: Payoff functions of classical two-asset options and their $\mathcal{I}$-forms. With $K \geq 0$ and $K_1, K_2 > 0$.

option name	payoff	$\mathcal{I}$-form of z
spread option	$Z_T = \left(S_T^2 - S_T^1 - K\right)^+$	$z(s_1, s_2) = \int_0^\infty \mathbb{1}_{\{s_1 \leq x, s_2 \geq x+K\}} dx$
basket option	$Z_T = \left(K - \frac{1}{2}(S_T^1 + S_T^2)\right)^+$	$z(s_1, s_2) = \frac{1}{2}\int_0^{2K} \mathbb{1}_{\{s_1 \leq x, s_2 \leq 2K-x\}} dx$
max option	$Z_T = \left(\max\left(S_T^1, S_T^2\right) - K\right)^+$	$z(s_1, s_2) = \int_K^\infty \mathbb{1}_{\{s_1 \geq x, s_2 \geq x\}} dx$
double binary option	$Z_T = \mathbb{1}_{\left\{S_T^1 \geq K_1, S_T^2 \geq K_2\right\}}$	$z(s_1, s_2) = \mathbb{1}_{\{s_1 \geq K_1, s_2 \geq K_2\}}$

The price of a two-asset contract whose payoff function z belongs to the first form of the $\mathcal{I}$-class is given by

$$\begin{aligned} Z_0 &= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\int_L^U \mathbb{1}_{\left\{\pm S_T^1 \leq g_1(x), \pm S_T^2 \leq g_2(x)\right\}} dx\right] \\ &= e^{-rT}\int_L^U \mathbb{Q}\left(\left\{\pm S_T^1 \leq g_1(x), \pm S_T^2 \leq g_2(x)\right\}\right) dx. \end{aligned} \tag{6}$$

Where we assume the conditions are met so that we can switch the integral and the expectation operators. When the payoff function z belongs to the second form of the class $\mathcal{I}$, the price is given by

$$\begin{aligned} Z_0 &= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\left\{\pm S_T^1 \leq K_1, \pm S_T^2 \leq K_2\right\}} dx\right] \\ &= e^{-rT}\mathbb{Q}\left(\left\{\pm S_T^1 \leq K_1, \pm S_T^2 \leq K_2\right\}\right). \end{aligned} \tag{7}$$

For basket options, the $\mathcal{I}$-class representation is obtained as follows. The intuition comes from the integral representation of a call contract payoff: $(s-K)^+ = \int_K^{+\infty} \mathbb{1}_{\{s \geq x\}} dx$, for $K \geq 0$ and $\forall s \geq 0$. The price of a put on the equally weighted basket can be written

$$e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\left(K - \frac{1}{2}(S_T^1 + S_T^2)\right)^+\right] = e^{-rT}\frac{1}{2}\mathbb{E}^{\mathbb{Q}}\left[\left(2K - S_T^1 - S_T^2\right)^+\right].$$

Following Dhaene and Goovaerts (1996), we have, for $K \geq 0$ and $\forall s_1, s_2 \geq 0$

$$(2K - s_1 - s_2)^+ = \int_0^{2K} \mathbb{1}_{\{s_1 \leq x, s_2 \leq 2K-x\}} dx.$$

$$\begin{aligned} \text{So that} \quad \mathbb{E}^{\mathbb{Q}}\left[\left(2K - S_T^1 - S_T^2\right)^+\right] &= \int_0^{2K} \mathbb{Q}\left(S_T^1 \leq x, S_T^2 \leq 2K - x\right) dx \\ &= \int_0^{2K} H\left(x, 2K - x\right) dx. \end{aligned}$$

For spread and min-max options, the $\mathcal{I}$-class representation of their payoffs is obtained following the same steps and lead to the following expressions for their prices at $t = 0$

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-rT}\left((S_T^2 - S_T^1) - K\right)^+\right] = e^{-rT}\int_0^{+\infty}\left(F_1(x) - H(x, x+K)\right)dx,$$

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-rT}\left(\max\left(S_T^1, S_T^2\right) - K\right)^+\right] = S_0^2 + e^{-rT}\left(\int_0^{+\infty} F_2(x)dx - \int_K^{+\infty} H(x,x)dx - K\right),$$

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-rT}\left(\min\left(S_T^1, S_T^2\right) - K\right)^+\right] = C_0^1(K,T) + P_0^2(K,T) + e^{-rT}\left(\int_K^{+\infty} H(x,x)dx - \int_0^{+\infty} F_2(x)dx\right).$$

3.2. Pricing with the Gaussian copula

Marginal distributions F_1 and F_2, recovered from market prices of options, can be combined with a bivariate Gaussian copula to form the joint distribution H. This copula is the function corresponding to the dependence structure of a bivariate Gaussian distribution. It has one parameter $\rho \in [-1, 1]$ and we denote it by C_ρ^G. For $(u_1, u_2) \in [0, 1]^2$, the Gaussian copula is written, for $\rho \in]-1, +1[$

$$C_\rho^G(u_1, u_2) = \frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{\Phi^{-1}(u_1)}\int_{-\infty}^{\Phi^{-1}(u_2)}\exp\left(-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right)dxdy, \tag{8}$$

where Φ is the cumulative distribution function of the normal distribution. An extension by continuity with respect to ρ leads to

$$C_{\rho=+1}^G(u_1, u_2) = \min(u_1, u_2) \quad \text{and} \quad C_{\rho=-1}^G(u_1, u_2) = \max(u_1 + u_2 - 1, 0).$$

We refer to Meyer (2013) for a detailed review containing these formulas and other relevant properties of the bivariate Gaussian copula.

For a bivariate derivative contract Z, we denote its price by $Z_0^G(\rho)$ when the chosen joint distribution H is built with the Gaussian copula C_ρ^G with a correlation parameter $\rho \in [-1, +1]$. In the notation $Z_0^G(\rho)$, the superscript G stresses the choice of a Gaussian copula to model the dependence under $\mathbb{Q}$. However, it implies no specific form for the marginals F_1 and F_2.

3.3. The notion of implied correlation

Given that the price of a two-asset derivative contract belonging to the $\mathcal{I}$-class is observed in the market, the corresponding implied correlation is defined as the value of the correlation parameter to be used in (6) or (7), with the Gaussian copula, so that it reproduces the given price.

For a contract belonging to the class $\mathcal{I}$, it is also possible to obtain an analytic expression for the sensitivity of the price with respect to the correlation parameter ρ when the Gaussian copula is used for pricing.

The core results of the paper are given in the proposition below.

Proposition 2. *Let Z be a two-asset contract with payoff function z in the $\mathcal{I}$-class, written on the risky assets S^1 and S^2, and with maturity T. We assume that the marginals of S_T^1 and S_T^2 under $\mathbb{Q}$ are known and denoted by F_1 and F_2. Let $Z_0^G(\rho)$ be the price of the two-asset contract Z obtained with the Gaussian copula C_ρ^G.*

1. *Let Z_0^{obs} be an observed arbitrage-free price for a two-asset contract Z. There exists a unique $\rho^* \in [-1,+1]$ such that*

$$Z_0^G(\rho^*) = Z_0^{obs}. \tag{9}$$

2. *For $\rho \in]-1,+1[$, the sensitivity of the two-asset contract price with respect to ρ has the form*

$$\frac{\partial}{\partial\rho} Z_0^G(\rho) = \pm \frac{e^{-rT}}{2\pi\sqrt{1-\rho^2}} \int_L^U \exp\left(-\frac{1}{2}\frac{h_1(x)^2 - 2\rho h_1(x)h_2(x) + h_2(x)^2}{1-\rho^2}\right) dx, \tag{10}$$

where $h_1(x) = \Phi^{-1}(F_1(\pm g_1(x)))$ and $h_2(x) = \Phi^{-1}(F_2(\pm g_2(x)))$.

The proof of the proposition is detailed in Appendix A.

With this result, the notion of implied correlation for contracts with payoffs belonging to class $\mathcal{I}$ appears to be a well-defined concept in the sense that it exists and is unique as soon as the observed price is arbitrage-free. As such, implied correlation or implied dependence can be seen as a key concept for agents managing these derivative contracts. It corresponds to the level of Gaussian dependence between asset prices expected by market operators and embedded in an observed contract price. If the observed price leads to an arbitrage, the notion of implied correlation is meaningless because there is no true distribution able to reproduce the observed price.

For a spread option (call), the sensitivity to ρ is negative and, from the proposition above, it is written as

$$\frac{\partial}{\partial\rho} Z_0^G(\rho) = -\frac{e^{-rT}}{2\pi\sqrt{1-\rho^2}} \int_0^{+\infty} \exp\left(-\frac{1}{2}\frac{y_1(x)^2 - 2\rho y_1(x)y_2(x) + y_2(x)^2}{1-\rho^2}\right) dx < 0, \tag{11}$$

with $y_1(x) = \Phi^{-1}(F_1(x))$ and $y_2(x) = \Phi^{-1}(F_2(x+K))$.

For a call option on the equally weighted basket, this sensitivity is positive and written

$$\frac{\partial}{\partial\rho} Z_0^G(\rho) = \frac{e^{-rT}}{4\pi\sqrt{1-\rho^2}} \int_0^{2K} \exp\left(-\frac{1}{2}\frac{y_1(x)^2 - 2\rho y_1(x)y_2(x) + y_2(x)^2}{1-\rho^2}\right) dx > 0, \tag{12}$$

with $y_1(x) = \Phi^{-1}(F_1(x))$ and $y_2(x) = \Phi^{-1}(F_2(2K-x))$.

4. Numerical illustration

This section presents a numerical illustration of the results obtained above. The risky assets are the S&P500 and Nasdaq 100 indices. We use market data of June 2015 and initial asset prices are normalized at 100 USD. Marginal distributions F_1 and F_2 are fitted to the volatility smile and forward price of each asset. To do so, we used Normal Inverse Gaussian distributions for the log-returns and obtained the corresponding parameters by minimizing the squared errors with respect to the volatility smiles. The interest rate is $r = 0.80\%$.

The copula we use to produce results is a Power Student t copula (PST) obtained by applying an asymmetric power transformation to the bivariate Student t copula. This power transformation is introduced and studied in Liebscher (2008, 2011) and Andersen and Piterbarg (2010). This copula is denoted by $C^{\text{PST}}_{\rho,\nu,\delta,\theta}$ and works with four parameters, namely ρ, ν, δ and θ. It is written as, for $(u_1, u_2) \in [0, 1]^2$

$$C^{\text{PST}}_{\rho,\nu,\delta,\theta}(u_1, u_2) = u_1^{1-(\delta+\theta)} u_2^{1-(\delta-\theta)} T_2 \left(T_\nu^{-1} \left(u_1^{\delta+\theta} \right), T_\nu^{-1} \left(u_2^{\delta-\theta} \right), \rho, \nu \right),$$

with $T_2(.,.,\rho,\nu)$ the cumulative distribution function of the bivariate Student t distribution with parameters ρ and ν, and T_ν^{-1} the inverse cumulative distribution function of the univariate Student t distribution with parameter ν.

In Figure 1 we plot the implied correlation smile computed for spread option calls written on the S&P500 and Nasdaq 100 indices struck at different levels below and above zero and with 1-year maturity. We also plot sensitivity to implied correlation of these bivariate contracts. This sensitivity is to be understood as the dollar change in contract price when the correlation changes by one percentage point. This sensitivity is heavily dependent on the design of the product and reaches its highest values (in absloute terms) for strikes close to zero.

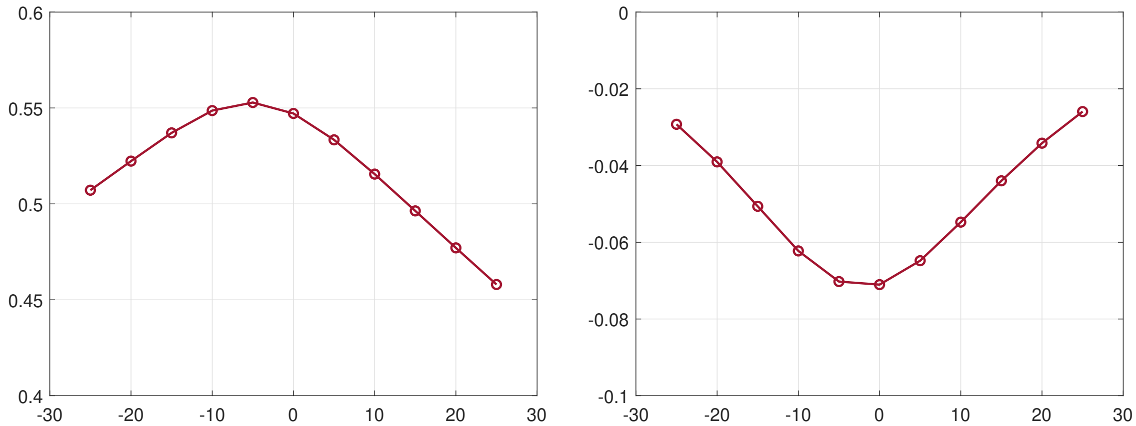

Figure 1: Implied correlation smile obtained with the PST copula for spread option calls with different strikes and written on the Nasdaq and S&P indices (*left*). Sensitivity to implied correlation of spread option calls with different strikes and written on the Nasdaq and S&P indices (*right*).

5. Conclusion

In this note we have formalized a rigorous framework in which the notion of implied correlation has been defined for a large class of two-asset derivative contracts. We have obtained an analytic expression for the sensitivity of a contract's price to the correlation parameter. These results have been illustrated with equity indices market data. In particular, we showed how the level of sensitivity to implied correlation evolves with the contract design, namely its strike.

Appendix A. Proof of Proposition 2

There are four cases for equation (6), depending on the signs in the payoff function:

$$\begin{aligned}
\int_L^U \mathbb{Q}\left[+S_T^1 \le g_1(x), +S_T^2 \le g_2(x)\right] dx &= \int_L^U H(g_1(x), g_2(x))dx,\\
\int_L^U \mathbb{Q}\left[-S_T^1 \le g_1(x), +S_T^2 \le g_2(x)\right] dx &= \int_L^U \left(F_2(g_2(x)) - H(-g_1(x), g_2(x))\right) dx,\\
\int_L^U \mathbb{Q}\left[+S_T^1 \le g_1(x), -S_T^2 \le g_2(x)\right] dx &= \int_L^U \left(F_1(g_1(x)) - H(g_1(x), -g_2(x))\right) dx,\\
\int_L^U \mathbb{Q}\left[-S_T^1 \le g_1(x), -S_T^2 \le g_2(x)\right] dx &= \int_L^U \left(1 - F_2(g_2(x)) - F_1(g_1(x))\right) dx\\
&+ \int_L^U \left(H(-g_1(x), -g_2(x))\right) dx,
\end{aligned}$$

so that the price has the following form

$$Z_0^G(\rho) = c \pm \int_L^U C_\rho^G(F_1(\pm g_1(x)), F_2(\pm g_2(x)))dx, \tag{A.1}$$

where the constant c is independent from the dependence structure, and hence, from ρ.

Now, given the Fréchet-Hoeffding bounds, this implies that under the absence of arbitrage,

$$\min\left(Z_0^G(+1), Z_0^G(-1)\right) \le Z_0^G(\rho) \le \max\left(Z_0^G(+1), Z_0^G(-1)\right). \tag{A.2}$$

If the observed price Z_0^{obs} lies outside the bounds, it leads to an arbitrage and the notion of implied correlation is not defined. For details on these no arbitrage bounds, we refer to Dhaene and Goovaerts (1996), Tavin (2015) and references therein.

For $\rho \in]-1, +1[$, we recall the Plackett formula for the derivative of the Gaussian copula obtained in Plackett (1954) for C_ρ^G:

$$\frac{\partial}{\partial\rho} C_\rho^G(u_1, u_2) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2}\frac{\Phi^{-1}(u_1)^2 - 2\rho\Phi^{-1}(u_1)\Phi^{-1}(u_2) + \Phi^{-1}(u_2)^2}{1-\rho^2}\right),$$

which has a bivariate Gaussian density form (easy to bound) and the Leibniz rule applied to (A.1) gives

$$\begin{aligned}
\frac{\partial}{\partial\rho} Z_0^G(\rho) &= \pm \int_L^U \frac{\partial}{\partial\rho} C_\rho^G(F_1(\pm g_1(x)), F_2(\pm g_2(x)))dx\\
&= \pm \frac{1}{2\pi\sqrt{1-\rho^2}} \int_L^U \exp\left(-\frac{1}{2}\frac{h_1(x)^2 - 2\rho h_1(x)h_2(x) + h_2(x)^2}{1-\rho^2}\right)dx,
\end{aligned}$$

where $h_1(x) = \Phi^{-1}(F_1(\pm g_1(x)))$ and $h_2(x) = \Phi^{-1}(F_2(\pm g_2(x)))$. Accordingly, $\frac{\partial}{\partial\rho} Z_0^G(\rho)$ is different from zero and either always positive or always negative (so that the price of the option is strictly monotonous in ρ). Since the integrand is continuous in ρ, so is the integral and hence the implied correlation exists and is unique. The proof follows exactly the same steps when the price is given by expression (7). □

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