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Ranking tail risk across international stock markets

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Abstract

The importance of extreme events in finance is increasingly recognized but the existing methods for estimating the tail risk are not very satisfactory. The Hill estimator which is the most commonly used in the literature, suffers severe deficiencies. As an alternative, we propose to estimate the relative tail risk and not the absolute tail risk. This allows us to revisit a larger pool of tail index estimators, including stable law estimators, which have been discarded because of their bias when estimating the absolute tail risk. From a database of 74 international stock market indices, and using both simulated and empirical data, we find that the McCulloch estimator gives an estimate of the relative tail risk up to 50\% more precise than the Hill estimator. These results are confirmed by the White (2000) and Hansen (2005) superior predictive ability tests.

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1 Introduction

In many fields, researchers have documented extreme risks with heavy-tailed distributions (Gabaix 2016). These distributions are characterized by slowly decaying power law functions in their tails. The speed at which they decay is determined by the exponent of these power laws. The latter is called the tail index and indicates the degree of extreme risk. A lot of efforts have been devoted for estimating the value of the tail index but the existing methods are not very satisfactory (Clauset et al. 2009). In this paper, We revisit this issue by noting that, often, the absolute level of risk is of little interest and only the relative level of risk is of importance. We take advantage of the fact that we do not try to estimate the absolute value of the tail index. Therefore we are not interested in the accuracy of an estimator but only in its precision. This allows us to consider a larger pool of tail index estimator, including estimators from the stable law family. We compare the performance of the Hill estimator, which is the most commonly used tail estimator with the McCulloch and the GMM stable estimators. We focus on the unconditional distribution of stock market returns.

Our analysis is based on an extensive data set of 74 international stock market indexes. First, we extend the work of Malevergne et al. (2005) and find that international stock market returns are not Pareto distributed in their tails. Since the Hill estimator is not robust to departures from the Pareto distribution, this result casts doubt on the use of the Hill method. Then we study the performance of the Hill, the McCulloch and the GMM estimators, both using a Monte Carlo analysis and based on empirical data. We show that the McCulloch estimator is the best able to track extreme risks measured by the kurtosis, left and right tail of the stock market returns. Conversely, the GMM estimator performs very poorly. Our finding is consistent across both the simulation study and the empirical study based on the bootstrap analysis of our 74 country stock indexes. In the empirical study, the McCulloch estimator outperforms the Hill estimator between 30% and 50% in the estimation of the extreme risk measures. These results are confirmed by the Hansen (2005) and White (2000) test for superior predictive ability. Of the three estimators, the McCulloch method is the most precise in estimating the relative tail risk and in establishing their ranking across international stock markets.

The paper proceeds as follows: Section 2 reviews the extreme value distributions of international stock market returns. Section 3 presents the tail index estimators. Section 4 compares the performance of the Hill, the McCulloch and the GMM estimators. Section 5 concludes.

2 Which domain of attraction for the distribution of international stock market returns?

Extreme Value Theory (EVT)[1] establishes that, assuming the existence of norming constants μ (location parameter) and σ (scale parameter), the standardized series of extreme values generated by a cumulative distribution function F is represented by the *Generalized Extreme Value* (GEV) distribution which reads

$$F_{\xi,\mu,\sigma}(x) = \begin{cases} exp\left\{-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}\right\}, & \xi \neq 0 \text{ and } 1+\xi\frac{x-\mu}{\sigma} > 0 \\ exp\left\{-exp\left(-\frac{x-\mu}{\sigma}\right)\right\}, & \xi = 0 \end{cases} \quad (1)$$

ξ is the shape parameter. $\xi > 0$ corresponds to the *Maximum Domain of Attraction* (MDA) of the Fréchet law, $\xi = 0$ corresponds to the MDA of the Gumbel law and $\xi < 0$ corresponds to the MDA of the Weibull law[2]. Distributions of the Fréchet MDA have heavy tails and decays as power laws with exponent $\alpha = \frac{1}{\xi}$. They include the Cauchy, Pareto, Student, Fisher, Fréchet and non-gaussian stable distributions. The value of the tail index α characterizes the degree of extreme risk. The Fréchet MDA is a necessary condition for many tail index estimators including the Hill estimator which is the most commonly used in he literature.

[1] See for instance Rocco (2014) or Gomes and Guillou (2015) for a review of extreme value theory.

[2] An alternative parametric approach of EVT focuses on the excess distribution over the threshold u and finds that its asymptotic cumulative distribution function F is the *generalized Pareto distribution* (GPD). Both GEV and GPD approaches have the same shape parameter ξ.

Many empirical distributions in finance seem to belong to the Fréchet MDA and follow a power law, at least in their higher quantiles (Gabaix et al. 2003). However Malevergne et al. (2005) found that for the U.S. stock markets, the tails decay faster than power laws, which questions the ability of the Hill estimator to correctly evaluate the tail risk. We extend Malevergne et al. (2005) work to a large panel of international stock markets. We compiled the daily returns of all stock market indexes across the world that are consistently available in the Bloomberg database from 1994 to 2014. This gives a database of 89 countries. We retained the stock markets indexes with at least 70 observations in the positive and negative highest deciles, and obtained 74 stock market indexes. We study to which extent some theoretical distributions fit the empirical data. We use the Anderson-Darling test and compare the empirical distribution with the Pareto distribution, the Weibull distribution and the GEV distribution, over the highest decile (Table 2). In the first case, when considering the full spectrum of the distribution, we find that the Pareto or Weibull distributions are good approximation of the empirical distribution (Table 1) for most countries. However, when focusing on the highest decile, we find that all countries reject the Pareto or Weibull distribution. This confirms Malevergne et al. (2005)'s results and shows that the tail of stock market return distribution is not well represented by Pareto distributions. The MDA over the top decile of the distribution of returns across international stock market is not of Fréchet type. Consequently, this questions the ability of the usual tail index estimators, such as the Hill estimator, to measure tail risk.

3 Tail index estimators

For investors who are only interested in the *relative* riskiness of an investment and not in their absolute value , it does not matter if the estimator misses the absolute value of risk provided it preserves the relative ordering. In that case, the performance of *relative* tail risk estimators only depends on its variance not on its bias. Consequently, when estimating *relative* tail risk, one must judge the merits of estimators only on the basis of their precision, not on their accuracy. This allows us to consider a larger pool of estimators, including biased ones. In particular, we are interested in revisiting the stable law estimators, which had been discarded after a consensus emerged against the stable distribution for financial series ((Longin 1996; Vilasuso and Katz 2000)). In this paper, we compare the Hill estimator with two alternative stable distribution estimators, the McCulloch (1986)'s estimator (MCC) and the generalized method of moments estimator (GMM).

3.1 The Hill estimator

The Hill estimator, also known as the maximum likelihood estimator (MLE), is the most popular tail exponent estimator. It was proposed by Hill (1975) as

$$\hat{\alpha}_{k,n} = \left[\frac{\sum_{j=1}^{k} (\log X_{j,n} - \log X_{k,n})}{k} \right]^{-1} \tag{2}$$

with $X_{1,n} \geq ... \geq X_{n,n}$ the order statistics of the data. k is the threshold from which start the $k+1$ upper exceedances corresponding to the most extreme observations. As non-parametric tail estimator, it has the advantage that one does not need to know the exact distribution of the data generating process (DGP). The Hill estimator only needs to assume that cumulative distribution function (CDF) is in the Fréchet MDA and consequently has a Pareto-type tail. In that case, the Hill estimator is generally the most efficient tail index estimator ((McNeil and Frey 2000)).
However this estimator has two important drawbacks. First it is non-robust to departure from the Pareto DGP. When the CDF is not in the MDA of a Fréchet distribution, the Hill estimator is strongly biased and even small departures from the Pareto DGP can have large effects on the estimates ((Brzezinski 2016)). The second drawback is its sensitiveness to the choice of the cut-off parameter k. This latter parameter corresponds to the threshold where the CDF converges to a Pareto distribution. The choice of k entails a trade-off between bias and variance. If one chooses a large k, the bias will be important since the sample will include observations that are not from a Pareto DGP. Conversely, if k is too small, the sample will be more Pareto-like but the variance of the estimates will increase dramatically. Unfortunately, since the optimal threshold depends on both the sample size and the unknown parameters of the distribution

Table 1: Goodness of fit of empirical stock market return distribution over the full distribution (Q0-Q100) with respectively Pareto, Weibull and GEV distribution. The figures give the confidence level of the Anderson-Darling statistics.

	Negative tail			Positive tail				Negative tail			Positive tail		
	Pareto	Weibull	GEV	Pareto	Weibull	GEV		Pareto	Weibull	GEV	Pareto	Weibull	GEV
Argentina	0.97	0.86	0.00	0.00	0.03	0.02	Malta	0.04	0.02	0.00	0.00	0.06	0.00
Australia	0.06	0.24	0.00	0.00	0.38	0.00	Mauritius	0.01	0.00	0.03	0.01	0.00	0.01
Austria	0.08	0.04	0.00	0.00	0.05	0.01	Mexico	0.47	0.78	0.00	0.00	0.06	0.00
Bahrain	0.53	0.15	0.01	0.99	0.98	0.01	Mongolia	0.88	0.00	0.00	0.96	0.00	0.00
Belgium	0.60	0.55	0.00	0.00	0.06	0.00	Morocco	0.62	0.58	0.00	0.16	0.07	0.00
Brazil	0.01	0.21	0.00	0.00	0.01	0.00	Namibia	0.04	0.52	0.06	0.01	0.44	0.05
Bulgaria	0.00	0.00	0.01	0.03	0.00	0.02	Netherlands	0.19	0.22	0.00	0.00	0.05	0.00
Canada	0.48	0.19	0.00	0.00	0.04	0.00	New Zealand	0.00	0.22	0.00	0.00	0.65	0.00
Chile	0.01	0.04	0.00	0.00	0.13	0.00	Nigeria	0.00	0.00	0.00	0.94	0.71	0.00
China	0.18	0.05	0.00	0.11	0.01	0.00	Norway	0.01	0.02	0.01	0.00	0.27	0.00
Colombia	0.18	0.06	0.00	0.10	0.07	0.00	Oman	0.68	0.00	0.00	0.00	0.00	0.01
Cote d'Ivoire	NA	NA	NA	0.54	0.28	0.00	Pakistan	0.01	0.17	0.00	0.00	0.49	0.00
Croatia	0.09	0.02	0.02	0.01	0.01	0.09	Palestine	0.00	0.00	0.00	0.26	0.05	0.00
Cyprus	0.39	0.30	0.00	0.16	0.12	0.00	Peru	0.20	0.05	0.00	0.01	0.06	0.01
Czech Republic	0.01	0.08	0.00	0.00	0.27	0.00	Philippines	0.01	0.07	0.01	0.00	0.23	0.01
Denmark	0.01	0.24	0.00	0.00	0.61	0.00	Poland	0.28	0.08	0.00	0.02	0.11	0.00
Egypt, Arab Rep.	0.76	0.68	0.00	0.06	0.57	0.00	Portugal	1.00	0.64	0.00	0.04	0.64	0.00
Finland	0.28	0.19	0.00	0.04	0.15	0.00	Qatar	0.39	0.00	0.00	0.61	0.00	0.00
France	0.05	0.36	0.00	0.00	0.33	0.00	Romania	0.10	0.02	0.01	0.26	0.08	0.00
Germany	0.29	0.70	0.00	0.00	0.65	0.00	Russia	0.63	0.32	0.00	0.01	0.09	0.00
Greece	0.08	0.10	0.00	0.11	0.49	0.00	Saudi Arabia	0.00	0.00	0.00	0.04	0.00	0.00
Hong Kong, China	0.96	0.87	0.00	0.07	0.20	0.00	Singapore	0.52	0.72	0.00	0.00	0.21	0.02
Hungary	0.00	0.01	0.01	0.05	0.53	0.00	Slovenia	0.79	0.02	0.00	0.16	0.06	0.00
Iceland	0.18	0.00	0.00	0.03	0.41	0.00	South Africa	0.01	0.20	0.00	0.00	0.10	0.02
India	0.00	0.36	0.00	0.00	0.38	0.00	Spain	0.00	0.55	0.00	0.00	0.34	0.00
Indonesia	0.09	0.05	0.00	0.00	0.01	0.01	Sri Lanka	0.14	0.01	0.01	0.20	0.19	0.00
Ireland	0.07	0.03	0.00	0.00	0.03	0.01	Sweden	0.00	0.85	0.00	0.00	0.21	0.00
Israel	0.07	0.66	0.00	0.00	0.42	0.00	Switzerland	0.02	0.21	0.00	0.00	0.17	0.00
Italy	0.18	0.51	0.00	0.00	0.96	0.00	Taiwan	0.86	0.82	0.00	0.01	0.69	0.00
Jamaica	0.00	0.03	0.00	0.09	0.00	0.00	Thailand	0.63	0.80	0.00	0.01	0.16	0.00
Japan	0.00	0.37	0.00	0.00	0.44	0.00	Tunisia	0.12	0.02	0.05	0.15	0.17	0.00
Kazakhstan	NA	NA	NA	0.00	0.18	0.00	Turkey	0.03	0.13	0.00	0.01	0.16	0.00
Kenya	0.07	0.01	0.01	0.01	0.00	0.04	Ukraine	0.01	0.04	0.00	0.04	0.01	0.00
Korea, Rep.	0.36	0.23	0.00	0.03	0.08	0.00	U.A.E.	0.07	0.03	0.03	0.10	0.12	0.02
Kuwait	0.25	0.06	0.07	0.05	0.42	0.09	United Kingdom	0.02	0.12	0.00	0.00	0.18	0.00
Latvia	0.08	0.03	0.02	0.02	0.02	0.03	United States	0.76	0.77	0.00	0.01	0.04	0.00
Malaysia	0.04	0.00	0.01	0.02	0.00	0.00	Vietnam	0.02	0.12	0.00	0.81	0.67	0.00

Table 2: Goodness of fit of empirical stock market return distribution highest quantiles (Q90-Q100) with respectively Pareto, Weibull and GEV distribution. The figures give the confidence level of the Anderson-Darling statistics.

	Negative tail			Positive tail				Negative tail			Positive tail		
	Pareto	Weibull	GEV	Pareto	Weibull	GEV		Pareto	Weibull	GEV	Pareto	Weibull	GEV
Argentina	0.00	0.00	0.50	0.00	0.00	0.25	Malta	0.00	0.00	0.27	0.00	0.00	0.22
Australia	0.00	0.00	0.18	0.00	0.00	0.31	Mauritius	0.00	0.00	0.23	0.00	0.00	0.29
Austria	0.00	0.00	0.30	0.00	0.00	0.09	Mexico	0.00	0.00	0.08	0.00	0.00	0.11
Bahrain	0.00	0.00	0.30	0.00	0.00	0.64	Mongolia	0.00	0.00	0.10	0.00	0.00	0.07
Belgium	0.00	0.00	0.27	0.00	0.00	0.55	Morocco	0.00	0.00	0.39	0.00	0.00	0.51
Brazil	0.00	0.00	0.22	0.00	0.00	0.41	Namibia	0.00	0.00	0.83	0.00	0.00	0.38
Bulgaria	0.00	0.00	0.42	0.00	0.00	0.39	Netherlands	0.00	0.00	0.15	0.00	0.00	0.50
Canada	0.00	0.00	0.31	0.00	0.00	0.04	New Zealand	0.00	0.00	0.39	0.00	0.00	0.56
Chile	0.00	0.00	0.36	0.00	0.00	0.18	Nigeria	0.00	0.00	0.63	0.00	0.00	0.75
China	0.00	0.00	0.11	0.00	0.00	0.26	Norway	0.00	0.00	0.23	0.00	0.00	0.17
Colombia	0.00	0.00	0.36	0.00	0.00	0.20	Oman	0.00	0.00	0.29	0.00	0.00	0.58
Cote d'Ivoire	NA	NA	NA	0.00	0.00	0.50	Pakistan	0.00	0.00	0.40	0.00	0.00	0.19
Croatia	0.00	0.00	0.44	0.00	0.00	0.71	Palestine	0.00	0.00	0.22	0.00	0.00	0.38
Cyprus	0.00	0.00	0.45	0.00	0.00	0.17	Peru	0.00	0.00	0.14	0.00	0.00	0.17
Czech Republic	0.00	0.00	0.21	0.00	0.00	0.34	Philippines	0.00	0.00	0.33	0.00	0.00	0.33
Denmark	0.00	0.00	0.31	0.00	0.00	0.57	Poland	0.00	0.00	0.22	0.00	0.00	0.24
Egypt, Arab Rep.	0.00	0.00	0.48	0.00	0.00	0.32	Portugal	0.00	0.00	0.27	0.00	0.00	0.59
Finland	0.00	0.00	0.24	0.00	0.00	0.19	Qatar	0.00	0.00	0.37	0.00	0.00	0.03
France	0.00	0.00	0.17	0.00	0.00	0.22	Romania	0.00	0.00	0.34	0.00	0.00	0.16
Germany	0.00	0.00	0.05	0.00	0.00	0.17	Russia	0.00	0.00	0.42	0.00	0.00	0.26
Greece	0.00	0.00	0.22	0.00	0.00	0.16	Saudi Arabia	0.00	0.00	0.22	0.00	0.00	0.18
Hong Kong, China	0.00	0.00	0.38	0.00	0.00	0.48	Singapore	0.00	0.00	0.65	0.00	0.00	0.23
Hungary	0.00	0.00	0.30	0.00	0.00	0.67	Slovenia	0.00	0.00	0.26	0.00	0.00	0.13
Iceland	0.00	0.00	0.50	0.00	0.00	0.47	South Africa	0.00	0.00	0.53	0.00	0.00	0.07
India	0.00	0.00	0.37	0.00	0.00	0.20	Spain	0.00	0.00	0.40	0.00	0.00	0.57
Indonesia	0.00	0.00	0.48	0.00	0.00	0.24	Sri Lanka	0.00	0.00	0.61	0.00	0.00	0.63
Ireland	0.00	0.00	0.26	0.00	0.00	0.09	Sweden	0.00	0.00	0.26	0.00	0.00	0.10
Israel	0.00	0.00	0.19	0.00	0.00	0.22	Switzerland	0.00	0.00	0.32	0.00	0.00	0.24
Italy	0.00	0.00	0.04	0.00	0.00	0.39	Taiwan	0.00	0.00	0.19	0.00	0.00	0.24
Jamaica	0.00	0.00	0.24	0.00	0.00	0.12	Thailand	0.00	0.00	0.79	0.00	0.00	0.20
Japan	0.00	0.00	0.28	0.00	0.00	0.62	Tunisia	0.00	0.00	0.42	0.00	0.00	0.24
Kazakhstan	NA	NA	NA	0.00	0.00	0.56	Turkey	0.00	0.00	0.19	0.00	0.00	0.19
Kenya	0.00	0.00	0.11	0.00	0.00	0.07	Ukraine	0.00	0.00	0.39	0.00	0.00	0.25
Korea, Rep.	0.00	0.00	0.34	0.00	0.00	0.05	U.A.E.	0.00	0.02	0.78	0.00	0.00	0.38
Kuwait	0.00	0.02	0.42	0.00	0.00	0.39	United Kingdom	0.00	0.00	0.63	0.00	0.00	0.39
Latvia	0.00	0.00	0.35	0.00	0.00	0.21	United States	0.00	0.00	0.76	0.00	0.00	0.15
Malaysia	0.00	0.00	0.46	0.00	0.00	0.53	Vietnam	0.00	0.04	0.12	0.00	0.06	0.34

((Hall and Welsh 1985)), the methods for determining this threshold are purely empirical which makes this method difficult to implement practically.

3.2 The stable law estimators

Non-gaussian stable laws have no analytical CDF but their characteristic function can be written as:

$$\log \rho(u) = i\delta u - \gamma |u|^{\alpha}\left[1 + i\beta \frac{u}{|u|} W(\alpha, u)\right],$$
$$\text{with } 0 < \alpha \leq 2, -1 \leq \beta \leq 1, \delta \in \mathbb{R}, \gamma \geq 0, \text{ and } W(\alpha, u) = \begin{cases} \tan \frac{\pi\alpha}{2} \text{ if } \alpha \neq 1 \\ -\frac{2}{\pi} \log |u| \text{ if } \alpha = 1 \end{cases} \quad (3)$$

Stable law implies a tail index α between 0 and 2. This is at variance with the empirical findings and it is now widely accepted that the tail index of financial asset returns is between 2 and 5 ((Cont 2001)). Consequently, using stable law estimators to estimate stock market returns is bound to generates strongly biased results. However, since we are not interested in the absolute value of tail risk but only in its relative value, we are not concerned by the bias. Rather we are interested in the dispersion and we investigate if stable law estimators could return relative tail estimates with a greater precision than the Hill estimator.
We study the performance of two stable law estimators, the McCulloch estimator ((McCulloch 1986)) and the generalized method of moments (GMM) estimator ((Hansen 1982)). The McCulloch method is very simple and consists in estimating quantiles ν_α and ν_β:

$$\nu_\alpha = \frac{\hat{x}_{95} - \hat{x}_{05}}{\hat{x}_{75} - \hat{x}_{25}}, \quad (4)$$

$$\nu_\beta = \frac{\hat{x}_{95} + \hat{x}_{05} - 2\hat{x}_{50}}{\hat{x}_{95} - \hat{x}_{05}}, \quad (5)$$

and then recovering the parameters α and β of the stable distribution by interpolating these estimated quantiles in the tables calculated by McCulloch (1986).

4 Comparing the Hill, the McCulloch and the GMM estimators

4.1 Methodology

We study the tail indexes estimated with the Hill estimator, which is based on a power law distribution, and tail indexes estimated from the McCulloch estimation method (MCC) and the generalized method of moments (GMM), which are both based on a stable law distribution. These tail index estimators will be compared to the relative ranking of various extreme risk measures, including both single and double extreme risk measures, namely the kurtosis, the left tail quantile and the right tail quantile. The detailed description of the extreme risk measures are presented in the appendix. Our focus is to establish the ability of the tail index in tracking the relative ranking of the above extreme risk measures. To do so, we calculate the Spearman rank correlation coefficient between the tail index estimates and the extreme risk measures. If the tail risk index is able to reflect the ranking of the extreme risk measure perfectly, then the Spearman correlation coefficient will be one. Conversely, a low Spearman correlation coefficient will indicate that the tail index tracks the relative ranking of the risk measure poorly. Denoting the Spearman correlation coefficient as $\rho_{j,k}$, where $j = (MCC, GMM, Hill)$ is the tail index estimator and $k = (kurtosis,\ left\ tail,\ right\ tail)$ is the extreme risk measure, we want to identify the tail index estimator j that yields $\rho_{j,k}$ that is (1) closest to one, and also (2) consistently high irrespective of the extreme risk measure k.

4.2 Simulation Study

In this section, we study the performance of the different tail index estimators in a simulation study. We generate distributions of stock returns based on common parametric distributions used to describe the fat tail distributions of stock returns, namely, the Pareto distribution (PD), the Weibull distribution to describe stretched exponential tail patterns ((Malevergne et al. 2005)) and the Generalized Extreme Value distribution to cover the other possible tail patterns (Fréchet, Gumbel etc.). The parametric density distributions used are described as follow:

1. The Pareto Distribution:

$$F(x) = \begin{cases} 1 - \left(\frac{\mu}{x}\right)^{\alpha} & x \geq \mu \\ 0 & x < \mu \end{cases} \tag{6}$$

where α is the tail parameter, and μ is the location parameter.

2. The Weibull Distribution:

$$F(x) = \begin{cases} 1 - \exp\left(\frac{x}{\lambda}\right)^{\kappa} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{7}$$

where κ is the shape parameter, and λ is the scale parameter.

3. The Generalized Extreme Value Distribution:

$$F(x) = \begin{cases} \exp\left\{-\left(1 - \frac{\xi(x-\mu)}{\sigma}\right)^{\frac{1}{\xi}}\right\} & \xi \neq 0 \\ \exp\left\{-\left(e^{\frac{-(x-\mu)}{\sigma}}\right)\right\} & \xi = 0 \end{cases} \tag{8}$$

where ξ is the shape parameter, μ is the location parameter, and σ is the scale parameter.

To simulate the heavy tail distributions, we first calibrate the above parametric distributions to reflect the actual stock market returns studied in the empirical study of section 4.3. From the 74 country's stock index returns, we extracted a range of estimated parameters for each type of parametric distribution and present them in table 3. We then vary the parameters over this range to define different returns distributions with different levels of heavy tails within each type of parametric distribution function denoted by $\Omega_{n,d}$. Where $n = 1, \cdots, 45$ denotes the n^{th} heavy tail distribution generated under the parametric distribution $d = (Pareto, Weibull, GEV)$.

Table 3: Range of Parameters for Heavy Tail Distributions

This table presents the range of parameters for the different parametric distributions, as extracted from the 74 country's stock index returns over the period of 1994 to 2014.

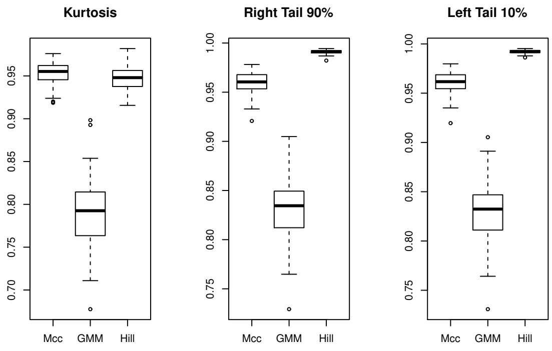
	α - tail	**μ - location**	
Pareto	1 - 9	0.05 - 0.30	
	κ - shape		**λ - scale**
Weibull	0.1 - 0.9		0.005 - 0.009
	ξ - shape	**μ - location**	**σ - scale**
GEV	0.1 - 0.9	0.002 - 0.009	0.001 - 0.009

For each defined distribution $\Omega_{n,d}$, we proceed to simulate 100 distributions (with 5000 observations each) in our simulation study. For each simulated distribution, we then compute the tail index estimators α_j, extreme risk measures γ_k and finally, the Spearman's correlation coefficient $\rho_{j,k}$. The range of Spearman's rank correlation coefficients $\rho_{j,k}$ computed for the tail indexes j and extreme risk measures k are presented in Figure 1 Panel A, B and C for respectively the simulated, Pareto, Weibull and GEV distributions.

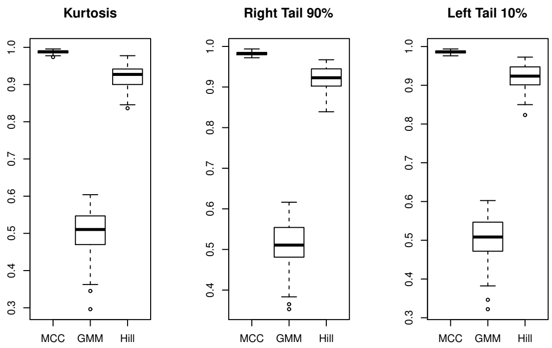
In Panel A of Figure 1, we see that for Pareto distribution, the Hill estimator exhibits the best performance for the tail index estimates, both in terms of performance and the tracking of the ranks. This is expected, as the Hill estimator is the most efficient estimator when the DGP is based on the Power law (McNeil and Frey (2000)). Meanwhile, the McCulloch estimator performs relatively well too, with a median of more than 0.95 and consistent precision for Kurtosis and both tail risk measures. When we turn to the Weibull and GEV distributions, the McCulloch estimator exhibits consistently the best

Figure 1: Boxplot of Spearman rank correlation coefficient for the simulated distributions

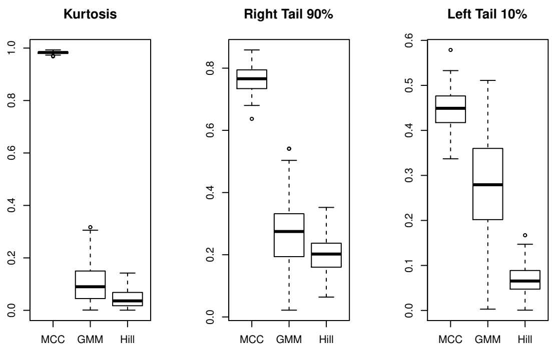
This set of graphs presents the boxplots of the Spearman rank correlation coefficient calculated between the tail indexes and extreme risk measures for the simulated distributions.



(a) Panel A: Pareto Distributions



(b) Panel B: Weibull Distributions



(c) Panel C: GEV Distributions

performance in terms of tracking the ranks of the various risk measures, and in term of precision in the estimation (Panel B and C). The Hill estimator shows less consistency and precision in tracking the ranking of the extreme risk measures with a median around 0.92 in the Weibull case (Panel B). For the GEV case (Panel C), the Hill estimator delivers some very poor results and cannot track any extreme risk measure. Last, in all cases, the GMM estimator performs very badly. Since the Pareto and Weibull distributions are strongly rejected for the highest decile (see Table 2), these results speaks in favor of using the McCulloch estimator.

In should be noted that the McCulloch estimator is able to track the kurtosis measure closely, as it also takes into account information from the body of the distribution (equations 4 and 5), which is similar to the kurtosis measure. This could account for its outperformance compared to the Hill estimator, which only focuses on the tails of the distribution.

4.3 Empirical Study

After the simulation study, we analyze the performance of the tail index estimators on ranking the extreme risk measures in the actual stock market return distributions. To do so, we constructed the return distributions of 74 different stock market indexes, based on the corresponding daily stock index returns over 10 years. Table 5 presents the estimations of the tail index based on the Hill and McCulloch estimators, as well as the common extreme risk measure kurtosis for the 74 countries, while Table 4 reports the summary statistics of the tail index estimates.

We can see from Table 4 that the range of the tail index estimated based on the Hill and the McCulloch estimators, and the extreme risk measure kurtosis differs greatly. Upon Winsorizing the 5% outliers, the range of the McCulloch estimator is relatively small, from 1.16 and 1.70, while the Hill estimator is in the range 2.78 and 6.62. It should also be noted that the range of both indexes are way smaller than the range of the kurtosis (0.83 to 8.30) estimated from the stock index returns.

Table 4: Descriptive statistics of the tail risk index estimation.

Tail index estimates are obtained from 74 stock index return distributions, constructed using daily observations over the period of 1994 to 2014. The summary statistics are computed from the Winsorized sample eliminative the 5% outliers from the sample.

	Number of Observations	**Hill Estimator**	**McCulloch Estimator**	**Kurtosis**
Sample Average	4420	4.11	1.42	3.46
Minimum	2509	2.81	1.17	0.85
Maximum	5358	6.14	1.70	7.08
Standard Deviation[3]	1052	0.89	0.24	2.93

Figure 2 shows that the McCulloch estimates are very correlated to the kurtosis in a log-linear manner, while the Hill estimates exhibits a much looser correlation with the kurtosis. As we are concerned with the ranking of the country tail risk (*relative* tail risk), as long as the tail index tracks the kurtosis in a monotonous manner, a pure linear relationship is not necessary.

Similar to the simulation study (Table 1), we first estimate the tail indexes α_j for these 74 country stock index returns using the McCulloch, GMM and Hill estimators. After which, we estimated the extreme risk measures γ_k, namely the kurtosis, left and right tails as defined in section 4 for the same set of stock index returns. To measure the ability of the tail indexes to track the ranking of the extreme risk measures, we compute the Spearman's rank correlation coefficient $\rho_{j,k}$ between the tail indexes and the extreme risk measures. In addition, as a robustness check, we conduct a bootstrap analysis on the return distributions of the stock index returns. Following Patton and Timmermann (2010), to account for time series dependencies, we set the average block length in the bootstrap analysis to be 6 days, so as to preserve the limited time series dependencies at the daily horizon. The number of bootstrap replications is set to 500 in this study. For each bootstrap sample, we repeat the procedure of estimating the tail indexes α_j and extreme risk measures γ_k, and computing the Spearman's rank correlation coefficient between the tail index and extreme risk measures $\rho_{j,k}$.

For the bootstrap analysis, the McCulloch estimator exhibits the best performance in tracking all the three extreme risk measures: kurtosis, left and right tails. As the bootstrap samples are replicated from the actual data, which contains more noise than our simulated sample, the precision for tracking the left and right tail estimates by the McCulloch and the Hill methods is less satisfactory than in the simulated samples. Nevertheless, as compared to the Hill and GMM methods, the McCulloch method

Table 5: Tail risk estimates and extreme risk measure for 74 countries.

This tables shows the Hill and McCulloch tails risk estimates and the kurtosis. A higher tail risk estimates denotes a lower tail risk.

Country name	Stock market index name	Number of observations	Hill estimator	McCulloch estimator	Kurtosis
Argentina	Buenos Aires Stock Exchange Merval Index	5120	3.59	1.46	1.38
Australia	S&P/ASX 200	5245	4.33	1.69	0.86
Austria	Vienna Stock Exchange Austrian Traded Index	5138	4.3	1.7	1.27
Bahrain	Bahrain Bourse All Share Index	2509	3.78	1.42	1.60
Belgium	BEL 20 Index	5265	4.46	1.63	1.41
Brazil	Ibovespa Brasil Sao Paulo Stock Exchange Index	5129	3.9	1.61	1.02
Bulgaria	SOFIX Index	3447	3.04	1.36	2.80
Canada	S&P/TSX Composite Index	5262	3.59	1.7	1.36
Chile	Santiago Stock Exchange IGPA Index	5176	3.61	1.74	0.85
China	Shanghai Stock Exchange Composite Index	5038	3.76	1.46	1.72
Colombia	MSCI Equity Index Emerging Mrkts in Local Curr Colombia	5405	4.2	1.49	2.03
Croatia	Croatia Zagreb Stock Exchange Crobex Index	3008	3.85	1.45	3.46
Cyprus	FTSE/Cyprus Stock Exchange 20 Index	3437	3.88	1.45	2.14
Czech Republic	Prague Stock Exchange Index	5046	4.39	1.59	2.05
Denmark	OMX Copenhagen 20 Index	5192	4.29	1.69	1.04
Egypt	Egyptian Financial Group Hermes Stock Market Index	4706	5.53	1.47	44.18
Finland	OMX Helsinki Index	5198	3.47	1.47	2.84
France	CAC 40 Index	5267	5.28	1.67	1.47
Germany	Deutsche Boerse AG German Stock Index DAX	5255	5.23	1.61	0.90
Greece	Athens Stock Exchange General Index	5180	3.42	1.55	14.08
Hong Kong	Hong Kong Hang Seng Index	5166	4.58	1.47	1.44
Hungary	Budapest Stock Exchange Budapest Stock Index	5178	3.88	1.66	1.38
Iceland	Iceland Stock Exchange ICEX Main Index	4067	4.07	1.44	1.03
India	National Stock Exchange CNX Nifty Index	5142	4.69	1.57	1.79
Indonesia	Jakarta Stock Exchange Composite Index	5054	4.25	1.48	0.96
Ireland	Irish Stock Exchange Overall Index	5223	3.99	1.61	1.59
Israel	Tel Aviv 25 Index	5085	3.67	1.59	1.65
Italy	FTSE Italia MIB Storico Index EUR	5251	4.48	1.52	1.03
Ivory Coast	BRVM Comp Share Index	3614	4.02	1.27	1.08
Jamaica	Jamaica Stock Exchange Market Index	4846	3.25	1.3	2.73
Japan	Tokyo Stock Exchange Tokyo Price Index TOPIX	5099	4.62	1.64	0.67
Kazakhstan	Kazakhstan Stock Exchange Index KASE	3476	4.32	1.12	5.39
Kenya	Nairobi Securities Exchange Ltd 20 Index	5038	3.14	1.46	1.92
Korea, South	Korea Stock Exchange KOSPI Index	5358	8.89	1.44	1.75
Kuwait	Kuwait Stock Exchange Weighted Index	1914	2.92	1.47	2.09
Latvia	OMX Vilnius Index	3660	3.79	1.46	1.89
Malaysia	FTSE Bursa Malaysia KLCI Index - Kuala Lumpur Composite Index	5108	3.68	1.39	2.31
Malta	Malta Stock Exchange	3203	3.38	1.26	2.84
Mauritius	Mauritius Stock Exchange SEMDEX Index	4747	2.85	1.35	2.85
Mexico	Mexican Bolsa Index	5211	4.12	1.53	1.25
Mongolia	Mongolia Stock Exchange Top 20 Index	3767	2.23	1.07	7.08
Morocco	Morocco Casablanca Stock Exchange CFG 25	5024	4.66	1.42	2.27
Namibia	FTSE JSE Namibia Overall Index	2670	3.56	1.71	1.13
Netherlands	AEX-Index	5286	3.31	1.56	1.42
New Zealand	MSCI New Zealand Index	5305	4.35	1.69	0.72
Nigeria	Nigerian Stock Exchange All Share Index	3993	4.37	1.39	2.14
Norway	Oslo Stock Exchange OBX Price Index	5205	4.06	1.67	1.22
Oman	Muscat Securities MSM 30 Index	4798	2.89	1.2	3.20
Pakistan	Karachi Stock Exchange KSE100 Index	4986	5.14	1.38	1.93
Palestine	Palestine Stock Exchange Al Quds Index	2370	4.05	1.17	4.07
Peru	Bolsa de Valores de Lima General Sector Index	5169	3.24	1.53	1.60
Philippines	Philippines Stock Exchange PSEi Index	5124	4.14	1.61	1.12
Poland	Warsaw Stock Exchange WIG Total Return Index	5104	3.69	1.46	1.57
Portugal	PSI All-Share Index Gross Return	5198	5.11	1.58	1.68
Qatar	Qatar Exchange Index	4050	2.81	1.17	4.52
Romania	Bucharest Stock Exchange Trading Index	4191	3.43	1.46	1.88
Russia	Russian Trading System Cash Index	4726	4.16	1.45	1.62
Saudi Arabia	Tadawul All Share TASI Index	5754	2.61	1.33	3.49
Singapore	Straits Times Index STI	3789	5.39	1.57	1.28
Slovenia	Ljubljana Stock Exchange Composite Index	4129	3.15	1.42	2.00
South Africa	FTSE/JSE Africa All Share Index	4808	4.08	1.64	0.98
Spain	IBEX 35 Index	5237	4.51	1.66	1.02
Sri Lanka	Sri Lanka Colombo Stock Exchange All Share Index	4946	4.2	1.42	2.01
Sweden	OMX Stockholm 30 Index	5205	4.08	1.61	1.06
Switzerland	Swiss Market Index	5214	4.68	1.63	1.06
Taiwan	Taiwan Stock Exchange Weighted Index	5352	6.14	1.51	1.39
Thailand	Stock Exchange of Thailand SET Index	5078	3.92	1.46	1.14
Tunisia	Tunisia Stock Exchange TUNINDEX	3471	3.42	1.61	1.32
Turkey	Borsa Istanbul 100 Index	5172	4.49	1.47	1.32
Ukraine	Ukraine PFTS Index	4100	3.03	1.35	2.54
United Arab Emirates	Dubai Financial Market General Index	2681	3.35	1.47	1.90
United Kingdom	FTSE All-Share Index	5237	3.94	1.7	1.31
United States	S&P 500 Index	5220	4.19	1.55	1.50
Vietnam	Vietnam Ho Chi Minh Stock Index / VN-Index	3345	23.21	1.46	1.98

Figure 2: Scatter plot between the tail index estimates and extreme risk measure.

This scatter plot shows the relationship between the tail risk index and extreme risk measures estimated from 74 country stock indexes. The kurtosis measure is plotted on a natural log scale.

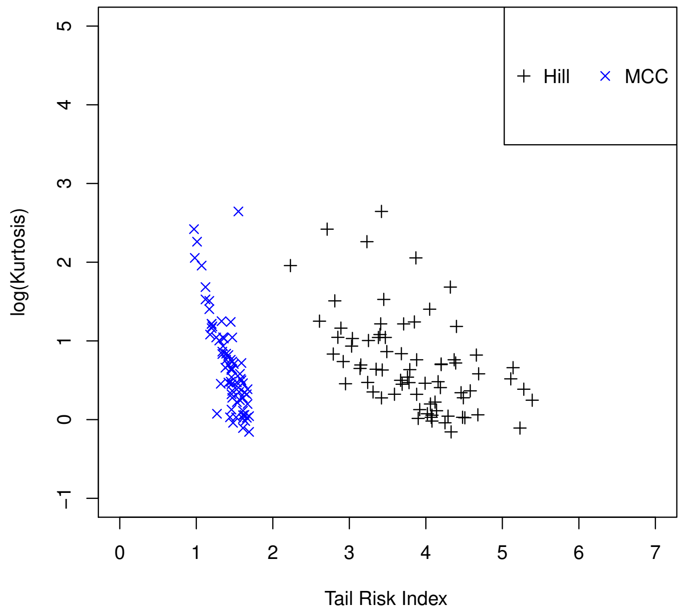
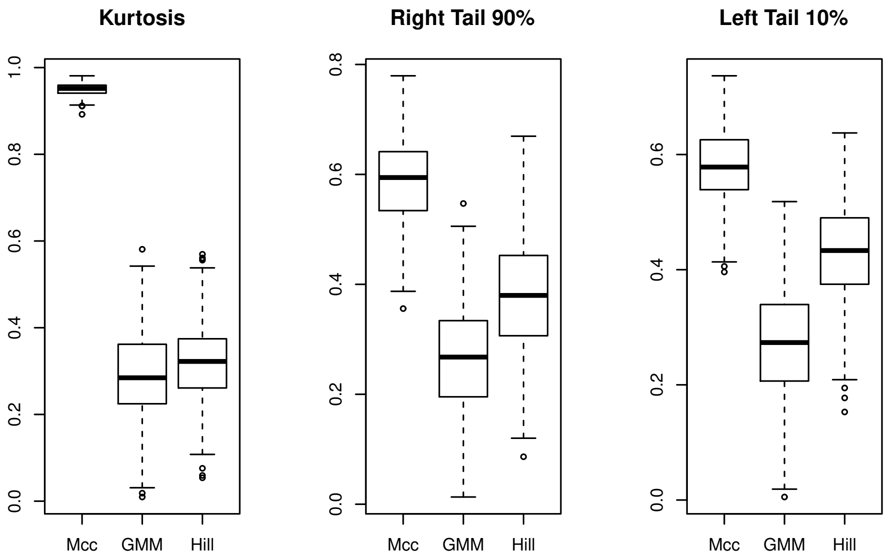


Figure 3: Boxplot of Spearman rank correlation coefficient for the bootstrapped stock return distributions

This set of graphs presents the boxplots of the Spearman rank correlation coefficient calculated between the tail indexes and extreme risk measures for the bootstrapped stock index return distributions. The bootstrap samples are replicated from 51 country stock index returns, with a block size of 6 days to account for time series dependencies. The number of replication is set to 500 in this study.



still shows a much better performance in tracking the different types of extreme risk measures observed in the different country's stock market returns. The McCulloch estimator is between 30% and 50% more precise than the Hill estimator for the tail risk measures and 170% more precise for the kurtosis.

4.4 Superior Predictive Ability Test for the Tail Index Estimators

To verify that the McCulloch estimation method produces the best comprehensive tail risk index, we use the superior predictive ability (SPA) test proposed by White (2000) and (Hansen 2005) to compare the three methods. The objective of the SPA test is to compare multiple models with a benchmark model, and determine if the alternative models outperform the benchmark model.

In our case, we set the benchmark model to be the McCulloch estimation method. As explained in the section 4, we use the Spearman's correlation coefficient $\rho_{j,k}$ calculated between the estimated tail index j and the extreme risk measures k to evaluate the performance of the tail index in reflecting the ranking of the extreme risk measure. If the tail index reflects perfectly the ranking of extreme risk measure, then $\rho_{j,k}$ will be equal to 1. We proceed to define the loss function of the tail index estimation methods as $L_j = (1 - |\rho_j|)^2$, where $j = (McCulloch, GMM, Hill)$.

Following White (2000), we define our null hypothesis that *the benchmark estimation method is not inferior to any alternative estimation methods*. Defining the vector of relative loss functions as $\mathbf{d} = L_0 - L$, where L_0 is the loss function of the benchmark McCulloch method, and $L \equiv [L_{GMM}, L_{Hill}]$ is the loss function of the alternative estimation methods (GMM and the Hill estimator respectively), we formulate the null hypothesis as

$$H_0 : E(\mathbf{d}) \leq 0 \qquad (9)$$

As the distribution of the vector of relative loss function $\mathbf{d}$ is unknown to us, it is not possible to deduce a critical value for the test statistic for the null hypothesis (Equation 9). To overcome this problem, we use the bootstrap methodology in section 4.3 to generate an extensive bootstrap sample of $\rho_{j,b}$ where b is the bootstrap sample, and use it to simulate the distribution of $\mathbf{d}$. This distribution is subsequently used to compute the significance (p-value) of our test statistic $\mathbf{d}^*$ which is a studentised value of $\mathbf{d}$.[4]

We run the SPA test with 500 bootstraps to compare the performance of the estimated tail indexes in tracking the actual kurtosis measured in the empirical data, with the McCulloch method as the benchmark. We also repeat the same tests on the tail indexes in tracking the left and right tails of the empirical data. The results are reported in Table 6 below:

Table 6: P-values of the superior predictive ability tests with McCulloch method as the benchmark method.

	Kurtosis	Left tail 10%	Right tail 10%
p-Value	0.82	0.52	0.54

The results in Table 6 is coherent with Figure 3. Irrespective of the extreme risk measures, we cannot reject the null hypothesis that the tail index estimated using the McCulloch method is not inferior to the tail index estimated through the GMM method nor the Hill estimator. We can see that the McCulloch estimated tail index outperforms the alternative estimators for the kurtosis with a p-value of 0.82. Even for tracking one-sided extreme risk measures, the McCulloch also outperforms the alternative methods with a p-value of 0.52 - 0.54.

5 Conclusion

The existing methods for measuring the *absolute* tail risk have not been conclusive in their findings. As an alternative, this paper proposes to estimate the *relative* tail risk by focusing only on the ranking of extreme risk measures. We compare the Hill, the McCulloch and the GMM tail estimators. We judge their performance only on their precision, independently of their possible bias. We apply this analysis to a data set of 74 international stock market returns. Of the three methods, we find that the McCulloch estimation method produces the most precise *relative* tail risk estimate, which tracks comprehensively the kurtosis, the left and the right tail measures of the stock market returns.

[4] Detail steps and description of the superior predictive ability test are presented in the Appendix.

Our finding is consistent across both the simulation study and the empirical study based on the bootstrap analysis of the 74 country stock index. In addition, the Hansen (2005) and White (2000) test for superior predictive ability establishes that the McCulloch estimator clearly outperforms the two other tail indexes. Given the results shown in this study, we propose to use the McCulloch estimator as a comprehensive measure for assessing relative country tail risks, in addition to common risk measures such as kurtosis and volatility (see Acemoglu et al. (2017)).

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