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A note on pollution and infectious disease

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Abstract

Within the present paper, we build a model from epidemiology and economics to study the impact of an infectious disease on the steady states and dynamic of an economy. More precisely, we embed a SIS model within a Ramsey growth model in a close framework with a tax where pollution comes from consumption. Firstly, we show that a consumption tax allocated to a depollution policy possesses an ambiguous effect on consumption and welfare, depending on the disease infectivity factor. Secondly, we point out that an increase in the spread of an infectious disease can't make a limit cycle (Hopf bifurcation) emerge near the endemic steady state.

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1 Introduction

In medical studies, air pollution is known to perturb the immune system which increases the spread of infectious diseases (Caren 1981; Bauer et al. 2012). This interplay between environmental pollutants and disease persistence is a major concern from an economic point of view since illness is recognized as one of the main sources of workers' absenteeism (Akazawa et al. 2003). Indeed, it is obvious that a lower labor supply is related to a lower growth potential but it can also be related to the possible emergence of economic cycles: a higher production level implies an increase of pollution which promotes the spread of infectious diseases and reduces labor supply and the production level in turn. That is, a higher production level is followed by a drop of production and so on: endogenous cycles arise.

The possible emergence of endogenous cycles due to the interplay between pollution and disease spread has been considered by Bosi and Desmarchelier (2018). These authors have embedded a disease spread model (SIS) into an exogenous growth model *à la Ramsey*. The SIS model consists in dividing the population in two classes: the succeptibles (S) and the infected (I). A succeptible agent becomes infected with a fixed probability due to contacts with infected individuals, while an infected agent recovers from the disease after a fixed amount of time. The connection between the SIS model and the Ramsey model is two sided. First, the labor supply only consists of healthy people and second, pollution flow, coming from production, promotes infectious diseases through its effects on both the probability to contract the disease and the recovery time(Caren 1981; Bauer and al. 2012).

The configuration to obtain endogenous cycles in Bosi and Desmarchlier (2018) is restrictive, and endogenous cylces appear only with a very strong income effect. Moreover, we observe that in their analysis, the channel for cycles to emerge is the capital accumulation: because of the strong income effect, the consumption decision implies a variation of the next period pollution flow through the capital accumulation.

In connection with the source of pollution, questions arise when pollution is generated by consumption rather than production: how does this change in the source of pollution affect the existence of steady-state and the dynamic system. Intuitively, in this case, the intertemporal linkage between the consumption choice and the pollution flow induced by the strong income effect in Bosi and Desmarchelier (2018) may vanish. This may cast some doubts about the possible occurrence of endogenous cycles since pollution comes from consumption rather than from production. Motivated by this intuition, this paper aims to investigate the interplay between pollution and disease by considering a SIS model into a growth model *à la Ramsey* where pollution is generated by consumption. We also add a tax in order to apply a pollution control policy. The effect of this policy on the household's, pollution and welfare is analyzed as well.

Our analyses, firstly show that a change in source of pollution has no effects on the existence neither on the number of steady-states. More precisely, we demonstrate the coexistence of two steady states: a disease-free one and an endemic one. However, the origin of pollution alters substantially the dynamics of the model by avoiding the possible occurrence of a limit cycle around the endemic steady state. In other words, considering the endemic steady state, we show that there is no room for a Hopf bifurcation which shades the conclusion obtained by Bosi and Desmarchelier (2018). Secondly, by adding a tax, we observe an ambiguious effect depending on the disease infectivity parameters coming from the SIS model. This means that based on the infectivity rate of the disease, the effect of the tax will be different, more precisely the more infectious the disease is, the more the effect of the tax will be positive. Those results shows that it would be difficult to generalize the Bosi and Desmarchelier (2018)'s analysis.

The rest of the paper is organized as follows. In Section 2, we present the model and discuss the steady state existence and the tax effect. Local dynamics are studied in Section 3. Section 4 concludes.

2 The model

Within this paper, we consider a SIS model into a growth model *à la Ramsey*. In an economy with a constant population, a representative firm produces a final good using capital and labor. The latter input is provided only by healthy people. A tax on consumption is applied and tax revenue is used to finance a depollution policy. Pollution may affect the share of healthy and unhealthy groups in the population and we assume that pollution flow comes from consumption.

2.1 Infectious disease

The disease spread is described by a simple SIS model. The population N consists of two classes: susceptible (S) and infective (I). That is, $N = S + I$. To keep the framework as simple as possible, there is no birth, no death, implying that the population remains constant over time, namely $\dot{N} = 0$. Let β be the share of susceptibles who contract the disease after a contact with infected individuals while γ represents the share of infected individuals who recover from the disease. We can interpret β as the disease probability and γ as the recovery probability. The evolution of both S and I is simply given by:

$$\dot{S} = -\beta \frac{I}{N} S + \gamma I$$
$$\dot{I} = \beta \frac{I}{N} S - \gamma I$$

We observe that the considered disease has two main characteristics. First, the disease is not lethal and second, recovery from the disease does not confer immunity.

Let $s \equiv S/N$ the share of healthy (susceptible) people and $i \equiv I/N$ the share of unhealthy (infective) people in the population, the time evolution of the share of healthy people is simply given by:

$$\dot{s} = (1 - s)(\gamma - \beta s)$$

Following Caren (1981) or Bauer *et al.* (2012), pollution (P) increases the infectivity rate β while it reduces the recovery rate γ in each period. The following assumption sums up properties of β and γ.

Assumption 1 $\beta(P) : \mathbb{R}_+ \to \mathbb{R}_+$, *is* C^2 *with* $\beta'(P) > 0$, $\lim_{P\to 0} \beta(P) = 0$ *and* $\lim_{P\to +\infty} \beta(P) = +\infty$. $\gamma(P) : \mathbb{R}_+ \to \mathbb{R}_+$ *is also* C^2 *with* $\gamma'(P) < 0$, $\lim_{P\to 0} \gamma(P) = +\infty$ *and* $\lim_{P\to +\infty} \gamma(P) = 0$.

For further references, we define the following first order elasticities of β and γ with respect to pollution P.

$$\varepsilon_\beta \equiv \frac{P\beta'(P)}{\beta(P)} > 0 \text{ and } \varepsilon_\gamma \equiv \frac{P\gamma'(P)}{\gamma(P)} < 0$$

As in Bosi and Desmarchelier (2018) and Goenka et al. (2014), the labour supply (L) only consists of healthy people, that is $L = S$. It follows that the labour supply inherits the dynamics of the susceptible class. That is, the share of labour supply $l \equiv L/N$ evolves according to the following simple equation:

$$\dot{l} = (1-l)(\gamma - \beta l) \tag{1}$$

For a fixed level of pollution, we observe that equation (1) possesses two steady states: $l = 1$ and $l^* \equiv \gamma/\beta < 1$. The first one is a disease-free steady state while the disease persists at l^*. This latter is an endemic steady state.

2.2 Governement

The government implements a depollution policy. It uses all the tax revenues (τC) to finance depollution expenditures (G) according to a balanced budget rule :

$$G = \tau C \tag{2}$$

As we can see, depollution spending depends on a consumption tax.

2.3 Pollution

Within this paper, as in Bosi and Desmarchelier (2018), we assume that pollution is a flow. One of the main differences with their paper rests on the fact that pollution is assumed to come from consumption rather than from production. Another difference is the addition of a government which applies a depollution policy financed by a consumption tax. This allows us to examine the interplay between environmental aspect, infectious diseases, and public action.

The flow of pollution is written as below:

$$P = aC - bG \tag{3}$$

Where $a > 0$ captures the environmental impact of consumption C and $C = cN$ where c denotes the consumption per capita. Note that $b > 0$ captures the efficacity of depollution policy (G).

In that case we can rewrite equation (3) as follow :

$$P = (a - \tau b)C \tag{4}$$

Assumption 2 $\tau \leq a/b$

Assumption 2 ensures that $P \geq 0$.

2.4 The household

The household earns a capital income rh and an income ω, where r and h represent respectively the real interest rate and the individual wealth at time t. As said before the household is subject to a consumption tax τ. As usual, income is consumed and saved/invested according to the budget constraint:

$$\dot{h} \leq (r-\delta)h + \omega - (1+\tau)c \tag{5}$$

where δ represents the capital depreciation rate.

As in Bosi and Desmarchelier (2018), since only healthy people can work, a social security system is introduced such that healthy and ill agents earn the same income ω. That is, $\omega = wl$, where w represents the wage rate. It follows simply that $\omega N = wL$. For the sake of simplicity, we normalize the population to the unity ($N = 1$) which implies that aggregate demand for labor L is equal to l (i.e. $L = Nl = l$). If we note K as aggregate demand for capital, we will get $K = Nh = h$ and a link between capital supply from household (h) and firm demand for capital kl (i.e. $h = K/N = kl$).

As in a usual Ramsey model, the household's utility depends on the consumption level. Properties of this utility function are summed up in the following assumption.

Assumption 3 *Preferences are rationalized by a C^2 utility function $u : \mathbb{R}_+ \to \mathbb{R}_+$ with $u'(c) > 0$, $u''(c) < 0$, $\lim_{c\to 0} u'(c) = +\infty$ and $\lim_{c\to +\infty} u'(c) = 0$.*

The agent maximizes the intertemporal utility function $\int_0^\infty e^{-\rho t} u(c)\, dt$ under the budget constraint (5), where $\rho > 0$ is the rate of time preference. First order conditions give the following dynamic relations:

$$\dot{h} = (r-\delta)h + wl - (1+\tau)c \tag{6}$$

$$\dot{c} = \varepsilon (r-\delta-\rho) c \tag{7}$$

where $\varepsilon(c) \equiv -u'(c)/[cu''(c)] > 0$ is the elasticity of intertemporal substitution in consumption, jointly with the transversality condition $\lim_{t\to+\infty} e^{-\rho t} u'(c) h = 0$.

We notice that the intertemporal elasticity of substitution in consumption is a key element in Bosi and Desmarchelier (2018) since they have shown that a limit cycle can occur around the endemic steady-state l^* when $\varepsilon \to 0$. This conclusion may not hold when we consider pollution coming from consumption rather from production.

2.5 Firm

A representative firm maximizes the profit $F(K,L) - rK - wL$ taking as given prices (r and w). $Y \equiv F(K,L)$ is the aggregate production function while K and L represent respectively the aggregate demand for capital and labor. F possesses usual properties for a neoclassical production function as mentionned in Assumption 4.

Assumption 4 *The production function $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ is C^1, constant returns to scale, strictly increasing in both the arguments and concave. Standard Inada conditions hold.*

First order conditions for the profit maximization imply that the amount of capital and labor used to produce are such that marginal productivity of capital is equal to the real interest rate while marginal productivity of labor is equal to the wage rate:

$$r = f'(k) \equiv r(k) \text{ and } w = f(k) - kf'(k) \equiv w(k) \tag{8}$$

where $k \equiv K/L$ denotes the capital intensity and $f(k) \equiv F(k,1)$ the average productivity. For further references, we set α and σ respectively the capital share in total income and the elasticity of capital-labor substitution, namely:

$$\alpha \equiv \frac{kf'(k)}{f(k)} \text{ and } \sigma = \alpha \frac{w(k)}{kw'(k)}$$

2.6 Equilibrium and steady state

Since $N = 1$, considering Assumption 1 jointly with equation (3), equation (1) is rewritten as:

$$\dot{l} = (1-l)\left[\gamma((a-b\tau)c) - \beta\left((a-b\tau)c\right)l\right] \tag{9}$$

Considering (6),(7), (8) and (9) leads to the following three-dimensions dynamical system [1]

$$\dot{k} = f_1(k,l,c) = f(k) - \delta k - \frac{(1+\tau)c}{l} - g(c,l)k \tag{10}$$

$$\dot{l} = f_2(k,l,c) = g(c,l)l \tag{11}$$

$$\dot{c} = f_3(k,l,c) = \varepsilon(r(k) - \delta - \rho)c \tag{12}$$

such that $g(c,l) \equiv \dot{l}/l = (1-l)\left(\gamma((a-b\tau)c) - \beta\left((a-b\tau)c\right)l\right)/l$. A steady state for this economy consists in a triplet (k,l,c) such that $\dot{k} = \dot{l} = \dot{c} = 0$. Equation (12) gives the capital intensity k^* at the steady state:

$$k^* = r^{-1}(\rho + \delta) > 0 \tag{13}$$

Since, at the steady state, $g(c,l) = 0$, it follows from equation (10) that:

$$c^* = c(l) = [f(k^*) - \delta k^*]\frac{l}{1+\tau} \tag{14}$$

Where l can take two values $l = 1$ or $l = \gamma/\beta$. That is, for a given l , there is a unique positive consumption level at the steady state. Injecting (14) into (11) gives:

$$g(c(l),l) = (1-l)\phi(l) = 0 \tag{15}$$

with $\phi(l) \equiv [\gamma((a-b\tau)c(l)) - \beta((a-b\tau)c(l))l]/l$.

Proposition 1 *Let Assumptions 1, 2 and 4 hold:*

- If $\beta \geq \gamma$, there are two steady states, namely $(k,l,c) = (k^*,1,c^*)$ (disease-free) and $(k,l,c) = (k^*,l^*,c^*)$ (endemic steady state).

- If $\beta < \gamma$, there is a unique steady state, namely $(k,l,c) = (k^*,1,c^*)$ (disease-free).

Proof. See the Appendix. ■

We are now going to examine the effects of the tax at the steady state for each variables and following this, we also study its effects on household's welfare and pollution.

Proposition 2 *The effect of the tax rate τ on the steady state is given by:*

[1] Remark that, $h = kl$ as discussed before

$$\frac{\tau}{k^*}\frac{\partial k^*}{\partial \tau} = 0$$
$$\frac{\tau}{l^*}\frac{\partial l^*}{\partial \tau} = \frac{\tau}{1+\tau}\frac{\varepsilon_\beta - \varepsilon_\gamma}{1+\varepsilon_\beta - \varepsilon_\gamma}\frac{a+b}{a-b\tau} > 0$$
$$\frac{\tau}{c^*}\frac{\partial c^*}{\partial \tau} = -\frac{\tau}{1+\tau}\frac{a-b\tau-b\left(\varepsilon_\beta - \varepsilon_\gamma\right)(1+\tau)}{(1+\varepsilon_\beta - \varepsilon_\gamma)(a-b\tau)} \lesseqgtr 0 \tag{16}$$

Proof. See the Appendix. ■

Notice that $\varepsilon_\beta - \varepsilon_\gamma > 0$ (assumption 1) and $a - b\tau > 0$ (assumption 2) it follows that : $\frac{\tau}{l}\frac{dl}{d\tau} > 0$ and $\frac{\tau}{c}\frac{dc}{d\tau} \lesseqgtr 0$.

To propose a clear-cut interpretation, let:

$$\varepsilon_l \equiv \frac{Pl'(P)}{l} = -\left(\varepsilon_\beta - \varepsilon_\gamma\right) \text{ and } \pi \equiv \frac{a-b\tau}{b(1+\tau)}$$

From the previous proposition, we observe that:

$$\frac{\tau}{c^*}\frac{\partial c^*}{\partial \tau} > 0 \text{ if } |\varepsilon_l| > \pi \tag{17}$$
$$\frac{\tau}{c^*}\frac{\partial c^*}{\partial \tau} < 0 \text{ if } |\varepsilon_l| < \pi$$

This result can be simply explained. A higher tax rate has two opposite effects on consumption :

(1) Since τ is levied on the consumption level, a higher tax rate gives incentives to reduce consumption.

(2) A higher tax rate is also related to a higher level of depollution expenditure which tends to reduce the pollution level and increase labour supply (Assumption 1), increasing labour income and the consumption possibilities in turn.

If the pollution effect on labour supply is moderate ($|\varepsilon_l| < \pi$), effect (1) dominates effect (2) implying that a higher tax rate always lower the consumption level in the long run. Conversely, if the pollution effect on labour supply becomes excessive ($|\varepsilon_l| > \pi$), then effect (2) dominates effect (1), in this case a higher tax rate has a positive effect on the consumption level in the long run.

Let W be the welfare function at the steady state:

$$W(\tau) \equiv \int_0^{+\infty} e^{-\rho t} u(c^*(\tau))\, dt = \frac{u(c^*(\tau))}{\rho}$$

Since $u'(c) > 0$ (Assumption 2), the effect of τ on welfare is exactly the same than the one on consumption, interpretations are analogous as well: the tax rate is wellfare improving if and only if the labour supply is very sensitive to pollution ($|\varepsilon_l| > \pi$).

Focusing on (16), it follows that, at the endemic steady state:

$$\frac{\tau P^{*\prime}(\tau)}{P^{*}(\tau)} = -\frac{\tau}{1+\tau}\frac{a+b+b\tau\left(\varepsilon_{\beta}-\varepsilon_{\gamma}\right)}{\left(1+\varepsilon_{\beta}-\varepsilon_{\gamma}\right)(a-b\tau)} < 0 \tag{18}$$

Since Sinn (2008), it is well-known that a higher green tax can speed up the global warming in a simple Hotelling framework (1931). Such a counter intuitive relation between the green tax and the environment was also recovered at the steady state of a Ramsey model when a Laffer Curve occurs (Bosi and Desmarchelier, 2017). Considering (18), it appears that there is no room for a Green Paradox here because a higher green tax always lowers the pollution level in the long run.

3 Local dynamics

Let us now consider the dynamics of the economy. To do this, we linearize the dynamic system (10)-(12) at the endemic steady state when $(k,c,l) = (k^{*},c^{*},l^{*})$.We choose the endemic state because it is around this one that Bosi and Desmarchelier (2018) found a Hopf bifurcation. The Jacobian matrix is given by:

$$J \equiv \begin{bmatrix} \frac{\partial f_1}{\partial k} & \frac{\partial f_1}{\partial l} & \frac{\partial f_1}{\partial c} \\ \frac{\partial f_2}{\partial k} & \frac{\partial f_2}{\partial l} & \frac{\partial f_2}{\partial c} \\ \frac{\partial f_3}{\partial k} & \frac{\partial f_3}{\partial l} & \frac{\partial f_3}{\partial c} \end{bmatrix} = \begin{bmatrix} \rho & \frac{\beta}{\gamma}k\left(\frac{\rho+(1-\alpha)\delta}{\alpha}+\eta\right) & \frac{\beta(1+\tau)}{\gamma}\left(\frac{\alpha\eta\theta}{\rho+(1-\alpha)\delta}-1\right) \\ 0 & -\eta & -\frac{\eta\theta}{k}\frac{\alpha(1+\tau)}{\rho+(1-\alpha)\delta} \\ \varepsilon\frac{\rho+(1-\alpha)\delta}{\alpha(1+\tau)}\frac{(\alpha-1)(\rho+\delta)}{\sigma}\frac{\gamma}{\beta} & 0 & 0 \end{bmatrix}$$

With $\eta \equiv \beta-\gamma > 0$ (Since, at the endemic state $l^{*} = \gamma/\beta < 1$), and $\theta = -\varepsilon_{l} = \varepsilon_{\beta}-\varepsilon_{\gamma} > 0$. We found that the determinant (D), the sum of minors of order two (S) and the trace (T) are equal to :

$$T = \rho-\eta \tag{19}$$

$$S = \varepsilon\frac{(1-\alpha)(\delta+\rho)(\rho+\delta(1-\alpha))}{\alpha\sigma}\left(\theta\alpha\frac{\eta}{\rho-\delta(\alpha-1)}-1\right)-\eta\rho \tag{20}$$

$$D = \varepsilon\eta(\delta+\rho)(1-\alpha)(\rho+(1-\alpha)\delta)\frac{(1+\theta)}{\alpha\sigma} > 0 \tag{21}$$

Proposition 3 *There is no room for a Hopf bifurcation.*

Proof. See the Appendix ■

Bosi and Desmarchelier (2018) have pointed out the destabilizing role of the interplay between pollution generated by production and disease spread. Their main result showed a possible occurrence of a limit cycle near the endemic steady state through a Hopf bifurcation when the household's preferences exhibit a very strong income effect ($\varepsilon \to 0$). This result does not hold in our framework. It follows that our intuition, exposed in introduction, is verified: the intertemporal linkage between consumption and pollution induced by a very weak intertemporal elasticity of substitution in consumption, which is at the origin of the occurrence of the Hopf bifurcation in Bosi and Desmarchelier (2018), vanishes when pollution comes from consumption.

4 Conclusion

Our paper reconsiders the framework proposed by Bosi and Desmarchelier (2018) by embedding a SIS model into a Ramsey framework with a government and where a pollution externality affects both the probability to contract the disease and the recovery rate. Considering that the pollution flow is coming from consumption instead of production, we have shown that the possible occurrence of a limit cycle near the endemic steady state, as the main result in Bosi and Desmarchelier (2018), vanishes. This conclusion casts some doubts about the robustness of the limit cycle pointed out by Bosi and Desmarchelier (2018). Moreover, we have also introduced a pollution tax. We have pointed out that this green policy is welfare improving if and only if the labor supply is sufficiently sensitive to pollution and that a green paradox can't appears.

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Appendix

Proof of Proposition 1

Notice that the existence and the number of steady state for this economy depends on the number of $l \in [0,1]$ satisfying equation (15). We observe that (15) is verified when $l = 1$ and/or when $l = l^* = \gamma/\beta$. This later case is obtained when $\phi(l) = 0$. Considering Assumption 1, we observe that:

$$\phi'(l) = \varepsilon_\gamma \frac{\gamma}{l} - \beta\left(1 + \varepsilon_\beta\right) < 0$$

and:

$$\lim_{l \to 0} \phi(l) = +\infty \text{ and } \lim_{l \to +\infty} \phi(l) = -\infty$$

That is, there exists a unique $l > 0$, given by l^*, such that $\phi(l) = 0$. If $l^* > 1$, the disease-free steady state (given by $l = 1$) is unique while if $l^* < 1$, two steady state coexist, the disease free one (given by $l = 1$) and the endemic one, given by l^*. In this case, the infectivity rate β is higher or equal to the recovery rate γ. ■

Proof of Proposition 2

From (13), it follows that:

$$\frac{\tau}{k^*}\frac{\partial k^*}{\partial \tau} = 0 \tag{22}$$

At the endemic steady state, (10) and (11) give respectively:

$$f(k) - \delta k - \frac{(1+\tau)c}{l} = 0 \tag{23}$$

$$(\gamma(P) - \beta(P)\,l) = 0 \tag{24}$$

With $P = P(c,\tau) \equiv (a - b\tau)c$.

Let us differentiate (23) and (24) with respect to c, l and τ:

$$-\frac{(1+\tau)}{l}dc + \frac{(1+\tau)c}{l^2}dl - \frac{c}{l}d\tau = 0$$

$$(a - b\tau)\left[\gamma'(P) - \beta'(P)\,l\right]dc - \beta dl + \left[bc\beta'(P)\,l - bc\gamma'(P)\right]d\tau = 0$$

Since, at the endemic steady state, $l = \gamma/\beta$, it follows:

$$-\frac{dc}{c} + \frac{dl}{l} - \frac{\tau}{1+\tau}\frac{d\tau}{\tau} = 0$$

$$\frac{dc}{c} - \frac{1}{\varepsilon_\gamma - \varepsilon_\beta}\frac{dl}{l} - \frac{\tau b}{(a - b\tau)}\frac{d\tau}{\tau} = 0$$

That is,

$$\begin{bmatrix} \frac{\tau}{c}\frac{dc}{d\tau} \\ \frac{\tau}{l}\frac{dl}{d\tau} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -\frac{1}{\varepsilon_\gamma - \varepsilon_\beta} \end{bmatrix}^{-1} * \begin{bmatrix} \frac{\tau}{1+\tau} \\ \frac{\tau b}{a - b\tau} \end{bmatrix}$$

■

Proof of Proposition 3

Following Bosi and Desmarchelier (2019b), a Hopf bifurcation occurs if and only if $D = ST$ such that $S > 0$. Considering (21), $D > 0$, that is, a necessary, but not sufficient, condition for a Hopf bifurcation to occur is that $T > 0$ and then $\rho > \eta$. Focusing on (19), (20) and (21), $D = ST$ if and only if $\varepsilon = \varepsilon^H$ such that:

$$\varepsilon^H \equiv -\frac{\alpha\sigma\eta\rho\,(\rho-\eta)}{(\delta+\rho)\,(1-\alpha)\,(\theta\alpha\eta^2+\theta\eta\,(\delta+\rho)\,(1-\alpha)+\rho\,(\rho+(1-\alpha)\,\delta))}$$

However, since $\rho > \eta$, $\varepsilon^H < 0$, a contradiction appears since, by definition, $\varepsilon > 0$. That is, any Hopf bifurcation is ruled out. ■