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A Choice of an Index under Inefficiency: Tornqvist or Fisher Index? Evidence from Simulation

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Abstract

This study addresses a choice of quantity index when inefficiency exists. Performing simulations, we show that 1) the finding from the previous literature that the Fisher index and the Tornqvist index approximate each other is only a special case when inefficiency is negligible, 2) the Tornqvist index begins to deviate as inefficiency increases because the Tornqvist index uses the average cost shares as a weight in the calculation, and 3) the Tornqvist index eventually suffers from the Simpson's paradox, in which the average of two higher numbers are lower than the average of two lower numbers, when the factor substitutability is large. From these results, we conclude that the Fisher index is preferred to the Tornqvist index when any inefficient use of factors is suspected.

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1. Introduction

An important issue has been how an index is constructed to reflect accurately the economic phenomena of interest. To this end, various index formulae, e.g., Fisher index or Tornqvist index, have been proposed and widely used. To find a suitable index number, a number of desirable axioms or tests have been suggested and discussed by a number of economists, notably Fisher (1922), Eichhorn (1978) and Diewert (1993). However, as Snyder (1923) and Balk (2008) noted, no perfect index number is attainable.

While no perfect index is attainable, economic theory has been brought to bear, although implicitly, in the hope of improving the choice among indices. This economic approach assumes that the data observed are the result of optimizing behavior. For a price index, Konus (1924) proposed a price index as a ratio of two cost functions. For a quantity index, Malmquist (1953) proposed the ratio of two distance functions that represent the behavior of cost minimization or of revenue maximization. However, the Malmquist index, which is only theoretical because any functional form of the distance function is not specified, was not operational.

A way to operate the theoretical index is to use an index formula which approximates the theoretical index. An index is called exact when the index approximates the theoretical index. Further, an exact index is called superlative when the flexible functional form, e.g., translog, is specified for a distance function. Caves et al. (1982) showed that the Tornqvist index is superlative under certain conditions, and Diewert (1992) showed that the Fisher index is also superlative under the same conditions. The conditions, required for making them superlative, are i) the benchmark technology must be constant returns to scale, ii) the distance function must have identical second-order coefficients in the flexible functional form, and iii) production must be efficient in the allocation of resources. The first two conditions may not be too restrictive, but the last condition is restrictive as resources are often inefficiently allocated for many reasons. A failure to meet the last condition will be serious because it precludes the behavior of economic optimization (Fried et al., 2008). Therefore, there is no guarantee that an index formula, the Fisher index or the Tornqvist index, approximates the unobservable theoretical index when resources are not efficiently allocated in production.

Unfortunately, no study has investigated the degree to which a superlative index is compromised when allocative inefficiency exists. Therefore, we will investigate this issue and derive an implication for the choice of an index. For a choice of an index, Diewert (1978) showed that the Fisher and Tornqvist indexes numerically approximate each other, but Dumagan (2002) advocated the Tornqvist index because it requires less data. Yet, both studies are based upon the assumption that the behavior of economic optimization is achieved and thus there is no inefficiency in the use of resources.

Our investigation of how the Fisher and the Tornqvist indexes are compromised under inefficiency will be based upon simulations as the evaluation of an index is essentially numerical, not analytical. To this end, we perform simulations in a way that the degree of inefficiency is controlled.

2. Design of the Simulation

We designed the simulations to evaluate how a superlative index is compromised as the degree of inefficiency changes. The degree of inefficiency is defined as the ratio of an actual cost to a minimum cost;

$$Degree\ of\ Inefficiency\ = \frac{C_A}{C_M} \tag{1}$$

where C_A is an actual cost and C_M is a minimum cost. The actual cost is the cost achieved when resources are inefficiently allocated while the minimum cost is the cost achieved when resources are efficiently achieved. Obviously, the actual cost is higher than the minimum cost and the difference between the actual cost and the minimum cost is due to the misallocation of resources.

To generate the data for the evaluation of an index, we assume that a firm operates in a competitive market and further assume that a firm's production technology follows constant elasticity of substitution (CES) technology in which the percentage change in input ratio does not respond to a percentage change in factor prices. As the concept of the elasticity of substitution is not dependent on the behavior of cost minimization (Silberberg, 1990), we can generate the data on input factors by controlling the degree of inefficiency. That is, when we set the elasticity of substitution at a certain level, input factors are derived by the factor prices at the preset degree of the elasticity of substitution when input factors are efficiently allocated, i.e., when minimum cost is achieved. With the cost-minimizing input factors in hand, we can variate the value of cost-minimizing factors that reflect inefficient use of resources. By computing costs with these two sets of input factors, i.e., one for efficient use and the other for inefficient use, we can calculate the degree of inefficiency.

Consider a CES technology for the case of two inputs, x_1 and x_2, for simplicity. That is,

$$y = \left(\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho}\right)^{\frac{1}{\rho}} \tag{2a}$$

where the sum of distributive parameters is unity ($\alpha_1 + \alpha_2 = 1$), σ denotes the elasticity of substitution and thus $\rho = 1 - \frac{1}{\sigma}$. The cost-minimizing inputs x and minimum cost C_M in a competitive market where input prices are exogenously determined are obtained as below;

$$x_j = y\left(\alpha_1^{\frac{1}{1-\rho}} w_1^{\frac{-\rho}{1-\rho}} + \alpha_2^{\frac{1}{1-\rho}} w_2^{\frac{-\rho}{1-\rho}}\right)^{-\frac{1}{\rho}} \alpha_j^{\frac{1}{1-\rho}} w_j^{\frac{-\rho}{1-\rho}} \quad \text{for } j=1, 2, \tag{2b}$$

$$C_M = y\left(\alpha_1^{\frac{1}{1-\rho}} w_1^{\frac{-\rho}{1-\rho}} + \alpha_2^{\frac{1}{1-\rho}} w_2^{\frac{-\rho}{1-\rho}}\right)^{-\frac{1-\rho}{\rho}} \tag{2c}$$

Since x and C_M are affected only by ρ (thus input substitutability σ) and exogenously given factor prices (w), we generate x and C_M by controlling input substitutability σ.

Now consider generating input factors ($\hat{x}$) that are inefficiently allocated. For a simple two-factor production, we only need to control one factor (e.g., $\hat{x}_1$) as the other factor (e.g., $\hat{x}_2$) is determined as a residual. That is, the corresponding factor, $\hat{x}_2$, to the factor, $\hat{x}_1$, that we control is

$$\hat{x}_2 = [(y^{\rho} - \alpha_1(\hat{x}_1)^{\rho})/\alpha_2]^{\frac{1}{\rho}} \tag{2d}$$

With more than three inputs, a shadow price, an arbitrary price deviated from actual prices at which a firm tries to achieve its cost minimization, must be used to generate all input factors as inferred by (2b). Once two factors are generated, then it is trivial to calculate the actual cost that is higher than the minimum cost.

$$C_A = w_1\hat{x}_1 + w_2\hat{x}_2. \quad (2e)$$

Using (2c) and (2e), the degree of allocative inefficiency is calculated, given the input substitutability σ. If we control input substitutability, then we obtain different sets of input factors (x and $\hat{x}$), costs (C_M and C_A) and the degree of inefficiency.

With the data obtained above, we are ready to calculate two popular bilateral quantity indexes for two periods (*t* and *t*+1) the Fisher index and Tornqvist index. Because we investigate how an index changes as the degree of inefficiency changes, we assume that productive inefficiency exists in one period, e.g., at *t*+1. That is, the input factors at time *t* are cost-minimizing whereas the input factors at *t*+1 are inefficient. Assuming that productive inefficiency exists at time *t* does not matter as the index is only bilateral.

Because $\hat{x}$ denotes an inefficient input factor, whereas x denotes an efficient input factor, the Tornqvist quantity index (Q_T) is now calculated as

$$Q_T = \left(\frac{\hat{x}_1^{t+1}}{x_1^t}\right)^{\frac{S_1^t+\hat{S}_1^{t+1}}{2}} \cdot \left(\frac{\hat{x}_2^{t+1}}{x_2^t}\right)^{\frac{S_2^t+\hat{S}_2^{t+1}}{2}} \quad (3a)$$

where S_1^t and S_2^t are the cost shares of factor 1 and factor 2 at time *t*, respectively. $\hat{S}_1^{t+1}$ and $\hat{S}_2^{t+1}$ are the cost shares of factors 1 and 2 at time *t*+1, respectively, when factors are inefficiently used. When we take the log of (3a), we have the log of the Tornqvist index,

$$log(Q_T) = \left(\frac{S_1^t+\hat{S}_1^{t+1}}{2}\right) log\left(\frac{\hat{x}_1^{t+1}}{x_1^t}\right) + \left(\frac{S_2^t+\hat{S}_2^{t+1}}{2}\right) log\left(\frac{\hat{x}_2^{t+1}}{x_2^t}\right), \quad (3b)$$

which is calculated by the weighted average of the growth rates of all factors where an average cost share of each factor is used as a weight. This implies that the index is more weighted by a factor if the cost share of the factor is larger. Hence, when a weighted cost share of a factor, inflated by the inefficient use of an input factor, overwhelms the change in input ratio, the impact of inefficiency is more reflected in the index. In particular, the impact of inefficient use of a factor on the index is larger when a use of a factor over the other factor is more easily made in the production (e.g., where the input substitutability is greater). If the impact is particularly large, the index may suffer from the Simpson's paradox, in which the average of two higher numbers are lower than the average of two lower numbers.[1] We will present a numerical example in the next section.

The Fisher quantity index (Q_F) is calculated as the geometric mean of a Laspeyres quantity index (Q_L) and a Paasche index (Q_P);

$$Q_F = (Q_L \cdot Q_P)^{\frac{1}{2}} \quad (3c)$$

[1] Simpson's paradox is a statistical paradox in which the result of a group is reversed when data from the groups are combined. Suppose that Player *A* has higher batting averages than Player *B* for two consecutive years. It is possible that Player *A*'s weighted two-year batting average becomes lower than Player *B*'s (Blyth, 1972).

where $Q_L = \frac{w_1^t \hat{x}_1^{t+1} + w_2^t \hat{x}_2^{t+1}}{w_1^t x_1^t + w_2^t x_2^t}$ and $Q_P = \frac{w_1^{t+1} \hat{x}_1^{t+1} + w_2^{t+1} \hat{x}_2^{t+1}}{w_1^{t+1} x_1^t + w_2^{t+1} x_2^t}$. The log of the Fisher index becomes the simple mean of the log of the Laspeyres index and the log of the Paasche index;

$$log(Q_F) = \frac{1}{2}[log(Q_L) + log(Q_P)]. \quad (3d)$$

From (3c) and (3d), it is interesting to note that the Laspeyres index and the Paasche index are identical as long as the factor prices are invariant over time no matter how inefficiently factors are used. Following Reinsdorf et al. (2002, p58), the Laspeyres index and the Paasche index can be written as the function of input growth and cost shares,

$$log(\,Q_L) = \sum_{i=1}^{2} \frac{S_i^t m(r_i, Q_L)}{S_1^t m(r_1, Q_L) + S_2^t m(r_2, Q_L)} log(\,r_i) \quad (3e)$$

and

$$log(\,Q_P) = \sum_{i=1}^{2} \frac{\hat{S}_i^{t+1} m\left(\frac{1}{r_i}, \frac{1}{Q_P}\right)}{\hat{S}_1^{t+1} m\left(\frac{1}{r_1}, \frac{1}{Q_P}\right) + \hat{S}_2^{t+1} m\left(\frac{1}{r_2}, \frac{1}{Q_P}\right)} log(\,r_i) \quad (3f)$$

where $r_i = \frac{x_i^{t+1}}{x_i^t}$ and m is the logarithmic mean function which indicates $m(a, b) = \frac{(a-b)}{(log\, a - log\, b)}$. (3e) and (3f) indicates that the logarithmic values of Laspeyres index and the Paasche index can be viewed as the weighted average of the input growths weighted by the ratio of the cost shares that are adjusted by the logarithmic mean function.

3. Simulation Results

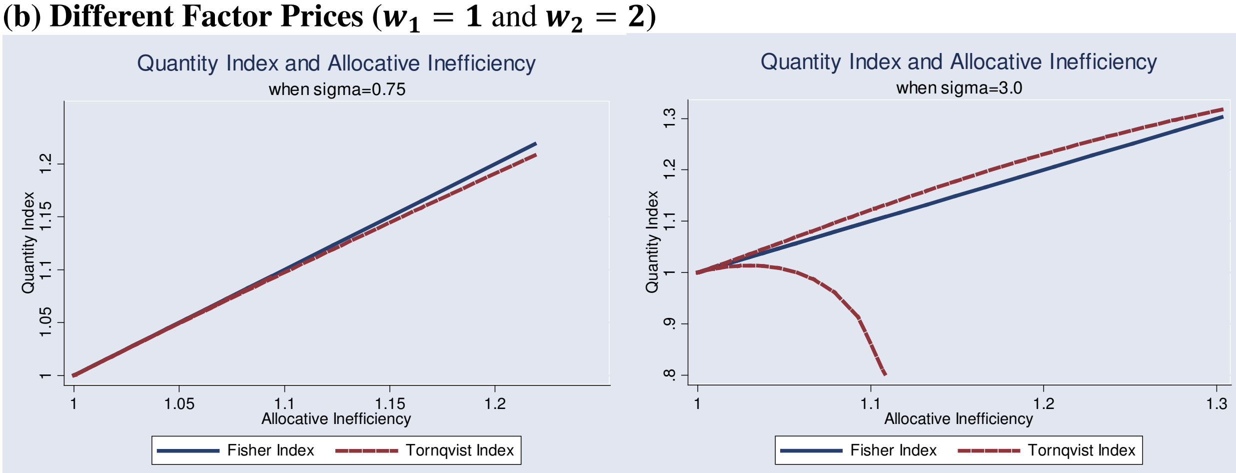
In the simulations, we consider two cases, i.e., same factor prices (e.g., $w_1 = w_2 = 1$) and different factor prices (e.g., $w_1 = 1$ and $w_2 = 2$), and set $\alpha_1 = \alpha_2 = 0.5$ and $y = 1$ for simplicity. Under these values, the input factors and the minimum cost (C_M) are derived using the equations (2b) and (2c) as long as the factor substitutability is arbitrarily given. We consider two different sets of factor substitutability: lower substitutability (e.g., $\sigma = 0.75$) and higher substitutability (e.g., $\sigma = 3$). As our objective is to evaluate the indexes under inefficiency, we need to consider input factors that are not efficiently used in the production process. For this purpose, we arbitrarily set the value of one factor $\hat{x}_1$ (and thus the other factor $\hat{x}_2$ is automatically derived from the equation (2d) in the two-factor production). To generate the value of the factor $\hat{x}_1$, we use the uniform distribution, the interval of which ranges from 0.5 to 1.5.

Following the equations (3a) and (3c), the Tornqvist index and the Fisher index are calculated once two sets of factors, cost-minimizing factors x_1 and x_2 and inefficiently used factors $\hat{x}_1$ and $\hat{x}_2$, are generated. We assume that the cost-minimizing factors are for time *t* and that the inefficiently used factors are for time *t*+1. As the actual cost (C_A) in the equation (2e) is calculated with $\hat{x}_1$ and $\hat{x}_2$, we calculate the degree of inefficiency in the equation (1). Then, we can see how two indexes change as the degree of inefficiency changes in the different level of factor substitutability. Figure 1 summaries the simulation results.

For the case of the same factor prices (see the upper panel), both the Fisher index and the Tornqvist index approximate the degree of inefficiency (and thus the quantity of factors used) pretty well. However, as the degree of inefficiency is larger, the Tornqvist index begins to deviate from the Fisher index that exactly approximates inefficiency. By comparing the left and the right panels, we see that the deviation of the Tornqvist index amplifies as the factor substitutability becomes larger. For the case of the different factor prices (see the lower panel), both the Fisher index and the Tornqvist index approximate the inefficiency well when the factor substitutability is low. However, the Tornqvist index shows dramatic deviations by which it understates and overstates the degree of inefficiency when the factor substitutability is large (σ =3). These deviations are due to the use of the average cost shares as a weight in its calculation. This problem is illustrated as the Simpson's paradox.

Figure 1: Quantity Index and Degree of Inefficiency
(a) Same Factor Prices ($w_1 = w_2 = 1$)

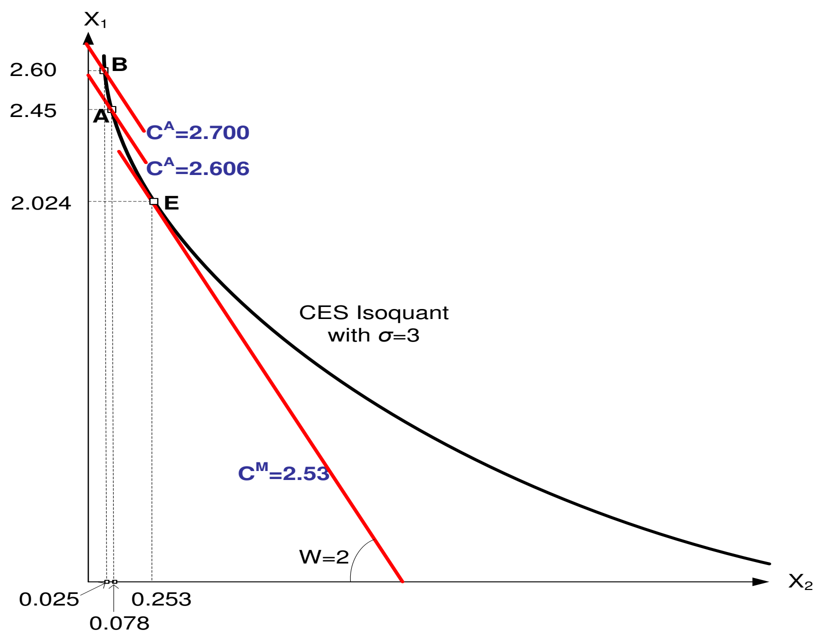
(b) Different Factor Prices ($w_1 = 1$ and $w_2 = 2$)



The Simpson's paradox encountered in the simulations is depicted in Figure 2 in which a firm's use of resources is hypothesized in a way that the firm uses input factors efficiently at time *t*, but inefficiently at time *t*+1. Under a CES production technology with $\sigma = 3$, the point *E* shows the efficient use of factors at time *t* whereas the points *A* and *B* show the inefficient use of factors at time *t*+1. Among the points *A* and *B*, the use of factors at the point *B* is more inefficient than the use of factors at the point *A* as the actual cost at point *B* (C_A at point *B*=2.7) is higher than the actual cost at point *A* (C_A at point *A*=2.606) when the factor prices are given at w_1=1 and w_2=2. To be confirmed, we calculated the degree of inefficiency using the equation (1). The degree of inefficiency at the point *B* is 1.067 while the degree at the point *A* is 1.03. Thus, point *B* indicates more inefficient use and thus more resources are used.

In this hypothetical example, we expect that any quantity index must exhibit a higher number for the point *B* as input factors are more inefficiently used at the point *B* (that is, more factors are used at the point *B*). However, the Tornqvist index calculated using the equation (3a) shows the opposite, illustrating the Simpson's paradox. The Tornqvist index is 1.014 at the point *A* and 0.987 at the point *B*, meaning that the quantity index using less quantities (i.e., the point *A*) shows higher score than that using more quantities (i.e., the point *B*).

Figure 2: An Example of Simpson's Paradox



4. Conclusion

We have performed the simulations to address how a quantity index is compromised when efficient production, one condition for being superlative, is violated. The simulations were designed to generate the quantity of input factors used for the CES production technology under the behavioral assumption of cost minimization when the factor prices and the level of output are exogenously given. Then, the two indexes, the Fisher index and the Tornqvist index, are calculated from the data generated in the simulations.

The simulation results show that both indexes approximate the degree of inefficiency (and thus the quantity of factors used) well when inefficiency is small. Yet, the Tornqvist index begins to deviate as inefficiency increases. Thus, the finding from the previous literature that the Fisher index and the Tornqvist index approximate each other is only a special case when inefficiency is negligible. This deviation from the inefficiency amplifies when the factor substitutability is larger. The deviation is due to the use of the average cost shares as a weight in its calculation. We showed that the Tornqvist index eventually suffers from the Simpson's paradox, the average of two higher numbers are lower than the average of two lower numbers, when the factor substitutability is large.

From these results, we conclude that the Fisher index is preferred to the Tornqvist index when any inefficient use of factors is suspected.

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