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On the All Commodities Surplus Theorem

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Abstract

The purpose of this note is to propose the All Commodities Surplus Theorem (ACST) which is derived from the Hawkins-Simon (HS) condition. According to the Commodity Exploitation Theorem (CET), the HS condition implies the exploitation of only one commodity arbitrarily chosen. In contrast to the CET, the ACST emphasizes that the HS condition implies the exploitation of all commodities (or the existence of the surpluses in all commodity sectors). Based on the ACST, we derive the physical and price surplus determining equations, and integrating these two equations, we seek for the macro-identity (total amount of surplus products equals total amount of profits) in value theoretical contexts. These equations of this note have not appeared in the related literature. Because the economists in this field have studied by using the single commodity theory of value, it is impossible to derive the equations obtained in this note by their methods. We must use the ACST.

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1. The All Commodities Surplus Theorem (ACST)

In this section, we propose the ACST which is derived from the HS condition within the framework of the simple Leontief model. At first, we consider the value determining equations system. There exist $n (n \geq 2)$ kinds of the ordinary commodities and one type of labor. Let A represent a given $n \times n$ indecomposable physical input matrix. A_{kk} is the $(n-1) \times (n-1)$ coefficient matrix obtained by eliminating the k-th row and the k-th column vector from A. Let us make the following assumption.

Assumption HS condition (Hawkins and Simon (1949)): The matrix $I - A$ satisfies the HS condition[1], that is, all the principal minors of $I - A$ are positive, where I is the identity matrix according to the dimension.

Choose the commodity k as the standard of value. The values of commodities $j\ (j = 1, \cdots, n)$ can be expressed in terms of the total amount of commodity k included in one unit of them. As the matrix A is indecomposable, the all commodities become the standard value (ACST).

The value determining equations system is as follows.[2]

$$\lambda_{kk} = a_{kk} + \boldsymbol{\Lambda}_k^k \tilde{\boldsymbol{a}}_k^T, \tag{1}$$

$$\boldsymbol{\Lambda}_k^k = \hat{\boldsymbol{a}}_k + \boldsymbol{\Lambda}_k^k A_{kk}, \tag{2}$$

where a_{kk}: the amount of k directly needed to produce one unit of commodity k, $\boldsymbol{\Lambda}^k = (\lambda_{k1}, \cdots, \lambda_{kn})$: the n-dimensional row value vector, λ_{kj}: the total amount of commodity k directly and indirectly necessary to produce one unit of commodity j, and $\boldsymbol{\Lambda}_k^k$: the $n-1$ row value vector obtained by eliminating λ_{kj} from $\boldsymbol{\Lambda}^k$, that is, $\boldsymbol{\Lambda}_k^k = (\lambda_{k1}, \cdots, \lambda_{k,k-1}, \lambda_{k,k+1}, \cdots, \lambda_{kn})$. Moreover, $\tilde{\boldsymbol{a}}_k^T$ is the $n-1$ column vector obtained by eliminating a_{kk} from the k-th column vector, and $\hat{\boldsymbol{a}}_k$ is the $n-1$ row vector obtained by eliminating a_{kk} from the k-th row vector, that is, $\tilde{\boldsymbol{a}}_k^T = (a_{1k}, \cdots, a_{k-1,k}, a_{k+1,k}, \cdots, a_{nk})^T, \hat{\boldsymbol{a}}_k = (a_{k1}, \cdots, a_{k,k-1}, a_{k,k+1}, \cdots, a_{kn})$.

By the Cramer's rule, we obtain the value of each commodity from (2).

$$\lambda_{ki} = \frac{|(I - A_{kk})_i|}{|I - A_{kk}|}\ (i \neq k), \tag{3}$$

where $(I - A_{kk})_i$ is the matrix obtained by replacing the i-th row for $i < k$ and $(i-1)$-th row for $i > k$ by the row vector $\hat{\boldsymbol{a}}_k$ respectively. From (1) and (3), we obtain

1 For the economic meaning of the HS condition, see Fujimoto and Fujita (2008, p. 531).

2 See also Fujita (2006) and Fujimoto and Fujita (2008).

$$
\begin{aligned}
1-\lambda_{kk} &= 1-a_{kk}-\frac{\sum_{i\neq k} a_{ik}|(I-A_{kk})_i|}{|I-A_{kk}|} \\
&= \frac{(1-a_{kk})\cdot|I-A_{kk}|-\sum_{i\neq k} a_{ik}|(I-A_{kk})_i|}{|I-A_{kk}|}.
\end{aligned}
\tag{4}
$$

Since the numerator of (4) is equal to the cofactor expansions of $|I-A|$ with respect to the k-th column, we get

$$
1-\lambda_{kk}=\frac{|I-A|}{|I-A_{kk}|}. \tag{5}
$$

From the assumption of HS condition we have $|I-A|>0$ and $|I-A_{kk}|>0$, (5) implies that $\lambda_{kk}<1$, and $\boldsymbol{\Lambda}^k>0$ holds. In this case, the surplus value $(1-\lambda_{kk})$ of one specific commodity k (e.g. labor) is positive.

The HS condition, however, requires all the principal minors of $|I-A|$ to be positive. In the relation with the principal minors of $n-1$ order,

$$
|I-A_{kk}|>0 \text{ for all } k,
$$

and the above relation must hold *simultaneously*.

Considering this result, the equation (5) must be replaced by the following one.

$$
1-\lambda_{kk}=\frac{|I-A|}{|I-A_{kk}|} \text{ for all } k, \tag{5'}
$$

and (5') must hold *simultaneously*, that is, ACST holds: HS condition $\Rightarrow \lambda_{kk}<1(k=1,\cdots,n) \Leftrightarrow \boldsymbol{\Lambda}^k>0\ (k=1,\cdots,n)$.

2. The Surplus Determining Equations in Physical and Price System

At first, we derive the relationship between each component of B and the value of each commodity. $B=(b_{ij})$ is the Leontief inverse matrix $(I-A)^{-1}$. Abridging the detailed explanation,[3] we obtain the following relations:

$$
\lambda_{ki}=\frac{b_{ki}}{b_{kk}}\ (i\neq k), \lambda_{kk}=\frac{b_{kk}-1}{b_{kk}}\ (k=1,\cdots,n). \tag{6}
$$

From (6), we obtain n dimensional row vector of B in term of the value

$$
\left(\frac{\lambda_{k1}}{1-\lambda_{kk}},\frac{\lambda_{k2}}{1-\lambda_{kk}},\cdots,\frac{1}{1-\lambda_{kk}},\cdots,\frac{\lambda_{kn}}{1-\lambda_{kk}}\right).
$$

Thus, B can be expressed as follows:

[3] For details, see Fujita (2006) and Fujimoto and Fujita (2008).

$$B = \begin{pmatrix} \frac{1}{1-\lambda_{11}} & \frac{\lambda_{12}}{1-\lambda_{11}} & \cdots & \frac{\lambda_{1n}}{1-\lambda_{11}} \\ \frac{\lambda_{21}}{1-\lambda_{22}} & \frac{1}{1-\lambda_{22}} & \cdots & \frac{\lambda_{2n}}{1-\lambda_{22}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\lambda_{n1}}{1-\lambda_{nn}} & \frac{\lambda_{n2}}{1-\lambda_{nn}} & \cdots & \frac{1}{1-\lambda_{nn}} \end{pmatrix}.$$

If we use the single commodity (including labor) theory of value, we can obtain only one row vector.

Firstly we consider the Leontief physical system:

$$\boldsymbol{X}^T = A\boldsymbol{X}^T + \boldsymbol{F}^T, \tag{7}$$

where, $\boldsymbol{X}^T = (x_1, \cdots, x_{n-1}, N)^T$ is the n column vector of $n-1$ commodities and one kind of labor, and $\boldsymbol{F}^T = (f_1, \cdots, f_{n-1}, 0)^T$ is the n column vector of the surplus product of each commodity sector. From (7), $\boldsymbol{X}^T = B\boldsymbol{F}^T$. Premultiplying the k-th equation by $(1-\lambda_{kk})$ in order, we get the following n equations:

$$(1-\lambda_{kk})x_k = f_k + \sum_{j \neq k} \lambda_{kj} f_j \ \ (k = 1, \cdots, n), \tag{8}$$

where $x_n = N, f_n = 0$.

Okishio (1963, p.292), adopting the labor theory of value, gets the next surplus determining equation:

$$(1-\lambda_{nn})N = \sum_{j} \lambda_{nj} f_j \ \ (N\text{: total amount of labor}). \tag{9}$$

He asserts that the total amounts of surplus labor produce the total amounts of value of the surplus products. However, his equation (9) expresses only the n-th equation of (8) and he does not discuss the $n-1$ remaining equations of (8). By the labor theory of value, we cannot grasp the whole structure of surplus productions in the economy. To examine the entire structure of surplus theory, we must use the ACST.

Now, we consider the economic meaning of (8). The left-hand side (LHS) of (8) is the total amounts of surplus. The 1st term of the right-hand side (RHS) is the net surplus product of each commodity sector, and the 2nd term of RHS is the amount of commodity k directly and indirectly necessary to produce commodity j.

Next, we consider the price system:

$$\boldsymbol{P} = \boldsymbol{P}A + \boldsymbol{V}, \qquad \boldsymbol{P} = \boldsymbol{V}B,$$

where $\boldsymbol{P} = (\boldsymbol{P}_k, w)$, $\boldsymbol{P}_k$ is the $n-1$ price row vector and w is the money wage, $\boldsymbol{V} = (\boldsymbol{V}_k, 0)$, $\boldsymbol{V}_k$ is the $n-1$ profit row vector. Concerning each diagonal element of B, we substitute $\frac{1}{1-\lambda_{kk}}$ for b_{kk}, that is, principal diagonal is expressed by $\frac{1}{1-\lambda_{kk}}$ for all

k.

Hence, we get the following equations system

$$\boldsymbol{P} = \boldsymbol{V}B = \boldsymbol{V}\begin{pmatrix} \frac{1}{1-\lambda_{11}} & b_{12} & \cdots & b_{1n} \\ b_{21} & \frac{1}{1-\lambda_{22}} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & \frac{1}{1-\lambda_{nn}} \end{pmatrix}.$$

Premultiplying the k-th equation by $(1-\lambda_{kk})$ in order, we get the following equations system:

$$(1-\lambda_{kk})p_k = v_k + \sum_{i \neq k} v_i b_{ik}(1-\lambda_{kk}) \quad (k = 1, \cdots, n). \tag{10}$$

As far as we know, the equations system (10) as well as (8) has never appeared in the related literature. Let us explain the economic meaning of the equations system (10). The LHS is the total amount of prices of surplus of commodity k sector. The 1st term of the RHS is the profit of k sector. The 2nd term of the RHS is the total prices of k sector directly and indirectly necessary to produce commodity i.

Additional Note 1: In Section 2, we get the surplus determining equations (8) and (10) in physical and price system. In this additional note, we point out that the above equations (8) and (10) have another characteristic. That is, in (8) and (10) with respect to the commodities (except labor), $\lambda_{kk} < 1$ is equivalent to $f_k, v_k > 0$ for all k. It is reasonable to call this equivalent relation "the generalized commodity surplus theorem," and this theorem puts an end to the single commodity surplus theorem (or the commodity exploitation theorem).

3. The Value Theoretical Foundation of Macro-identity

In previous sections, we derived two surplus determining equations (8) and (10). In this section, using the simple Leontief model, we consider "the macro-identity"

$$\boldsymbol{X}_n^T = A_{nn}\boldsymbol{X}_n^T + \boldsymbol{F}_n^T, \quad \boldsymbol{P}_n = \boldsymbol{P}_n A_{nn} + \boldsymbol{V}_n, \tag{11}$$

where A_{nn} is $(n-1) \times (n-1)$ coefficient matrix obtained by eliminating the n-th row and the n-th column vector from A, $\boldsymbol{X}_n^T = (x_1, \cdots, x_{n-1})^T$, $\boldsymbol{P}_n = (p_1, \cdots, p_{n-1})$, $\boldsymbol{F}_n^T = (f_1, \cdots, f_{n-1})^T$, $\boldsymbol{V}_n = (v_1, \cdots, v_{n-1})$.

From (11), we obtain

$$\boldsymbol{P}_n\boldsymbol{F}_n^T = \boldsymbol{V}_n\boldsymbol{X}_n^T, \tag{12}$$

that is, total prices of surplus products equal total profits. We call (12) the macro-identity, which we borrow from Hollander (2008, p.53), though he uses it in a different context.

Multiplying (8) by p_k from LHS in order and (10) by x_k from RHS in order, we get the following two equations:

$$\sum_k p_k(1-\lambda_{kk})x_k = \sum_k p_k f_k + \sum_{j\neq k} \lambda_{kj} p_k f_j, \tag{13}$$

$$\sum_k p_k(1-\lambda_{kk})x_k = \sum_k v_k x_k + \sum_{j\neq k} v_j b_{jk}(1-\lambda_{kk})x_k. \tag{14}$$

From (13) and (14), we get the next equation in vector form:

$$\begin{aligned} &\boldsymbol{P}_n\boldsymbol{F}_n^T + \boldsymbol{P}(\boldsymbol{\Lambda}_1^1\boldsymbol{F}_1 \cdots \boldsymbol{\Lambda}_n^n\boldsymbol{F}_n)^T \\ &\quad = \boldsymbol{V}_n\boldsymbol{X}_n^T \\ &\quad + (\boldsymbol{V}_1(1-\lambda_{11})\boldsymbol{B}_1^T \cdots \boldsymbol{V}_n(1-\lambda_{nn})\boldsymbol{B}_n^T)\boldsymbol{X}^T, \end{aligned} \tag{15}$$

where $\boldsymbol{B}_k^T = (b_{k1}, \cdots, b_{k,k-1}, b_{k,k+1}, \cdots, b_{kn})^T$. We divide (15) into the three parts.

$$\boldsymbol{P}_n\boldsymbol{F}_n^T = \boldsymbol{V}_n\boldsymbol{X}_n^T.$$

$$\begin{aligned} &\boldsymbol{P}(\boldsymbol{\Lambda}_1^1\boldsymbol{F}_1 \cdots \boldsymbol{\Lambda}_{n-1}^{n-1}\boldsymbol{F}_{n-1})^T \\ &\quad = \big(\boldsymbol{V}_1(1-\lambda_{11})\boldsymbol{B}_1^T \cdots \boldsymbol{V}_{n-1}(1 \\ &\quad - \lambda_{n-1,n-1})\boldsymbol{B}_{n-1}^T\big)\boldsymbol{X}_n^T. \end{aligned} \tag{16}$$

$$w\boldsymbol{\Lambda}_n^n\boldsymbol{F}_n^T = \boldsymbol{V}_n(1-\lambda_{nn})\boldsymbol{B}_n^T N. \tag{17}$$

The first equation is the macro-identity mentioned above (eq. (12)). Equation (16) is related to each commodity sector and (17) is related to labor household sector. It is somewhat difficult to interpret the exact meaning of (16) and (17). However, it is clear that the LHS and the RHS of eq. (16) are the total costs of intermediate input to produce $\boldsymbol{P}_n\boldsymbol{F}_n^T$ and $\boldsymbol{V}_n\boldsymbol{X}_n^T$ respectively,[4] while in eq. (17), it is certain that the LHS as well as the RHS is not so, because $f_n = v_n = 0$ in macro-identity (12).

The LHS of eq. (17) is the total wages of labor included in each commodity, which the labor-household sectors receive from the commodity sectors, and the RHS is the total prices (in terms of profit) directly and indirectly necessary to produce the total amount of surplus labor, which the labor-household sectors pay to the commodity sectors.

The macro-identity is easily derived from the Leontief physical and price system. But to hold this identity we need such eqs. (16) and (17) which have not appeared in the literature of mathematical Marxian economists (including Okishio (1963), Morishima (1973), and others).

The achievements we established in this note using the ACST within the framework

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[4] See Additional Note 2.

of the simple Leontief model, for example, physical and price surplus determining equations (8), (10) and the macro-identity (12) in value theoretical context and so on, would play an important role in the field of value theory.

The equations derived in this note have never appeared in the field of value theory. Because the authors who are interested in value theory have investigated by using the single commodity theory of value, it is impossible to derive the equations obtained in this note by their methods. We must use the ACST to make the principle of value clear.

Additional Note 2: In this Note, we show that the LHS and the RHS of (16) is the total costs of intermediate inputs to produce $\boldsymbol{PF}^T$ and $\boldsymbol{VX}^T$. The total prices of surplus commodities can be expressed as $\sum_{k=1}^{n-1} p_k(1-\lambda_{kk})x_k$, or in matrix form,

$$\boldsymbol{P}_n\begin{pmatrix}1-\lambda_{11} & & 0\\ & \ddots & \\ 0 & & 1-\lambda_{nn}\end{pmatrix}\boldsymbol{X}_n^T.$$

Using $\boldsymbol{X}_n^T = B\boldsymbol{F}_n^T, \boldsymbol{P}_n = \boldsymbol{V}_n B$, we get

$$\boldsymbol{P}_n\begin{pmatrix}1-\lambda_{11} & & 0\\ & \ddots & \\ 0 & & 1-\lambda_{nn}\end{pmatrix}\boldsymbol{X}_n^T = \boldsymbol{P}_n\begin{pmatrix}1-\lambda_{11} & & 0\\ & \ddots & \\ 0 & & 1-\lambda_{nn}\end{pmatrix}B\boldsymbol{F}_n^T =$$

$$\boldsymbol{V}_n B\begin{pmatrix}1-\lambda_{11} & & 0\\ & \ddots & \\ 0 & & 1-\lambda_{nn}\end{pmatrix}\boldsymbol{X}_n^T.$$

Therefore, taking the above relation into account we obtain the following two equations.

$$\boldsymbol{P}_n\begin{pmatrix}1-\lambda_{11} & & 0\\ & \ddots & \\ 0 & & 1-\lambda_{nn}\end{pmatrix}\begin{pmatrix}\frac{1}{1-\lambda_{11}} & \frac{\lambda_{12}}{1-\lambda_{11}} & \cdots & \frac{\lambda_{1n}}{1-\lambda_{11}}\\ \frac{\lambda_{21}}{1-\lambda_{22}} & \frac{1}{1-\lambda_{22}} & \cdots & \frac{\lambda_{2n}}{1-\lambda_{22}}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\lambda_{n1}}{1-\lambda_{nn}} & \frac{\lambda_{n2}}{1-\lambda_{nn}} & \cdots & \frac{1}{1-\lambda_{nn}}\end{pmatrix}\boldsymbol{F}_n$$

$$-\boldsymbol{P}_n(\boldsymbol{\Lambda}_1^1\boldsymbol{F}_1^T \cdots \boldsymbol{\Lambda}_{n-1}^{n-1}\boldsymbol{F}_{n-1}^T)^T = \boldsymbol{P}_n\boldsymbol{F}_n^T.$$

$$\boldsymbol{V}_n\begin{pmatrix}1-\lambda_{11} & & 0\\ & \ddots & \\ 0 & & 1-\lambda_{nn}\end{pmatrix}\begin{pmatrix}\frac{1}{1-\lambda_{11}} & b_{12} & \cdots & b_{1n}\\ b_{21} & \frac{1}{1-\lambda_{22}} & \cdots & b_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ b_{n1} & b_{n2} & \cdots & \frac{1}{1-\lambda_{nn}}\end{pmatrix}\boldsymbol{X}_n^T$$

$$-\left(\boldsymbol{V}_1(1-\lambda_{11})\boldsymbol{B}_1^T \cdots \boldsymbol{V}_{n-1}\left(1-\lambda_{n-1,n-1}\right)\boldsymbol{B}_{n-1}^T\right)\boldsymbol{X}_n^T = \boldsymbol{V}_n\boldsymbol{X}_n^T.$$

Using two commodities economy, we certify that the above equations hold as follows.

$$(p_1 \quad p_2)\begin{pmatrix} 1-\lambda_{11} & 0 \\ 0 & 1-\lambda_{22} \end{pmatrix}\begin{pmatrix} \dfrac{1}{1-\lambda_{11}} & \dfrac{\lambda_{12}}{1-\lambda_{11}} \\ \dfrac{\lambda_{21}}{1-\lambda_{22}} & \dfrac{1}{1-\lambda_{22}} \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} - (p_1\lambda_{12}f_2 + p_2\lambda_{21}f_1)$$

$$= (p_1 \quad p_2)\begin{pmatrix} 1 & \lambda_{12} \\ \lambda_{21} & 1 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} - (p_1\lambda_{12}f_2 + p_2\lambda_{21}f_1) = p_1 f_1 + p_2 f_2.$$

$$(v_1 \quad v_2)\begin{pmatrix} \dfrac{1}{1-\lambda_{11}} & b_{12} \\ b_{21} & \dfrac{1}{1-\lambda_{22}} \end{pmatrix}\begin{pmatrix} 1-\lambda_{11} & 0 \\ 0 & 1-\lambda_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$- (v_2(1-\lambda_{11})b_{21}x_1 + v_1(1-\lambda_{22})b_{12}x_2)$$

$$= (v_1 \quad v_2)\begin{pmatrix} 1 & b_{12}(1-\lambda_{22}) \\ b_{21}(1-\lambda_{11}) & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$- (v_2(1-\lambda_{11})b_{21}x_1 + v_1(1-\lambda_{22})b_{12}x_2) = v_1 x_1 + v_2 x_2.$$

References

Fujimoto, T. and Y. Fujita (2008), "A Refutation of the Commodity Exploitation Theorem," *Metroeconomica* **50**, 530-540.

Fujita, Y. (2006), "On the All Commodities Surplus Theorem," (in Japanese), mimeo, Fukuoka University.

Hawkins, D and H. Simon (1949), "Note: Some Conditions of Macroeconomic Stability," *Econometrica* **17**, 245-248.

Hollander, S. (2008), *The Economics of Karl Marx: Analysis and Application*, Cambridge: Cambridge University Press.

Morishima, M. (1973), *Marx's Economics: A Dual Theory of Value and Growth*, Cambridge: Cambridge University Press.

Okishio, N. (1963), "A Mathematical Note on Marxian Theorems," *Weltwirtschaftliches Archiv* **91**, 287-299.