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Co-monotonic allocation with heterogeneous agents and non-linear-discounting utility: an example

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Abstract

Many dynamic macroeconomics models employ time additively separable utility functions although capital ownership accrues to the most patient consumer. To modify this result, previous studies employing recursive utility show consumption and saving are co-monotonic to the degree of each consumer's time preference in the long run. Although they run in continuous time model, we demonstrate equivalent results in discrete time model.

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1 Introduction

Ramsey (1928) asserts the society will be divided by two classes, profligate laborers and thrifty capitalists. Mitra and Sorger (2013) establish formal proof in continuous time after Becker's (1980) verification in discrete time. They employ *Time Additively Separable* (TAS) utility $U(c_0, c_1 \ldots) = \sum_{t=0}^{\infty} \beta^t v(c_t)$ where the constant β represents consumer's discount factor. Accordingly, the most patient agent with the highest discount factor is the economy's only saver. This unrealistic result is due to TAS's exogenous discount factor[1].

Koopmans (1960) recommends a *Recursive Utility*, which describes the life-time utility from today as $U(c_0, c_1 \ldots) = W[c_0, U(c_0, c_1 \ldots)]$ where c_0 is the consumption today and $U(c_0, c_1 \ldots)$ is the lifetime utility from tomorrow. We call the function W *aggregator*. Letting $W(c, u) = v(c) + \beta u$, TAS utility satisfies recursivity. The *time perspective* $D_u W(c, u)$ represents the magnitude of consumer's discounting preference, and its high value means that he/she is much patient. TAS assumes constant time perspective β, however non-TAS Recursive Utility allows the time perspective to vary depending on the consumption path, avoiding extreme allocation. Lucas and Stokey (1984), and Epstein (1987) demonstrate that all consumers save in the equilibrium, nevertheless amount saved is not clear, In particular, the latter shows that consumers' time preferences and the amount of consumption and savings are co-monotonic.

The novelty of this model is summarized in Table 1.[2] In this paper we investigate the stationary equilibrium with the KDW (Koopmans, Diamond and Williamson) utility function which is one of non-TAS and recursive utility. The main result is that co-monotonic allocation occurs in the stationary equilibrium. A consumer with higher time preference parameter saves and enjoys consumption larger. Even the least patient consumer saves, given sufficiently high interest rate.

[1] Considering perpetual growth, gradual allocation may occur under TAS. It is shown by Nakamura(2014).

[2] Besides time structure, this model is different from Epstein and Hynes in the meaning of "patience". With EH utility, a consumer becomes patient as his/her immediate utility increased: whereas the KDW utility assumes a consumer becomes patient as he/she obtains larger future utility increases.

Table 1: This model and previous studies

Utility/ Time	Continuous	Discrete
TAS	Mitra and Sorger (2013)	Becker (1980)
Recursive (non-TAS)	Epstein and Hynes (1983)	**This Model**

2 Model and Assumptions

We assume one sector model. Time is discrete with index $t = 0, 1, 2 \ldots$. There are H consumers with index $h = 1, 2 \ldots H$, and one producer. This economy has capital and products markets. Although labor supply is absent in this model, if we assume identical labor supply as many macro economic models do, the main results are maintained.

Let $\mathbf{c}^h = (c_0^h, c_1^h, \ldots) \in \mathbb{R}_+^\infty$ and $\mathbf{k}^h = (k_0^h, k_1^h \ldots) \in \mathbb{R}_+^\infty$ be the h^{th} consumer's consumption and saving path, respectively. The consumption path from period t to s is represented by ${}_t\mathbf{c}_s^h = (c_t^h, c_{t+1}^h \ldots c_s^h)$. Omitting the left (right) subscript means that the path starts from period 0 (continues infinitely). And let $\bar{\mathbf{c}}^h = (\bar{c}^h, \bar{c}^h \ldots)$ and $\bar{\mathbf{k}}^h = (\bar{k}^h, \bar{k}^h \ldots)$ be constant paths.

Consumer h has initial endowment k_0^h in period 0. He/she lends saving k_t^h to the producer with the interest rate $r_t (> 0)$ in each period. If the interest rate is zero, the Euler equation does not hold. Therefore main results may be changed. Both the capital and products markets are assumed to be perfectly competitive. They can foresee future interest rates, their behavior is described by the solution to the dynamic Utility Maximization Problem (UMP):

$$\begin{aligned} &\max_{\mathbf{c}^h, \mathbf{k}^h} U^h(\mathbf{c}^h) \\ &\text{sub.to } ({}^\forall t = 0, 1 \ldots)\ 0 \leq c_t^h, k_t^h \\ &\qquad ({}^\forall t = 0, 1 \ldots)\ c_t^h + k_{t+1}^h \leq (1 + r_t) k_t^h \\ &\qquad ({}^\forall t = 0, 1, \ldots)\ r_t \in \mathbb{R}_{++} :\ \text{given} \\ &\qquad k_0^h \in \mathbb{R}_+ :\ \text{given} \end{aligned}$$

Each consumer is assumed to have the KDW utility function ;

$$U^h(\mathbf{c}) = (c_0)^a + \lim_{t \to \infty} d_h \log[1 + (c_1)^a + d_h \log\{1 + (c_2)^a \ldots + d_h \log(1 + (c_t)^a)\}]$$

where $a, d_h \in (0, 1)$

defined over $\mathbb{R}_+^\infty$. The second right side term is monotonically increasing. Therefore the limit value exists if $+\infty$ is permitted.

The KDW utility has the aggregator function $W^h(c,u) = c^a + d_h \log(1+u)$ with time perspective $D_u W^h = d_h/(1+u)$, which is decreasing in u. That is, a consumer who is promised high future utility will increase not saving but immediate consumption if they obtain additional one unit of income. Each consumer is assumed to have his/her own time preference parameter d_h and common felicity function $v(c) = c^a$ in order to focus on the difference of time preference among consumers. A consumer with a higher d_h exhibits more impatience when all consumers take the same consumption path.

By adopting the chain rule repeatedly to the KDW utility, $U^h(\mathbf{c})$ is partial differentiable with respect to c_t for all $t = 0, 1 \ldots$. And marginal utility is described by

$$D_{c_t} U^h(\mathbf{c}^h) = \left(\prod_{s=1}^{t} \frac{d_h}{1 + U^h({}_s\mathbf{c}^h)} \right) a \left(c_t^h\right)^{a-1} \tag{1}$$

for any $t = 0, 1, \ldots$ and $\mathbf{c} \in \mathbb{R}_+^\infty$. From concavity of the KDW utility function, the inner solution to the UMP is characterized with the Euler equation. Following equation (1), the Euler equation is

$$\left(\prod_{s=1}^{t+1} \frac{d_h}{1 + U^h({}_s\mathbf{c}^h)} \right) a(c_{t+1}^h)^{a-1}(1 + r_t) = a(c_t^h)^{a-1} \left(\prod_{s=1}^{t} \frac{d_h}{1 + U^h({}_s\mathbf{c}^h)} \right)$$

We assume $d_h > 0$ for all $h = 1, 2, \ldots H$. If the stationary consumption level of the consumer h, $\bar{c}^h$ is strictly positive, then the Euler equation is reduced to simple form

$$\frac{d_h}{1 + U^h(\bar{\mathbf{c}}^h)}(1 + \bar{r}) = 1 \tag{2}$$

The producer hires capital $K_t \in \mathbb{R}_{++}$ from consumers with interest rate $r_t \in \mathbb{R}_{++}$ at the beginning of period t. To analyze consumption and saving distribution, we exclude the case of zero input and zero products. He/she is assumed to have the linear production function $F(K) = AK$ for simplicity. Co-monotonic allocation depends only on the order of time preference. Therefore any technology do not change the main results of this model. Under linear production function, the interest rate is always $r = A - 1$. Otherwise, a solution to the Profit Maximization Problem (PMP) does not exist. Under this interest rate, the profit is always zero. Therefore the producer never accumulates capital. The producer's behavior is described by the solution to the PMP in each period:

$$\max_{K_t} AK_t - (1+r_r)K_t$$
$$sub.to\ r_t \in \mathbb{R}_{++} : \text{given}$$
$$K_t \in \mathbb{R}_{++}$$

3 Stationary Competitive Equilibrium

The combination $< H, \mathbf{r}, (\mathbf{c}^h, \mathbf{k}^h)_{h=1}^H, \mathbf{K} >$ is the *Competitive Equilibrium* if the following conditions are satisfied;

(A) (Strict) Positiveness of Variables

:In all periods t and for all consumers h, consumption c_t^h and saving k_t^h are weakly positive. And capital input K_t and the interest rate r_t are strictly positive for all $t = 0, 1, \ldots$.

(B) Consumers' Optimality

:For all $h = 1, 2 \ldots H$, $(\mathbf{c}^h, \mathbf{k}^h)$ is the solution to the h^{th} consumer's *UMP*.

(C) Producer's Optimality

:For all $t = 0, 1 \ldots$, K_t is the solution to the *PMP*.

(D) Balance of Capital Market

:For all $t = 0, 1 \ldots$, it holds that $\sum_{h=1}^H k_t^h = K_t$.

(E) Balance of Products Market

For all $t = 0, 1 \ldots$, it holds that $\sum_{h=1}^H c_t^h + \sum_{h=1}^H k_{t+1}^h = F(K_t)$.

If the Competitive Equilibrium $< H, \mathbf{r}, (\mathbf{c}^h)_{h=1}^H, (\mathbf{k}^h)_{h=1}^H, \mathbf{K} >$ satisfies (F) besides (A)-(E), this is the *Stationary Competitive Equilibrium*.

(F) Constancy of Consumption, Capital, Input and Interest Rate

The paths $\mathbf{r}$, $(\mathbf{c}^h, \mathbf{k}^h)_{h=1}^H$ and $\mathbf{K}$ are constant.

Given concavity of the production function and the utility function, we can guess that the Stationary Competitive Equilibrium is characterized by the first order condition, the budget constraint and the balance of market;

(I) The Interest Rate:

$$\bar{r} = A - 1$$

(II) Euler Equation at the Constant Consumption Path:

$$(^\forall h = 1, 2, \ldots H)\ \frac{d_h}{1 + U^h(\bar{\mathbf{c}}^h)}(1 + \bar{r}) = 1$$

(III) The Budget Constraint with Equality

For all $h = 1, 2, \ldots H$, it holds that

$$\bar{k}^h = k_0^h,\ \bar{c}^h + \bar{k}^h = (1 + \bar{r})\bar{k}^h$$

(IV) The Clearing Capital Market

$$\sum_{h=1}^{H} \bar{k}^h = \bar{K}$$

Notice that products market is automatically balanced by Walras' Law.

4 Results

Lemma 4.1 indicates the sufficient condition for UMP[3]:

Lemma 4.1. *Given a consumer with KDW utility, then* $(\bar{\mathbf{c}}^h, \bar{\mathbf{k}}^h) \in \mathbb{R}^\infty_{++} \times \mathbb{R}^\infty_{++}$ *satisfying (II) and (III) is the solution to the* h^{th} *consumer's UMP under the initial capital stock* $k_0^h > 0$ *and the constant interest rate* $\bar{r} > 0$.

Theorem 4.1. *Assume the production function* $F(K) = AK$, *and that each consumer has KDW utility. Let*

$$\frac{1}{A - \log A} \leq d_1 < d_2 < \ldots < d_H \tag{3}$$

Then $< H, \bar{\mathbf{r}}, (\bar{\mathbf{c}}^h, \bar{\mathbf{k}}^h)_{h=1}^H, \bar{\mathbf{K}} >$ *satisfying (I)-(IV) is the unique inner Stationary Competitive Equilibrium. Moreover this equilibrium generates a co-monotonic allocation to* d_h.

We can grasp co-monotonic consumption and saving allocation intuitively as follows: When all consumers take just same consumption paths, and if they get additional one unit of income, patient one tends to save even if less patient one stops saving. As a result, he/she saves and consumes more.

The inequality (3) implies that all consumers are sufficiently patient to meet the Euler equation given the interest rate $\bar{r} = A - 1$.

Theorem 4.1 asserts the uniqueness of "inner" and "Stationary" Competitive Equilibrium. Thus, a corner-stationary equilibrium or non-stationary equilibrium may exist.

[3] The proof is available on request from the author.

Proof);

Firstly, we shall find a stationary consumption level satisfying (II). Choose $h \in \{1, 2, \dots H\}$ arbitrary. If c is sufficiently high, then it holds $c^a > Ad_h - 1$. Therefore some number $\tilde{u}^h (> Ad_h - 1)$ is included in the range of $U^h(\mathbf{c})$. By $A > 1$ and (3), we obtain

$$\frac{d_h}{\tilde{u}^h + 1} < \frac{1}{A} < \frac{1}{A - \log A} \leq d_h = \frac{d_h}{0 + 1} \tag{4}$$

By the intermediate value theorem, there exists a number $u^{h'} \in (0, \tilde{u}^h)$ satisfying $d_h/(u^{h'} + 1) = 1/A$, giving us $u^{h'} = d_a A - 1$. Take $\bar{c}^h$ to satisfy $u^{h'} = (\bar{c}^h)^a + d_h \log(u^{h'} + 1)$. Then, it follows that

$$\bar{c}^h = \{d_h A - 1 - d_h \log(d_h A)\}^{1/a} \tag{5}$$

By $d_h \in (0, 1)$, it holds that

$$d_h A - 1 - d_h \log(d_h A) > d_h A - 1 - d_h \log A \geq 0$$

By (4) and $A > 1$, it holds that $d_h \geq 1/(A - \log A) > 0$. Thus, we obtain

$$\left(\bar{c}^h\right) a > d_h A - 1 - d_h \log A \geq 0 \tag{6}$$

In addition, (II) is satisfied given $\bar{c}^h$.

Next we check co-monotonicity. Differentiate both sides of (5) with respect to d_h, it follows that

$$D_{d_h} \bar{c}^h = \frac{1}{a}\{d_h A - 1 - d_h \log(d_h A)\}^{1/a - 1}\{A - 1 - \log(d_h A)\}$$

(6) and $d_h \in (0, 1)$ brings

$$A - 1 - \log(d_h A) > A - \frac{1}{d_h} - \log(d_h A) > 0.$$

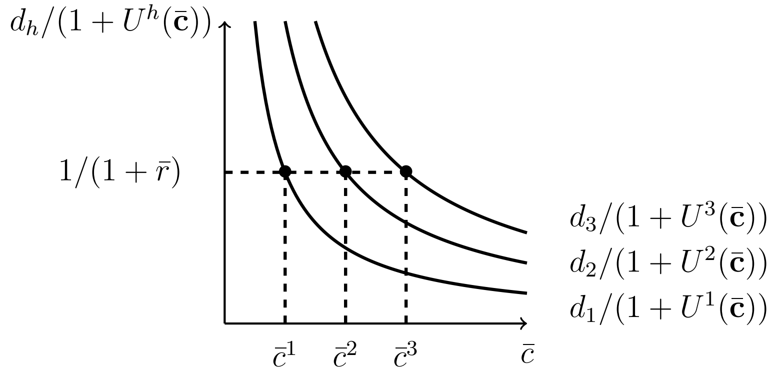
Therefore $\bar{c}^h$ is strictly increasing in d_h.

Letting $\bar{r} = A - 1, \bar{k}^h = \bar{c}^h/\bar{r}$ and $\bar{K} = \sum_{h=1}^H \bar{k}^h$, $< H, \bar{\mathbf{r}}, (\bar{\mathbf{c}}^h, \bar{\mathbf{k}}^h)_{h=1}^H, \bar{\mathbf{K}} >$ satisfies (I)-(IV). By (I), the PMP solution is $\bar{K}$. By Lemma 4.1, (II) and (III) imply that $(\bar{\mathbf{c}}^h, \bar{\mathbf{k}}^h)_{h=1}^H$ is the solution to the h^{th} consumer's UMP. From (III), the order of saving follows that of consumption.

□

In 3-consumers economy, each consumer's stationary consumption level is described by $\bar{c}^1, \bar{c}^2$ and $\bar{c}^3$ in Figure 1. Time perspective $d_h/(1 + U^h(\bar{\mathbf{c}}))$ is decreasing in $\bar{c}$. Therefore the each curve is down-sloping. And $d_3/(1+U^3(\bar{\mathbf{c}}))$ lays the highest level because of $d_1 < d_2 < d_3$.

Figure 1: Consumer's time perspective $D_u W(\bar{u}, U(\bar{\mathbf{c}}))$ in 3 consumer economy



5 Conclusion

This model runs in discrete time and it derives equilibrium allocation quantitatively. This advantage enables us to apply this model to computer simulations. The allocation in the real economic world that various people save is viewed as Co-monotonic allocation caused by differences of time preference. We can estimate the social distribution of discount rate from empirical consumption and saving data.

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