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# On Estimating Risk Premium With Flexible Fourier Form

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## Abstract

This paper proposes a semi-parametric estimate of risk premium using the Flexible Fourier From with a small number of low-frequency components. We provide an application to the forecast error decomposition based on the uncovered interest rate parity (UIP). Limited support is found for the omitted-variable explanation of the UIP puzzle.

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# Section 1. Introduction

The Flexible Fourier Form of Gallant (1981) has been applied to account for structural breaks, parameter instability and nonlinearity, see Becker et al. (2004), Enders and Lee (2012), Enders and Holt (2012), Enders and Li (2015) and Enders and Li (2018) for instance. This paper makes a contribution to the literature by proposing a simple Fourier estimate of risk premium, in an attempt to verify Fama (1984)'s hypothesis that the uncovered interest rate parity (UIP) puzzle can be explained by the omission of risk premium in the regression-based test for UIP.

Using Fourier form or trigonometric functions is motivated by the smoothness of risk premium. For instance, consider a structural model of risk premium in the foreign exchange market

$$E\left(\theta_t|\Omega_t\right) = \frac{\mathtt{cov}\left(\frac{u'(c_{t+1})}{p_{t+1}}\frac{p_t}{u'(c_t)}, \frac{S_{t+1}-S_t}{S_t} \mid \Omega_t\right)}{-E\left(\frac{u'(c_{t+1})}{p_{t+1}}\frac{p_t}{u'(c_t)} \mid \Omega_t\right)} \tag{1}$$

where $\theta_t$ denotes the risk premium, $u'$ is the first order derivative of utility function, $c_t$ is consumption, $p_t$ is domestic currency price of consumption good, $S_t$ is spot exchange rate defined as the domestic currency price of one unit of foreign currency, and $\Omega_t$ is information set. Equation (1) is basically the first order condition of a generalized Lucas (1978) intertemporal asset pricing model, see Mark (1985) for its derivation.

For our purpose, it suffices to note that the smoothness of risk premium stems from consumption smoothing $c_t \approx c_{t+1}$, sticky price $p_t \approx p_{t+1}$, and time-invariant utility function in (1). In the extreme case where price and consumption are fixed, covariance in the numerator becomes a constant of zero. Even if there may be heterogeneity in personal utility and information set, the central limit theorem is able to add another layer of smoothness at the aggregate level.

Restrictive assumptions such as the form of utility function are required[1] in order to obtain a parametric or "structural form" estimate of risk premium based on (1). For the sake of robustness and computational easiness, this paper advocates a semiparametric or "reduced-form" estimate of risk premium by applying the Fourier form to the Fama regression forecast error with the UIP imposed.

The economic literature on risk premium is vast, see Arnould and Nichols (1983), Mehra and Prescott (1988), and Hess and Kamara (2005) for discussion about risk premium of wage, equity, and interest rate. The proposed methodology can be applied to those topics provided that the risk premium is smooth. Other approaches of estimating the risk premium can be found in Linton and Perron (2003) and Collot and Hemauer (2021). This paper distinguishes itself by using the Flexible Fourier Form.

[1]For instance, the Constant Relative Risk Aversion (CRRA) utility function is commonly used.

# Section 2. Flexible Fourier Form

Let $u_t = \theta_t + e_t$ be a time series consisting of risk premium $\theta_t$ and idiosyncratic error $e_t$. This paper considers approximating unobserved $\theta_t$ with its Flexible Fourier Form (FFF)

$$u_t = FFF_t + \hat{e}_t \tag{2}$$

$$FFF_t = \mu + \sum_{k=1}^{p} \alpha_k \cos(2\pi kt/n) + \sum_{k=1}^{p} \beta_k \sin(2\pi kt/n) \tag{3}$$

where $k$ is index for frequency, $p$ is the number of frequencies, and $n$ is sample size. When $p = 0$, the risk premium is assumed to be constant. With rising $p$ we can allow for an increasingly complex pattern in risk premium. When $p = n/2$, the Fourier approximation is perfect for any absolutely integrable function, see Gallant (1981) for details. In light of the parsimony principle, this paper chooses the maximum value of $p = 4$. The coefficients $\alpha_k$ and $\beta_k$ are obtained by regressing $u_t$ onto sin and cos regressors. By construction, the FFF is a data-driven and endogenous approach of estimating the unknown risk premium. The Fourier series is able to yield a global approximation as long as the risk premium is bounded, and FFF works especially well when the risk premium is smooth.

We use simulation to illustrate in Figure 1 the performance of FFF approximation. The data are generated using a linear combination of Logistic Smooth Transition Autoregressive (LSTAR) models

$$y_t = \sum_{j=1}^{m} d_j (1 + \exp(-\gamma_j (t - \lambda_j n)))^{-1}, \quad (t = 1, \ldots, n) \tag{4}$$

where $\lambda$ determines the location of transition, and $\gamma$ controls the smoothness of transition. For instance, in panel A we set $m = 2$, $d_1 = 0.2, \gamma_1 = -0.15, \lambda_1 = 0.25$, $d_2 = 0.3, \gamma_2 = 0.05$, $\lambda_2 = 0.7$, $n = 200$. In each panel the solid line represents $y_t$ series; the dot line is the fitted value after using two frequencies and regressing $y_t$ onto $\cos(2\pi t/n), \sin(2\pi t/n)$, $\cos(4\pi t/n)$, and $\sin(4\pi t/n)$; the dash line is the fitted value with $\cos(6\pi t/n)$ and $\sin(6\pi t/n)$ being added.

## Figure 1: FFF Approximation of Smooth Functions

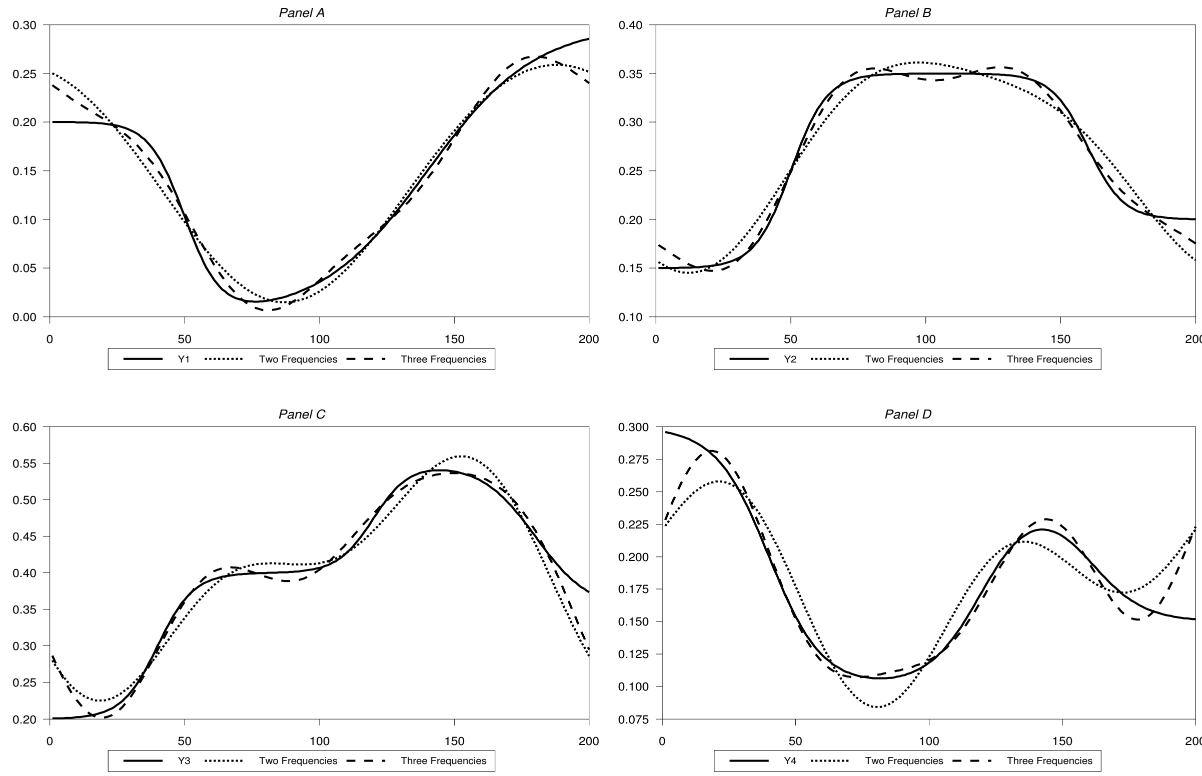



Figure 1 clearly shows that the FFF is able to approximate a smooth target reasonably well, and more frequencies improve the fit. In panel D, for instance, the adjusted R squared changes from 0.81 to 0.91 after the third frequency is included in the FFF.

# Section 3. Fourier Estimate of Risk Premium

Let $ID_t$ denote the home minus foreign nominal interest rate differential, and $\Delta s_{t+1} = \log S_{t+1} - \log S_t$ be the one-period change in log spot exchange rate defined as US dollars per one foreign currency. We are interested in the Fama regression

$$\Delta s_{t+1} = \beta_0 + \beta_1 ID_t + u_t \tag{5}$$

and testing the UIP hypothesis

$$H_0 : \beta_1 = 1. \tag{6}$$

Fama (1984) hypothesizes that the error term $u_t$ in (5) includes risk premium, which can be correlated with the interest differential and result in a biased estimate of $\hat{\beta}_1$. That is the omitted-variable explanation of the UIP puzzle—the historical data tend to reject (6) even if it is true. It is challenging to empirically verify Fama's theory since the risk premium is not directly observed. Nevertheless, this paper provides a solution by looking at (5) from a different angle—we impose the UIP condition $\beta_1 = 1$ and treat (5) as a forecast error decomposition of the change in log exchange rate

$$\Delta s_{t+1} = ID_t + u_t \tag{7}$$
$$u_t = \theta_t + e_t \tag{8}$$

where $\beta_0$ is absorbed in $\theta_t$. Decomposition (7) states that the interest differential $ID_t$ is the part of exchange rate movement explained by the UIP, whereas $u_t$ is the unexplained part or forecast error. Moreover, (8) divides the composite forecast error $u_t$ into the risk premium $\theta_t$ and an idiosyncratic error term $e_t$. Now it is straightforward to obtain the Fourier estimate of risk premium: first, $u_t$ is computed as $u_t = \Delta s_{t+1} - ID_t$; then the FFF given by (2) and (3) is applied to $u_t$.

# Section 4. Application

Monthly series are downloaded from FRED Economic Data, and the sample is from January 1974 to December 2020. To minimize the effect of government intervention, the nominal interest rates are 3-month Interbank Rates (percent per annum) in Australia, Canada, Switzerland, UK, and US. The spot exchange rates are US dollars per one foreign currency.

To duplicate the stylized fact reported in the literature[2], Figure 2 plots $\hat{\beta}_1$ in the Fama regression (5) after regressing 3-month change in log exchange rate onto the interest differential. Each window contains 60 or 5-year observations and every time we move the window forward by one observation. We see in all panels of Figure 2 the UIP hypothesis (6) is violated most of the time[3]. Note that the zero lower bound after 2008 adds volatility to $\hat{\beta}_1$ thanks to the diminishing variation of the regressor in the Fama regression (5).

[2] For instance, see Li et al. (2012), Lothian (2016), Ismailov and Rossi (2018) and Kumar (2019) for recent discussion about the UIP puzzle.

[3] $\beta_1 = 1$ is denoted by a horizontal line in Figure 2.

# Figure 2: Time-Varying Beta1

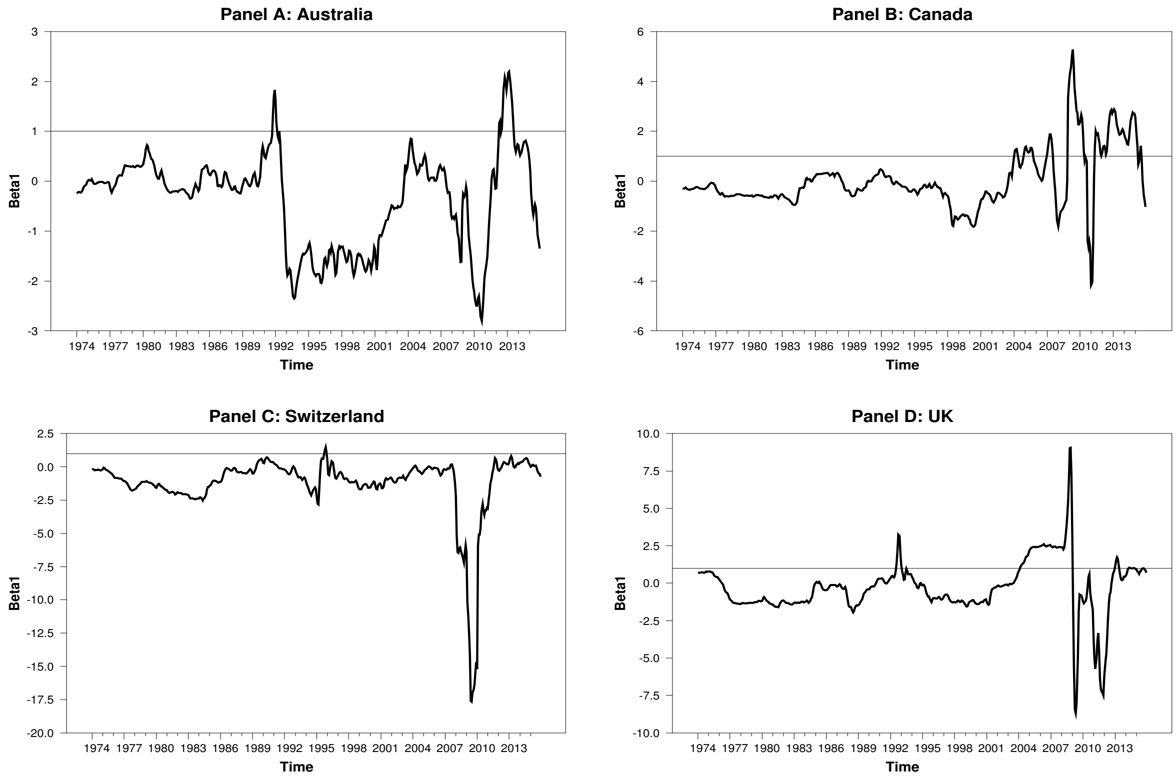
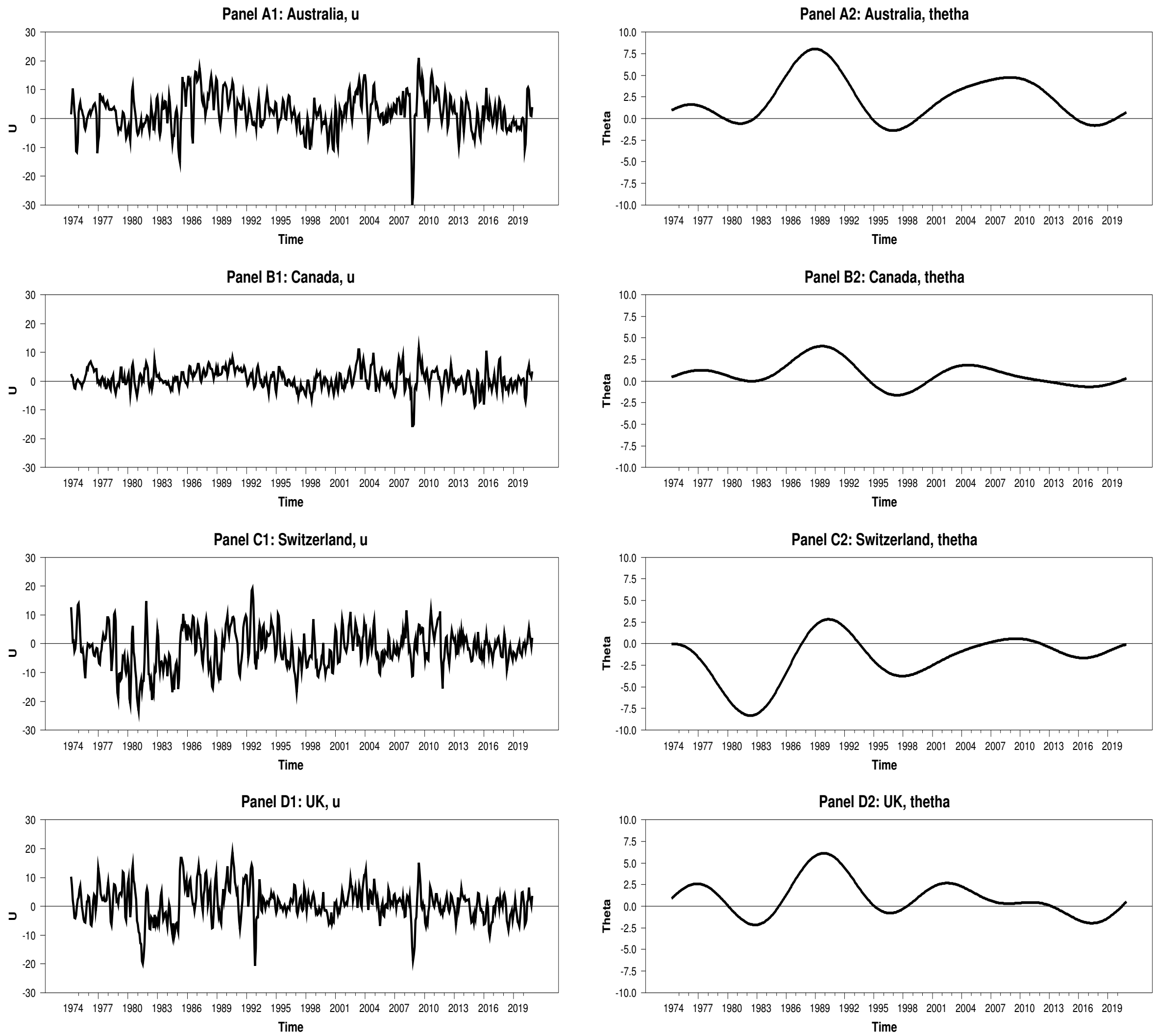


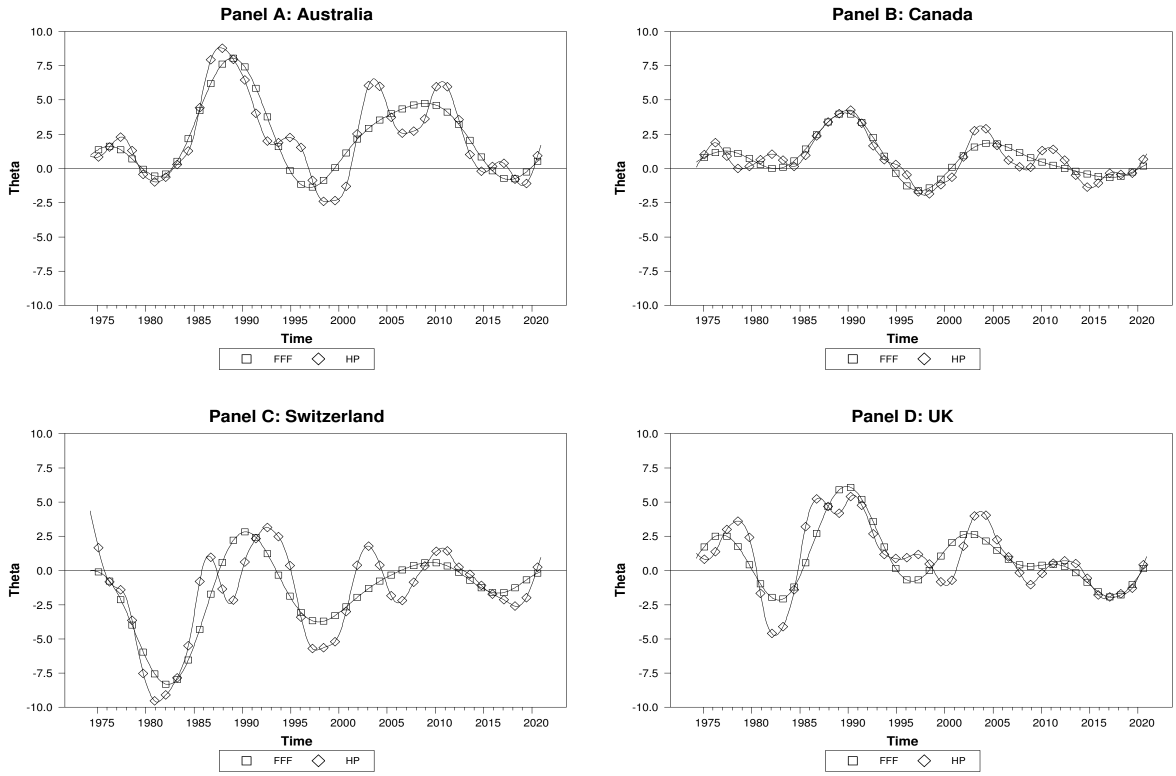

Figure 3 plots the composite forecast error $u_t = \Delta s_{t+1} - ID_t$, and the Fourier estimate of the risk premium using four frequencies $p = 4$ in (3). We find evidences for synchronization in the risk premium. For instance, there is a trough in the early 1980s followed by a peak around the early 1990s.

## Figure 3: Forecast Error and Fourier Risk Premium




For robustness check, Figure 4 compares the Fourier risk premium to the risk premium estimated by applying the Hodrick–Prescott (HP) filter to $u_t$ with the default tuning parameter for monthly data $\lambda = 144000$. The key message from Figure 4 is that the Fourier estimate of risk premium is largely consistent with the HP estimate: the magnitudes of the peak and trough are similar, and more importantly, locations of turning points match well.

**Figure 4: FFF and HP Risk Premia**

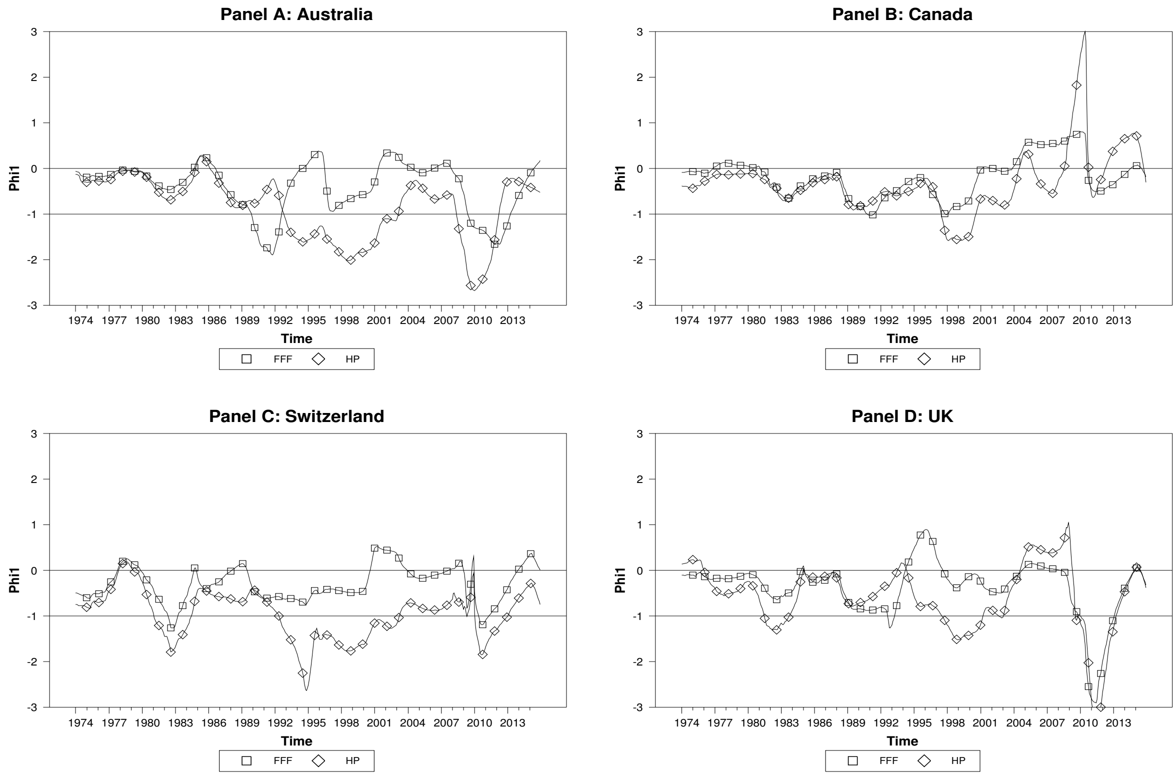



After risk premium is estimated by the Fourier approximation and HP filter, it is straightforward to verify the omitted-variable explanation of UIP puzzle—we only need to compare the covariance between the estimated risk premium and interest differential to zero (to explain $\hat{\beta}_1 < 1$), and the negative variance of interest differential (to explain $\hat{\beta}_1 < 0$). Equivalently, we can run the following auxiliary regression using the Fourier and HP estimates of the risk premium as dependent variables:

$$\hat{\theta}_{j,t} = \phi_0 + \phi_1 ID_t + \eta_t \qquad (j = \texttt{FFF, HP}) \tag{9}$$

where $\phi_1 = \frac{\texttt{cov}(\hat{\theta}_{j,t}, ID_t)}{\texttt{var}(ID_t)}$. It follows that the slope coefficient $\phi_1$ is negative if the covariance in the numerator is negative; while $\phi_1$ is less than $-1$ if the covariance is less than the negative variance of $ID_t$. Figure 5 plots $\hat{\phi}_1$ using the same rolling windows as Figure 2. We find that a majority of $\hat{\phi}_1$ are negative, supporting Fama's explanation of $\hat{\beta}_1 < 1$. Nevertheless, the evidence for Fama's explanation of $\hat{\beta}_1 < 0$ is not as strong as the explanation for $\hat{\beta}_1 < 1$ (see Panel B, Canada, in particular).

**Figure 5: Verifying Fama's Hypothesis**




# Section 5. Conclusion

The main contribution of this study is advocating a semi-parametric approach of estimating the risk premium based on the Flexible Fourier Form. The proposed methodology can be applied provided that the risk premium is smooth. We use the estimated risk premium to directly test the theory of Fama (1984) that the omission of risk premium can explain the uncovered interest parity puzzle. The covariance between the interest differential and the estimated risk premium is found to be mostly negative, which explains why the Fama beta coefficient is usually less than unity. However, there is less compelling evidence of the covariance being less than the negative variance of interest differential, which is needed to explain why the Fama coefficient is sometimes negative.

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