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Stability and Contractual Efficiency in Matching with Contracts and Lexicographic Preferences

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Abstract

We introduce an efficiency concept in matching markets with contracts called contractual efficiency. Contractual efficiency requires that each student is assigned to a school under her most preferred contractual term. We show that while in general it is not possible to have stable and contractually efficient matchings; if the preferences of each school are lexicographic, then there is a contractually efficient and stable matching. Moreover, we provided an algorithm, the Best-Term Deferred Acceptance (Best-Term DA) algorithm, that produces a contractually efficient and stable matching whenever schools' preferences are lexicographic. Finally, we turn to the question of whether a contractually efficient and stable matching can be implemented in dominant strategies. We show that in a two-stage matching market whereby contractual terms can be interpreted as pre-matching investments -each student first chooses her investment and then students and schools match according to the School-Optimal Stable Matching-, it is a dominant strategy for each student to choose the investment associated with the outcome of the Best-Term DA algorithm and the outcome of the two-stage market is precisely the outcome of the Best-Term DA algorithm.

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1 Introduction

We consider a one-to-one version of the matching with contracts model introduced by Hatfield and Milgrom (2005). Contracts are a way to model real-world matching markets in which individuals have preferences not only over their partners, but also over the contractual terms under which they are hired. Some applications of this model are the assignment of workers to firms when firms pay (discrete) salaries (Kelso and Crawford, 1982), the matching of U.S. Military Academy graduates to military branches where contracts differ by term length (Sönmez, 2013; and Sönmez and Switzer, 2013) and the participation of married couples in the market (Hatfield and Kojima, 2009; and Tello, 2016).

We interpret the model as a school choice problem: there are finite sets of students, schools and contractual terms. Each student can be matched to at most one school under the terms of an available contract. Each school is endowed with a single position. Each student has strict preferences over contracts, and each school has a strict priority ranking over the same contracts. Since the model we consider is one-to-one, there is a symmetry between students and schools, therefore we often refer to priority rankings as "schools' preferences", but it is important to keep in mind the interpretation of schools as resources that exist primarily to be of use for students and thus they are not considered neither in welfare nor in strategic notions.

In this note we introduce the concept of contractually efficient matchings. A matching is contractually efficient from the point of view of students, if each student gets the contractual term she prefers the most among all contractual terms between her and the school she is matched with. Contractual efficiency emphasizes the notion that schools are passive agents. To the best of our knowledge this is the first paper dealing with this efficiency concept.

Additionally to contractual efficiency, we focus on stability. Stability is a central concept in the matching literature. Specifically, a matching is stable if a student is denied a contract only if the school in question has filled its position with a higher priority contract.

Pakzad-Hurson (2014), introduced a restriction on each schools' preferences, called lexicographic preferences. Lexicographic preferences restrict the rankings within each school. A school's preferences are lexicographic if it orders all contracts associated with each student consecutively on its preference list. In this note we show that: if each school's preferences are lexicographic, then there is always a stable and contractually efficient matching. Our proof is constructive, we provide a new algorithm, the Best-Term Deferred Acceptance (Best-Term DA) algorithm that produces such a matching whenever schools' preferences are lexicographic. This algorithm reduces to the Gale and Shapley (1962) School-Proposing Deferred Acceptance algorithm when there is a single contract between each student and school.

Pakzad-Hurson (2014) argues that while restrictive there are real life examples of lexicographic preferences. His main example are admission processes in the United States, Canada and Scotland where students often declare an intended major in their applications, only to have the option to choose any desired major once enrolled.

Our second main result regards the possibility of implementing a stable and contractually efficient matching via a game form. Consider a two-stage game in which students first choose their contractual term, and then students and schools are matched according to the Gale and Shapley (1962) School-Optimal Stable Matching. Here we interpret contractual terms as investments that take place before a matching stage, so that the game has the interpretation of a prematching investment game. Our results state that (i) for each student, it is a dominant strategy to choose the investment prescribed by the matching produced by the Best-Term DA algorithm, and (ii) the outcome of the game coincides with such a matching.

Several authors have studied the conflict between stability and "efficiency" in several types of matching markets. Without contracts, Ergin (2002) shows that stability and efficiency are incompatible when school priorities have cycles between students which result in one student blocking a "trade" between two others. It is the domain of acyclic priorities that ensures stability, efficiency, and group strategy-proofness for Deferred Acceptance (Ergin, 2002). Unfortunately, this acyclicity condition is very restrictive, as cycles are often present in real-world priority structures.

Haeringer and Klijn (2009) show that acyclicity is a necessary and sufficient condition for Deferred Acceptance to result in stable and efficient equilibrium outcomes when students cannot list all schools in their submitted preferences. Pakzad-Hurson (2014) generalizes the results of Ergin (2002) to the many-to-one matching with contracts model, giving the maximal domain of priorities over which Deferred Acceptance is stable, efficient and group strategy-proof. This domain is given by priority structures that satisfy a generalization of Ergin acyclicity and lexicographicity.

To properly place our paper with respect to the close literature described above, we remark that our concept of efficiency (contractual efficiency) is weaker than Pareto efficiency, as we only require that any student gets the best contract at the school she gets admitted. Therefore, the trade off is only within the school not across schools. This points to the fact that in order to obtain stability and contractual efficiency we do not need to impose acyclicity restrictions *a là* Ergin that prevent "conflicts" across schools, but only a lexicographic restriction *a là* Pakzad-Hurson that prevents "conflicts" within each school.

2 Model

Let J and S be two disjoint sets of **students** and **schools**. There is a finite set T that contains all possible **contractual terms** that may exist between each student $j \in J$ and each school $s \in S$. Let $N = J \cup S$ denote the set of agents. A contract specifies a partnership between a student and a school under some contractual term. Formally, a **contract** is a triplet $(j, s, t) = x \in X$ where

$$X \subseteq \mathcal{X} \equiv J \times S \times T.$$

Additionally, there is a "null contract" which represents the prospect of holding no contract (or some outside option), and is denoted by $\varnothing$.

We only consider sets of contracts that contain at least one contract between each student and each school. Moreover, we assume that no agent (student or school) is assigned more than one contract.

Let $X \subseteq \mathcal{X}$ be a set of contracts. We write $j(x)$, $s(x)$ and $t(x)$ to denote the student, school and contractual term associated with contract $x \in X$, respectively. For each $X \subseteq \mathcal{X}$,

$$X(j) \equiv \{x \in X : \ j(x) = j\} \ \text{ and } \ X(s) \equiv \{x \in X : \ s(x) = s\}$$

denote the sets of contracts within X involving student j and school s, respectively.

For any $X \subseteq \mathcal{X}$, each agent $i \in N$ has a complete, transitive and strict **preference relation** P_i over the set $X(i)$ and the null contract $\varnothing$. For simplicity we refer to schools' priority rankings as schools' preferences. For $x, x' \in X(i) \cup \{\varnothing\}$ we write $x \, P_i \, x'$ if agent i prefers x to x' ($x \neq x'$), and $x \, R_i \, x'$ if i finds x at least as good as x', i.e., $x \, P_i \, x'$ or $x = x'$. If $x \in X(i)$ is such that $x \, P_i \, \varnothing$, then we call x an **acceptable** contract for agent i. We denote profiles of students' and profiles of schools' preferences by $P_J = (P_j)_{j \in J}$ and $P_S = (P_s)_{s \in S}$, respectively. Let $P = (P_J, P_S)$ be a preference profile. We represent agents' preferences by ordered lists of contracts; for example, $P_s \; : \; (j, s, t), \, (j', s, t'), \, \varnothing \ldots$ means that (j, s, t) is s's most preferred contract, (j', s, t') is s's second most preferred contract and any other contract is unacceptable to s.

We fix J, S and T. Therefore, a **market** is completely described by a non-empty set of contracts $X \subseteq \mathcal{X}$ and a preference profile P. We denote a market by a pair (X, P).

A **matching** μ for (X, P) is a mapping from N to X such that

(i) for each $i \in N, \;\; \mu(i) \in X(i)$,

(ii) for each $j \in J$ and each $s \in S$, if $x \in X(j) \cap X(s)$, then $\mu(j) = x$ if and only if $\mu(s) = x$.

A matching μ is individually rational if no agent would be better off by breaking a current contract. Formally, a matching μ is **individually rational** if for each $i \in N$, $\mu(i) \, R_i \, \varnothing$. A matching μ is **blocked** by $j \in J, \;\; s \in S$ and $x \in X(j) \cap X(s)$ if (1) $x \, P_j \, \mu(j)$ and (2) $x \, P_s \, \mu(s)$. A matching is **stable** if it is individually rational and it is not blocked.

Finally, a matching μ is **contractually efficient**, if for each student $j \in J, \;\; \mu(j)$ is an acceptable contract for j and it is the best contract for j among all contracts between j and $s(\mu(j))$.

The next example illustrates that in general stable matchings are not contractually efficient.

Example 1 (*Tension between Stability and Contractual Efficiency*)**.** Consider a market with $J = \{j_1, \, j_2\}, \;\; S = \{s_1, \, s_2\}, \;\; T = \{t_1, \, t_2\}$, and preferences P given by the columns in Table 1. Vertical dots mean that preferences can be arbitrary.

Table 1: Preferences P in Example 1

P_{s_1}	P_{s_2}	P_{j_1}	P_{j_2}
$\boxed{(j_1, \, s_1, \, t_1)}$	$\boxed{\boldsymbol{(j_2, \, s_2, \, t_2)}}$	$\boldsymbol{(j_1, \, s_1, \, t_2)}$	$(j_2, \, s_1, \, t_2)$
$(j_2, \, s_1, \, t_1)$	$\vdots$	$\boxed{(j_1, \, s_1, \, t_1)}$	$(j_2, \, s_1, \, t_1)$
$\boldsymbol{(j_1, \, s_1, \, t_2)}$		$(j_1, \, s_2, \, t_2)$	$\boxed{\boldsymbol{(j_2, \, s_2, \, t_2)}}$
$(j_2, \, s_1, \, t_2)$		$(j_1, \, s_2, \, t_1)$	$(j_2, \, s_2, \, t_1)$

There is a unique stable matching at $(\mathcal{X}, P)$ given by

$$\mu : \begin{array}{cc} j_1 & j_2 \\ | & | \\ s_1 & s_2 \\ | & | \\ t_1 & t_2 \end{array}$$

which is the boxed matching in Table 1. This can be verified by running the student and the school proposing DA algorithms with input $(\mathcal{X}, P)$ and observing that the outcome of both algorithms is the same[1]. However, matching μ is not contractually efficient. The reason being that student j_1 can be better off by going to s_1 under contractual term t_2, this is shown with the boldface matching in Table 1.

3 Results

We assume that the preferences of each school are lexicographic, that is, each school ranks the contracts involving some student consecutively. Formally, the preferences of a school $s \in S$ are **lexicographic** if for each acceptable student $j \in J$ and any two contracts $x, y \in X(j) \cap X(s)$ with $x \, P_s \, y$, there is no $z \in \{X \cup \{\varnothing\}\} \setminus X(j)$ such that $x \, P_s \, z \, P_s \, y$.

We present an algorithm that, under lexicographic preferences, is well defined and always produces a matching that is both stable an contractually efficient.

First we describe the algorithm, then we will present a series of lemmas that complete the proof.

3.1 Best-Term Deferred Acceptance algorithm

We proceed to describe the Best-Term DA algorithm. Let $X \subseteq \mathcal{X}$ be a non empty set of contracts. For each $i \in N$, let $\mathsf{Ch}_i(X, P_i)$ be i's most preferred contract in $X(i)$, i.e.,

$$\mathsf{Ch}_i(X, P_i) = \{x \in X(i) \,|\, x \, P_i x' \text{ for all } x' \in X(i) \setminus \{x\}\}.$$

When it is clear from the context we suppress the dependence of Ch_i on P_i.

Best-Term DA Algorithm
Input: A market (X, P) such that P_s is lexicographic for each $s \in S$.
Step 1: An arbitrary school $s_1 \in S$ proposes her most preferred contract in $X(s_1)$. This contract involves some student say $j_1 \in J$. Let student j_1 hold her most preferred contract among contracts involving j_1 and s_1 and the null contract $\varnothing$. Denote such contract by x_1. Set $y_2(j_1) = x_1$ and $y_2(j) = \varnothing$ for each $j \neq j_1$.
Step *k*: Let I_k be the set of schools involved in a contract which is held by any student after Step $k-1$. An arbitrary school $s_k \in S \setminus I_k$ proposes her most preferred acceptable contract $x' \in X(s_k)$ which she has not proposed in a previous step. This contract involves some student $j_k \in J$. Let student j_k hold the contract

$$x_k = \mathsf{Ch}_{j_k}\Big(\varnothing \,\cup\, \big\{y_k(j_k)\big\} \,\cup\, \big\{X(j_k) \,\cap\, X(s_k)\big\}\Big),$$

and reject the other (if any). All other $j \neq j_k$ continue to hold the contract they held at the end of Step $k-1$. Set $y_{k+1}(j_k) = x_k$ and set $y_{k+1}(j) = y_k(j)$ for each $j \neq j_k$.

The algorithm terminates at some step K when no school proposes any new contract. The function $\mu(j) = y_K(j)$ gives the final matching.

Lemma 1. The Best-Term DA algorithm finishes in a finite number of steps.

[1]A description of the DA algorithm can be found in the Appendix.

Proof. At each step of the algorithm a school proposes a contract that she has not proposed yet. Since there is a finite number of schools and a finite number of contracts, the algorithm must finish in a finite number of steps. □

Lemma 2. The Best-Term DA algorithm always produces a matching.
Proof. By construction no student holds more than one contract. Then by the definition of a matching, this implies that also no school holds more than one contract. □

Lemma 3. The matching produced by the Best-Term DA algorithm is stable.
Proof. Let μ be the matching resulting from the algorithm. First, we show that μ is individually rational. Suppose that at some step of the algorithm a school s proposes some contract involving a student j. Since s's preferences are lexicographic the contract chosen by j must also be acceptable to s. Clearly such contract would also be acceptable to j. Now we show that μ cannot be blocked. Suppose there are a student $j \in J,$ a school $s \in S$ and a contract $x \in X(j) \cap X(s)$ such that (1) $x \, P_j \, \mu(j)$ and (2) $x \, P_s \, \mu(s)$. Since $x \, P_s \, \mu(s)$, there is some step in the algorithm in which s proposed x. Now we analyze two cases:

Case 1: $\mu(j)$ involves s. This contradicts that j chose her most preferred contract in $X(j) \cap X(s)$.

Case 2: $\mu(j)$ does not involve s. Since $x \, P_s \, \mu(s)$, there is some step k in the algorithm in which s proposed x. Thus we have $y_k(j) \, R_j \, x \, P_j \, \mu(j)$. Since the welfare of student j weakly increases at each step of the algorithm, we have that at the end of the algorithm (step K), $y_K(j) \, P_j \, \mu(j)$ which is a contradiction. □

Lemma 4. The matching produced by the Best-Term DA algorithm is contractually efficient for students.
Proof. By construction of the algorithm. □

Proposition 1. If the preferences of each school are lexicographic, then there is a stable and contractually efficient matching.
Proof. Lemmas 1 to 4 imply that if the preferences of each school are lexicographic, then there is a matching that is contractually efficient and stable. ■

3.2 An investment game

Consider a two-stage market where students first choose their contractual term, and then students and schools match according to the **School-Optimal Stable Matching**. Here we can think of elements of T as pre-matching investments or training taken prior to (the realization of) the matching. Following this interpretation we will refer to contractual terms as investments.

An investment profile $t = \{t_j\}_{j \in J}$ induces a set of contracts

$$X(t) \equiv \{(j, s, t_j) \in \mathcal{X} \, : \, j \in J \text{ and } s \in S\}.$$

We say that $(X(t), P)$ is the market induced by the investment profile t. In this market all available contracts for a student j involve the same investment t_j. Therefore, this market represents a situation where students' investments are fixed in the matching stage.

In the second stage, students and schools are matched according to the **School-Optimal Stable Matching** μ^S. Formally, for any matching market, μ^S is the stable matching that is weakly preferred for each school to any other stable matching and strictly preferred for some school to any other stable matching (Gale and Shapley, 1962). Students, in anticipation to the matching stage, choose investments strategically. Formally, they play a complete information normal form game $\Gamma(P) = (J, T, P)$ where J is the set of players and T is the set of strategies for each player. Given an investment (strategy) profile t the outcome of this game is the School-Optimal Stable Matching for market $(X(t), P)$. Each student j evaluates the outcome according to her true preferences P_j.

For each student $j \in J$ the investment t_j is a best response to the investments of all other students $t_{J\setminus\{j\}}$ if $\mu'(j)\, P_j\, \mu(j)$, where μ is the School-Optimal Stable Matching at $(X(t), P)$ and μ' is the School-Optimal Stable Matching at $(X(t'_j,\, t_{J\setminus\{j\}}), P)$. For each student $j \in J$ an investment $t_j \in T$ is a dominant strategy in the game $\gamma(P)$ if t_j is a best response to any profile $t_{J\setminus}\{j\}$.

In what follows we introduce and fix the following notation. Fix a market (X, P). Let μ^* denote the matching produced by the Best-Term DA algorithm for input (X, P), and let $t^* = \Big\{t\big(\mu^*(j)\big)\Big\}_{j\in J}$ be the investment profile associated with matching μ^*.

Proposition 2. Suppose each school has lexicographic preferences. If students choose t^* in the first stage, then the outcome of the two-stage market is μ^*.

Proof. Since investments were sunk in the first stage, there is a single contract between any student and school. Since schools' preferences are lexicographic, the way each school orders students is independent of the investment chosen by each student in the first stage. Therefore, the Best-Term DA algorithm reduces to the standard School-Proposing Deferred Acceptance algorithm, which delivers the School-Optimal Stable Matching. Thus the partnership between students and schools must be the same under the Best-Term DA algorithm than under the two-stage market. ■

Proposition 3. Suppose each school has lexicographic preferences. Then, it is a dominant strategy for each student $j \in J$ to choose the investment associated with the matching μ^*.

Proof. Since schools preferences are lexicographic, changing investment in the first stage does not change the match in the second stage. Therefore, each student can anticipate the school she will match in the second stage. This implies that for a given student, independently of the investments of all other students, the best she can do is to choose the investment she likes the most given the school she will be matched with. ■

4 Conclusions

The matching with contracts model is capable to describe the problem of matching students and schools at the same time it takes into account the more realistic feature of several possible contractual terms between each student and each school.

One of the most important contributions in this note is the introduction of a new efficiency concept: contractual efficiency. A matching is contractually efficient, if each student gets the contractual term she prefers the most among all contractual terms between her and the school she is matched with.

Taking stability as another critical desiderata, we are concerned with the possibility of assigning students (workers, doctors) to schools (firms, hospitals) in a stable way at the same time we make sure that each student is admitted under her most preferred contractual term. An example for medical students arises when at a given hospital there are several specialties and we wish that each medical student is hired by the hospital to her specialty of expertise. In this paper we show that while in general, it is not possible to have stable and contractually efficient matchings; if the preferences of each school are lexicographic (Pakzad-Hurson, 2014), then there is a stable and contractually efficient matching. Moreover, we provided an algorithm, the Best-Term DA algorithm, that produces a contractually efficient and stable matching, whenever each school's preferences are lexicographic. Finally, we show that a contractually efficient and stable matching can be implemented via a game form in dominant strategies. We show that in a game where contractual terms can be interpreted as prematching investments -each student first choose her investment and then students and schools are matched according to the School-Optimal Stable Matching-, it is a dominant strategy for each student to choose the investment associated with the outcome of the Best-Term DA algorithm, and the outcome of the game is precisely the outcome of the Best-Term DA algorithm.

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Appendix

Deferred acceptance

We describe a Student-Proposing Deferred Acceptance algorithm which is a generalization of Gale and Shapley's (1962) Deferred Acceptance algorithm to markets with contracts. Fleiner (2003) and Hatfield and Milgrom (2005) show that this algorithm produces a stable matching that Pareto dominates any other stable matching from the point of view of students. For the School-Proposing Deferred Acceptance one just have to reverse the order of schools and students. While in our paper we just need the description of the algorithm for one-to-one markets, here we give a more general description for many-to-one markets. The description of the algorithm below is based on Pakzad-Hurson (2014).

The Student-Proposing Deferred Acceptance (DA) algorithm

Input: A market (X, P).

Step 1: An arbitrary student $j_1 \in J$ proposes her most preferred contract in $X(j_1)$. This contract involves some school say $s_1 \in H$. Let school s_1 hold contract x_1. Set $y_2(h_1) = x_1$ and set $y_2(h) = \varnothing$ for each $s \neq s_1$.

Step *k*: Let I_k be the set of students involved in a contract which is held by any school after Step $k-1$. An arbitrary student $j_k \in J \setminus I_k$ proposes her most preferred contract $x_k \in X(j_k)$ which he has not proposed in a previous step. This contract involves some school $s_k \in H$. school s_k holds the contract $x \in \mathsf{Ch}_s\big(\{y_k(s_k)\} \cup \{x_k\}\big)$, and rejects the other (if any). All other $s \neq s_k$ continue to hold the contract they held at the end of Step $k-1$. Set $y_{k+1}(s_k) = \mathsf{Ch}_s(\{y_k(s_k)\} \cup \{x_k\})$ and set $y_{k+1}(s) = y_k(s)$ for each $s \neq s_k$.

The algorithm terminates at some step K when no student proposes any new contract. The function $\mu(s) = y_K(s)$ gives the final matching and this matching is called the **Student-Optimal Stable Matching** at (X, P).

The School-Proposing DA algorithm is defined symmetrically by exchanging the roles of students and schools in the Student-Proposing DA algorithm. Hatfield and Milgrom (2005) show that the School-Proposing DA algorithm produces a stable matching at (X, P) that is Pareto dominated from the point of view of students by any other stable matching at (X, P) i.e., the School-Optimal Stable Matching for market (X, P).