Economics Bulletin

Volume 42, Issue 1

A decomposition of strategy-proofness in discrete resource allocation problems

Hidekazu Anno
University of Tsukuba

Sui Takahashi
University of Tsukuba

Abstract

This note provides a characterization of strategy-proofness in discrete resource allocation problems. Based on it, we establish a theorem that has the following two corollaries: (i) the student-proposing deferred-acceptance (DA) rule for college admission problems is strategy-proof for students and (ii) the top-trading cycles (TTC) rule for housing markets is strategy-proof.

We would like to thank Morimitsu Kurino for his useful comments. Hidekazu Anno gratefully acknowledges JSPS KAKENHI Grant Number 18K12741. All remaining errors are our own.
Citation: Hidekazu Anno and Sui Takahashi, (2022) "A decomposition of strategy-proofness in discrete resource allocation problems", *Economics Bulletin*, Vol. 42 No. 1 pp. 49-59.
Contact: Hidekazu Anno - anno.hidekazu.fp@u.tsukuba.ac.jp, Sui Takahashi - s2020426@s.tsukuba.ac.jp.
Submitted: October 18, 2021. **Published:** February 20, 2022.

1 Introduction

In this paper, we provide a necessary and sufficient condition for a rule in discrete resource allocation problems to be strategy-proof for a fixed group of the market participants. The condition is a combination of three invariance properties under preference transformation. The characterization enable us to provide an elementary and transparent proof for strategy-proofness of one-sided optimal core rules for a class of generalized indivisible goods allocation problems (Sönmez, 1999). Since our model includes both college admission problems and housing markets, the theorem implies both (i) strategy-proofness for students of the student-proposing deferred-acceptance (DA) rule and (ii) strategy-proofness of the top-trading cycles (TTC) rule.[1] Moreover, the proof technique also works to prove strategy-proofness of the cumulative-offer process rule for matching with contracts under weakened substitutes conditions (Hatfield and Kojima, 2010).

2 Model and a characterization of strategy-proofness

We describe a version of the model in Sönmez (1999).[2] A generalized indivisible goods allocation problem (GIGAP) is a 4-tuple $(\mathcal{N}, \omega, \mathcal{A}^f, R)$: $\mathcal{N} = \{1, \ldots, n\}$ denotes the set of agents. For each $i \in \mathcal{N}$, let $\omega(i)$ be the object initially owned by i. For each $S \subseteq \mathcal{N}$, let $\omega(S) := \{\omega(i) | i \in S\}$. An allocation is a function from $\mathcal{N}$ to $2^{\omega(\mathcal{N})}$. Throughout the paper, fix a non-empty subset of agents $N \subseteq \mathcal{N}$ such that each member of N is necessarily unit-demand.[3] Let $\mathcal{A} := \{a : \mathcal{N} \to 2^{\omega(\mathcal{N})} \mid \forall i \in N, |a(i)| = 1\}$. For each $a \in \mathcal{A}$, we call $a(i)$ the assignment of $i \in \mathcal{N}$ at a. We abuse the notation $a(i)$ for $i \in N$ to denote both a singleton and the object contained in it, interchangably. The set of feasible allocations, $\mathcal{A}^f$, is a subset of $\mathcal{A}$. We assume that $\mathcal{A}^f$ contains an allocation such that each $i \in N$ receives her endowment $\omega(i)$. For each $i \in \mathcal{N}$, letting $X_i := \{a(i) \mid a \in \mathcal{A}^f\}$, let $\mathcal{R}_i$ be the set of complete, transitive and anti-symmetric binary relations on X_i.[4] Let $\mathcal{D}_i \subseteq \mathcal{R}_i$ be the

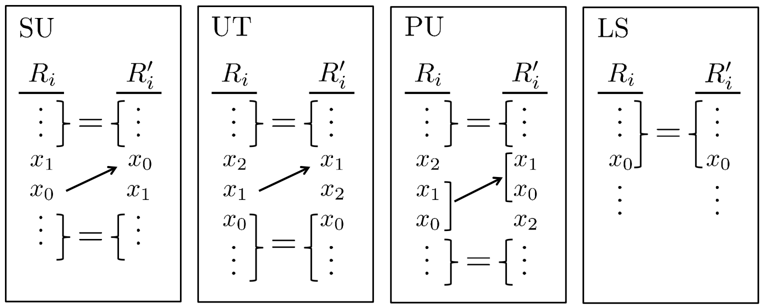
[1] Dubins and Freedman (1981) and Roth (1982a) first show that the men-proposing (resp. women-proposing) DA rule is strategy-proof for men (resp. women). Gale and Sotomayor (1985) and Hatfield and Milgrom (2005) provide an alternative proof for this result. The corresponding result for the TTC rule in housing market is first proved by Roth (1982b).

[2] The difference is summarized as follows. First, our model excludes externality and indifference in agents' preferences. Thus, our setup is a special case of Sönmez's model in this line. On the other hand, our model admits that some agents consume multiple units of objects while the Sönmez's setup focuses on the one-to-one cases. The main target of our model is college admission problems and housing markets described in Remark 1.

[3] In concrete problems, an interpretation is given to N. For example, N denotes the set of students in college admission problems. In housing markets, $N (= \mathcal{N})$ denotes the set of all agents in the economy.

[4] A binary relation R_i on X_i is complete *if* for each $\{x, y\} \subseteq X_i, x \; R_i \; y$ or $y \; R_i \; x$. A binary relation R_i on X_i is transitive *if* for each $\{x, y, z\} \subseteq X_i, [x \; R_i \; y$ and $y \; R_i \; z] \Rightarrow x \; R_i \; z$. A binary relation R_i on X_i is

Figure 1: Four types of preference transformation



Note: In each box, R_i' denotes the (a) transformed preference of R_i at x_0. The type of transformation is indicated on the upper left corner of the box.

set of feasible preference relations of i. We assume that $\mathcal{D}_i = \mathcal{R}_i$ for each $i \in N$. Letting $\mathcal{D} := \prod_{i \in \mathcal{N}} \mathcal{D}_i$, the symbol R denotes a preference profile belonging to $\mathcal{D}$. For notational simplicity, given $R = (R_1, \ldots, R_n) \in \mathcal{D}$ and $i \in N$, it is convenient to use the notation R_{-i} to represent the $(n-1)$-fold preference profile obtained by deleting R_i from R. Without any confusion, we use the notation (R_i, R_{-i}) to denote R even when $i \neq 1$. For each $i \in N$, each $x \in X_i$, and each $R_i \in \mathcal{D}_i$, let $UC(R_i, x) := \{y \in X_i | y \ R_i \ x\}$, $SUC(R_i, x) := UC(R_i, x) \backslash \{x\}$ and $r_{R_i}(x) := |UC(R_i, x)|$.

Hereafter, we fix $\mathcal{N}, \omega$ and $\mathcal{A}^f$. Thus a GIGAP, or a problem for short, is identified with a preference profile $R \in \mathcal{D}$.

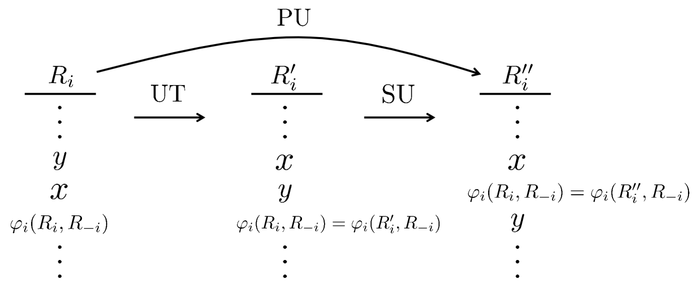
Remark 1. **A college admission problem** is a special case of GIGAPs. Let S and C be disjoint sets of students and colleges. Namely, the set of agents in this problem is $S \cup C$. Let $\omega(i) := i$ for each $i \in S \cup C$. Assume that a capacity vector $(q_c)_{c \in C} \in \mathbb{Z}^C_{++}$ is given. Assume also that for each $c \in C$, the set of feasible preferences of college c consists of responsive preferences on 2^S (Roth and Sotomayor, 1990). Let $\mathcal{A}^f := \{a \in \mathcal{A} | \forall (s, c) \in S \times C, a(s) \in C \cup \{s\}$ and $a(c) \in 2^S$ with $|a(c)| \leq q_c$ and $a(s) = c \Leftrightarrow a(c) \ni s\}$. Obviously, any college admission problem can be represented by the above specification along with a preference profile. **A housing market** is also a special case of GIGAPs. Letting $\omega(i) := i$ for each $i \in \mathcal{N}$, assume that $N = \mathcal{N}$. Let $\mathcal{A}^f = \mathcal{A}$. Obviously, any housing market can be represented by the above specification along with a preference profile. ◊

A rule is a function from $\mathcal{D}$ to $\mathcal{A}^f$. Our generic notation for a rule is φ. *Strategy-proofness* for N requires that no agent in N can profitably manipulate a rule by misreporting her preferences. Formally, a rule φ is **strategy-proof for N** *if* for each $i \in N$, each $(R_i, R_{-i}) \in \mathcal{D}$, and each $R_i' \in \mathcal{D}_i, \varphi_i(R_i, R_{-i}) \ R_i \ \varphi_i(R_i', R_{-i})$. To provide a characterization

anti-symmetric *if* for each $\{x, y\} \subseteq X_i, [x \ R_i \ y$ and $y \ R_i \ x] \Rightarrow x = y$.

Given $i \in \mathcal{N}$ and a preference relation R_i, the anti-symmetric part of R_i is denoted as P_i.

Figure 2: Proof of Remark 2.



of this property, we define four types of preference transformation. Let $i \in N, R_i \in \mathcal{D}_i$ and $x_0 \in X_i$.

- Assume that $r_{R_i}(x_0) \geq 2$. Let $x_1 \in X_i$ be such that $r_{R_i}(x_1) = r_{R_i}(x_0) - 1$. The preference $R'_i \in \mathcal{D}_i$ satisfying the following condition is called **the single-upgrade (SU) of $\mathbf{R_i}$ at $\mathbf{x_0}$**: For each $x \in X_i$, $r_{R'_i}(x) = r_{R_i}(x) - 1$ *if* $x = x_0, r_{R'_i}(x) = r_{R_i}(x) + 1$ *if* $x = x_1$, and $r_{R'_i}(x) = r_{R_i}(x)$ *if* $x \notin \{x_0, x_1\}$.

- Assume that $r_{R_i}(x_0) \geq 3$. Let $x_1 \in X_i$ and $x_2 \in X_i$ be such that $r_{R_i}(x_1) = r_{R_i}(x_0) - 1$ and $r_{R_i}(x_2) = r_{R_i}(x_0) - 2$. The preference $R'_i \in \mathcal{D}_i$ satisfying the following condition is called **the upper-transposition (UT) of $\mathbf{R_i}$ at $\mathbf{x_0}$**: For each $x \in X_i$, $r_{R'_i}(x) = r_{R_i}(x) - 1$ *if* $x = x_1, r_{R'_i}(x) = r_{R_i}(x) + 1$ *if* $x = x_2$, and $r_{R'_i}(x) = r_{R_i}(x)$ *if* $x \notin \{x_1, x_2\}$.

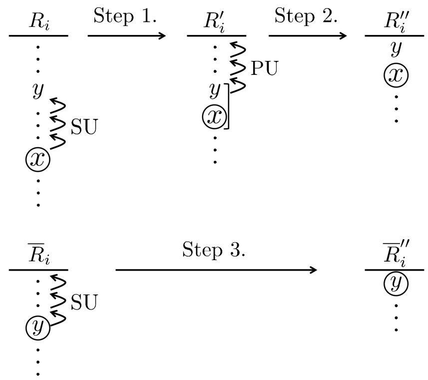
- Assume that $r_{R_i}(x_0) \geq 3$. Let $x_1 \in X_i$ and $x_2 \in X_i$ be such that $r_{R_i}(x_1) = r_{R_i}(x_0) - 1$ and $r_{R_i}(x_2) = r_{R_i}(x_0) - 2$. The preference $R'_i \in \mathcal{D}_i$ satisfying the following condition is called **the pairwise-upgrade (PU) of $\mathbf{R_i}$ at $\mathbf{x_0}$**: For each $x \in X_i$, $r_{R'_i}(x) = r_{R_i}(x) - 1$ *if* $x \in \{x_0, x_1\}, r_{R'_i}(x) = r_{R_i}(x) + 2$ *if* $x = x_2$, and $r_{R'_i}(x) = r_{R_i}(x)$ *if* $x \notin \{x_0, x_1, x_2\}$.

- A preference $R'_i \in \mathcal{D}_i$ is called **a lower-shuffle (LS) of $\mathbf{R_i}$ at $\mathbf{x_0}$** *if* for each $x \in UC(R_i, x_0), r_{R'_i}(x) = r_{R_i}(x)$.

We say that a rule φ satisfies the **single-upgrade invariance (SUI)** (resp. **upper-transposition invariance (UTI)**, **pairwise-upgrade invariance (PUI)**, **lower-shuffle invariance (LSI)**) **on N** *if* for each $i \in N$ and each $(R_i, R_{-i}) \in \mathcal{D}$, the assignment of i does not change when she changes her reporting to the SU (resp. the UT, the PU, a LS) of R_i at $\varphi_i(R_i, R_{-i})$. Note that SU, UT, PU and LS are special cases of Maskin monotonic transformation (Maskin, 1999). Thus, so-called **Maskin monotonicity** implies SUI, UTI, PUI and LSI. However, the converse is not true in general because our properties require invariance of the individual assignment while Maskin monotonicity does the allocation.

Note that the PU at the selected object is obtained by the combination of the UT and SU at the same object (Figure 2).

Figure 3: Proof of Theorem 1.



Note: Fixing other agents' reporting R_{-i}, the assignment for i is indicated by the circled object. In step 1, successive applications of SU to R_i leads to R'_i. Then, in step 2, successive applications of PU to R'_i leads to R''_i. In step 3, successive applications of SU to $\overline{R}_i$ leads to $\overline{R}''_i$. Since R''_i is a LS of $\overline{R}''_i$ at y, $\varphi_i(\overline{R}''_i, R_{-i}) = \varphi_i(R''_i, R_{-i})$, a contradiction.

Remark 2. Suppose that a rule φ satisfies SUI and UTI on N. Then, φ satisfies PUI on N.

Theorem 1. *A rule φ is strategy-proof for N if and only if φ satisfies SUI, UTI and LSI on N.*

Proof. We only show the sufficiency part. Suppose to the contrary that there are $i \in N$, $(R_i, R_{-i}) \in \mathcal{D}$ and $\overline{R}_i \in \mathcal{D}_i$ such that $\varphi_i(\overline{R}_i, R_{-i})\ P_i\ \varphi_i(R_i, R_{-i})$. Let $x := \varphi_i(R_i, R_{-i})$ and $y := \varphi_i(\overline{R}_i, R_{-i})$. Notice that $x \neq y$.

Step 1. If $r_{R_i}(y) = r_{R_i}(x) - 1$, let $R'_i := R_i$. Otherwise, apply SU to R_i at x succesively until x becomes the immediate successor of y. Then, define R'_i as the preference found at the very last step. Since φ satisfies SUI on N, $\varphi_i(R'_i, R_{-i}) = x$.

Step 2. If $r_{R'_i}(y) = 1$, let $R''_i := R'_i$. Otherwise, apply PU to R'_i at x succesively until y becomes the most preferred object. Then, define R''_i as the preference found at the very last step. Since φ satisfies PUI on N ($\because$ Remark 2), $\varphi_i(R''_i, R_{-i}) = x$.

Step 3. If $r_{\overline{R}_i}(y) = 1$, let $\overline{R}''_i := \overline{R}_i$. Otherwise, apply SU to $\overline{R}_i$ at y succesively until y becomes the most preferred object. Then, define $\overline{R}''_i$ as the preference found at the very last step. Since φ satisfies SUI on N, $\varphi_i(\overline{R}''_i, R_{-i}) = y$.

Note that R''_i is a LS of $\overline{R}''_i$ at y. Since φ satisfies LSI on N, $x = \varphi_i(R''_i, R_{-i}) = \varphi_i(\overline{R}''_i, R_{-i}) = y$, a contradiction. □

To demonstrate the usefulness of Theorem 1, let us consider a rule in housing markets which is a variant of the so-called "priority rule". By utilizing the characterization, we show that the rule is strategy-proof.

Example 1. Suppose that $n \geq 3$. For each $R \in \mathcal{D}$, the rule φ selects the allocation obtained by the following procedure with n steps. In the first step, let $\varphi_1(R) := \omega(1)$. For $t \in \{2, \dots, n\}$, let

$$\begin{aligned} \varphi_t(R) &:= \max_{R_t} \omega(N) \backslash \{\varphi_1(R), \varphi_2(R), \dots, \varphi_{t-1}(R)\} && \text{if } \omega(2)\ P_1\ \omega(n), \textit{ and} \\ \varphi_{n-t+2}(R) &:= \max_{R_{n-t+2}} \omega(N) \backslash \{\varphi_1(R), \varphi_n(R), \dots, \varphi_{n-t+3}(R)\} && \text{if } \omega(n)\ P_1\ \omega(2). \end{aligned}$$

In words, the rule always assigns $\omega(1)$ to agent 1 in the first step. Agent 1's preference for $\{\omega(2), \omega(n)\}$ plays the switch role in the later steps. If $\omega(2)\ P_1\ \omega(n)$, agents $2, \dots, n$ choose their assignments one by one from the remaining objects in this order. On the other hand, if $\omega(n)\ P_1\ \omega(2)$, agents $2, \dots, n$ choose their assignments one by one from the remaining objects in the reverse order.

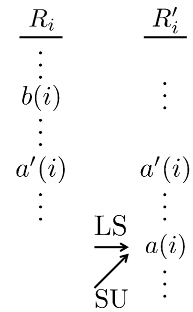
To prove the strategy-proofness of φ, we need to show that φ satisfies SUI, UTI and LSI. Let $R \in \mathcal{D}$ and $i \in N$ be arbitrary. Let $R'_i \in \mathcal{R}$ be one of the SU, UT or LS of R_i at $\varphi_i(R)$. First, suppose that $i \in N \backslash \{1\}$. Since the preference change of agent i does not affect the objects i can choose, it is clear that the best object under R'_i, which is one of the SU, UT or LS of R_i, remains the same. Thus, $\varphi_i(R'_i, R_{-i}) = \varphi_i(R)$. Finally, suppose that $i = 1$. Since $i = 1$ always receives $\omega(1)$, the SU, UT and LS of R_1 at $\varphi_1(R)$ do not change agent 1's assignment (although UT and LS may change other agents' assignments). $\diamondsuit$

Note that Theorem 1 is also useful when we show a rule is not strategy-proof. For example, the Boston mechanism in school choice problem (Abdulkadiroğlu and Sönmez, 2003) is not strategy-proof because it does not satisfy UTI. Moreover, the Japanese deceased-donor lung allocation rule (Anno and Kurino, 2017) is not strategy-proof because it does not satisfy LSI.

3 Strategy-proofness for students of the student-proposing DA rule and strategy-proofness of the TTC rule

As an application of Theorem 1, we prove that N-optimal core rules for GIGAPs are strategy-proof for N. This result implies both (i) strategy-proofness for students of the student-proposing DA rule and (ii) strategy-proofness of the TTC rule. Let $R \in \mathcal{D}$. An allocation $a \in \mathcal{A}^f$ is **blocked** at R by a coalition $S \in 2^{\mathcal{N}} \backslash \{\emptyset\}$ via an allocation $b \in \mathcal{A}^f$ *if* (i) $b(S) \subseteq \omega(S)$, (ii) $b(i)\ R_i\ a(i)$ for each $i \in S$, and (iii) $b(i)\ P_i\ a(i)$ for some $i \in S$. An allocation $a \in \mathcal{A}^f$ belongs to the **core** at R, written as $a \in \mathcal{C}(R)$, *if* it is not blocked at R by any coalition via any allocation. Letting $R_N \in \prod_{i \in N} D_i$, the common preference of N-agents,

Figure 4: For the last part of the proof of Claim 2 in Lemma 1.



written as $\geq_{R_N}$, is defined as follows: For each $\{a, a'\} \subseteq \mathcal{A}^f$, $a \geq_{R_N} a'$ *if and only if* $a(i)\ R_i\ a'(i)$ for all $i \in N$. Note that $\geq_{R_N}$ is a reflexive, anti-symmetric and transitive,[5] but not necessarily complete binary relation on $\mathcal{A}^f$.

Assumption 1. *[Existence of N-optimal core allocations]* $\forall R = (R_N, R_{-N}) \in \mathcal{D}, \exists a \in \mathcal{C}(R)$ *s.t.* $\forall a' \in \mathcal{C}(R), a \geq_{R_N} a'$.

Note that every N-optimal core allocation gives the same assignment for N-agents.[6] We call a rule that assigns an N-optimal core allocation for each $R \in \mathcal{D}$ an **N-optimal core (NOC) rule**.

Assumption 2. *[A version of rural hospital theorem]* $\forall R \in \mathcal{D}, \forall \{a, a'\} \subseteq \mathcal{C}(R), \forall i \in N, [a(i) = \omega(i) \Leftrightarrow a'(i) = \omega(i)]$.

Under Assumption 1 and 2, we show the following theorem.[7] A proof is given after we prove two lemmas.

Theorem 2. *Every NOC rule is strategy-proof for N.*

Lemma 1. *Every NOC rule satisfies SUI and LSI on N.*

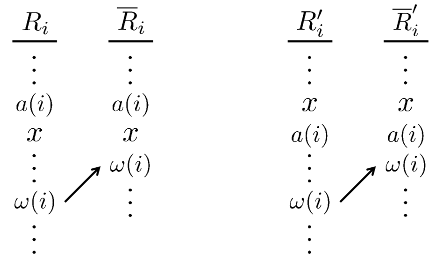
Proof. Let φ be an NOC rule. Let $i \in N$, $R = (R_i, R_{-i}) \in \mathcal{D}$ and $a := \varphi(R)$. Let $R'_i \in \mathcal{D}_i$ be the SU or a LS of R_i at $a(i)$. Let $R' := (R'_i, R_{-i})$ and $a' := \varphi(R')$. Suppose to the contrary that $a'(i) \neq a(i)$.

[5] This is a direct consequence of our setting where each agent in N has a complete, transitive and anti-symmetric preference.

[6] This is because $\geq_{R_N}$ is anti-symmetric. Note that N-optimal core allocation is unique if $\mathcal{A}^f$ consists of allocations that exclude a match between $(\mathcal{N} \backslash N)$-agents. College admission problems and housing markets are included in this class.

[7] A closely related paper by Takamiya (2003) establishes that any selection from the core is coalition strategy-proof under an assumption, called **essentially single-valuedness (ESV)** of the core, that requires that all core allocations be indifferent for all agents. Note that, in our setting with strict preferences, ESV of the core combined with the existence of a selection from the core implies Assumption 1 and 2. Since Takamiya's setting includes indifference and externalities in agents' preferences, his theorem and ours do not imply each other.

Figure 5: The construction of $\overline{R}_w$ and $\overline{R}'_w$ in the proof of Lemma 2.



Claim 1. $a \in \mathcal{C}(R')$. Suppose not. Then, there is $S \subseteq \mathcal{N}$ that blocks a at R' via $b \in \mathcal{A}^f$. Since the only difference between R and R' is i's preference, i must be a member of S who exibits $b(i)\ P'_i\ a(i)$. Since $SUC(R'_i, a(i)) \subseteq SUC(R_i, a(i))$, $b(i)\ P_i\ a(i)$. This implies that S blocks a at R via b, i.e., $a \notin \mathcal{C}(R)$, a contradiction. This completes the proof of Claim 1.

By Claim 1, $\{a, a'\} \subseteq \mathcal{C}(R')$. Since φ is an NOC rule, $a'(i)\ R'_i\ a(i)$. Thus $a'(i)\ P'_i\ a(i)$.

Claim 2. $a' \in \mathcal{C}(R)$. Suppose not. Then, there is $S \subseteq \mathcal{N}$ that blocks a' at R via $b \in \mathcal{A}^f$. Similar to Claim 1, i exibits $b(i)\ P_i\ a'(i)$. Since R'_i is either the SU or a LS of R_i at $a(i)$, the preference between $b(i)$ and $a'(i)$ is preserved, i.e., $b(i)\ P'_i\ a'(i)$ (See Figure 4).[8] Thus, S blocks a' at R' via b, i.e., $a' \notin \mathcal{C}(R')$, a contradiction. This completes the proof of Claim 2.

By Claim 2, $\{a, a'\} \subseteq \mathcal{C}(R)$. Since φ is an NOC rule, $a(i)\ R_i\ a'(i)$. On the other hand, $a'(i)\ P'_i\ a(i)$ and $SUC(R'_i, a(i)) \subseteq SUC(R_i, a(i))$ together imply that $a'(i)\ P_i\ a(i)$, a contradiction. □

Lemma 2. *Let φ be an NOC rule. Let $i \in N, R = (R_i, R_{-i}) \in \mathcal{D}$ and $a := \varphi(R)$. Let $x \in X_i$ be such that $r_{R_i}(x) = r_{R_i}(a(i)) + 1$. Then, for the SU of R_i at x, denoted as R'_i, $\varphi_i(R'_i, R_{-i}) \in \{a(i), x\}$.*

Proof. Let $R' := (R'_i, R_{-i})$ and $a' := \varphi(R')$. Suppose to the contrary that $a'(i) \notin \{a(i), x\}$. First note that $a(i)\ P'_i\ a'(i)$.[9] Thus we have $x\ P_i\ a'(i)\ R_i\ \omega(i)$ and $a(i)\ P'_i\ a'(i)\ R'_i\ \omega(i)$. Define $\overline{R}_i$ (resp. $\overline{R}'_i$) by upgrading $\omega(i)$ to be the immediate successor of x (resp. $a(i)$) at R_i (resp. R'_i) without affecting the preferences between other objects (Figure 5). Let $\overline{R} := (\overline{R}_i, R_{-i}), \overline{R}' := (\overline{R}'_i, R_{-i})$ and $\overline{a}' := \varphi(\overline{R}')$.

Claim 1.$\varphi_i(\overline{R}) = a(i)$.Since $\overline{R}_i$ is a LS of R_i at $a(i)$, $\varphi_i(\overline{R}) = \varphi_i(R)$ by Lemma 1.This completes the proof of Claim 1.

Claim 2. $\overline{a}'(i) = \omega(i)$. Suppose to the contrary that $\overline{a}'(i)\ \overline{P}'_i\ \omega(i)$.[10] Note that $\overline{a}'(i)\ \overline{P}'_i$

[8]Since $a'(i)\ P'_i\ a(i)$, $a'(i)\ P_i\ a(i)$. Thus, $b(i)$ and $a'(i)$ do not move at the transformation from R_i to R'_i.

[9]If this is not true, $a'(i)\ P'_i\ x$. However, in this case, R_i is a LS of R'_i at $a'(i)$. Thus, $a(i) = a'(i)$ by Lemma 1, a contradiction.

[10]Since φ is an NOC rule, $\overline{a}' \in \mathcal{C}(\overline{R}'_i, R_{-i})$. Thus, $\overline{a}'(i)\ \overline{R}'_i\ \omega(i)$.

Figure 6: The preference relations R_i and R'_i in the proof of Theorem 2.

R_i	R'_i
$\vdots$	$\vdots$
y	x
x	y
$a(i)$	$a(i)$
$\vdots$	$\vdots$

$\omega(i)\ \overline{R}'_i\ a'(i)$.[11] This implies that $\overline{a}'(i)\ P'_i\ a'(i)$ since the only difference between R'_i and $\overline{R}'_i$ is the position of $\omega(i)$. On the other hand, $\overline{a}' \in \mathcal{C}(R')$. This contradicts that a' is an N-optimal core allocation at R'. This completes the proof of Claim 2.

Claim 3. $\overline{a}' \in \mathcal{C}(\overline{R})$. Suppose to the contrary that $\overline{a}' \notin \mathcal{C}(\overline{R}_i)$. Then, there is $S \subseteq \mathcal{N}$ that blocks $\overline{a}'$ at $\overline{R}$ via $b \in \mathcal{A}^f$. Since the only difference between $\overline{R}$ and $\overline{R}'$ is i's preference, $i \in S$ exibits $b(i)\ \overline{P}_i\ \overline{a}'(i)$. By Claim 2, $\overline{a}'(i) = \omega(i)$. As $SUC(\overline{R}_i, \omega(i)) \subseteq SUC(\overline{R}'_i, \omega(i))$, $b(i)\ \overline{P}'_i\ \omega(i) = \overline{a}'(i)$. This implies $\overline{a}' \notin \mathcal{C}(\overline{R}')$, a contradiction. This completes the proof of Claim 3.

Now we complete the proof of Lemma 2. By Claim 1, the assignment of i at $\varphi(\overline{R}) \in \mathcal{C}(\overline{R})$ is $a(i) \neq \omega(i)$. On the other hand, by Claim 2 and 3, the assignment of i at $\overline{a}' \in \mathcal{C}(\overline{R})$ is $\overline{a}'(i) = \omega(i)$. This contradicts Assumption 2. □

Proof of Theorem 2. Let φ be an NOC rule. We show that φ satisfies UTI on N. Let $i \in N$ and $R = (R_i, R_{-i}) \in \mathcal{D}$. Letting $a := \varphi(R)$, let $R'_i \in \mathcal{D}_i$ be the UT of R_i at $a(i)$. Let $R' := (R'_i, R_{-i})$ and $a' := \varphi(R')$. Let $x, y \in X_i$ be such that $r_{R_i}(x) = r_{R_i}(a(i)) - 1$ and $r_{R_i}(y) = r_{R_i}(a(i)) - 2$. Suppose to the contrary that $a'(i) \neq a(i)$. We first show that $a'(i) = x$ through the following three claims.

Claim 1. $x\ R'_i\ a'(i)$. If $a'(i)\ P'_i\ x$, $a'(i) = a(i)$ ($\because R_i$ is a LS of R'_i at $a'(i)$), a contradiction. This completes the proof of Claim 1.

Claim 2. $a'(i) \neq y$.If $a'(i) = y$, $a'(i) = a(i)$ ($\because R_i$ is the SU of R'_i at $a'(i)$), a contradiction. This completes the proof of Claim 2.

Claim 3. $a'(i)\ R'_i\ a(i)$. Suppose to the contrary that $a(i)\ P'_i\ a'(i)$. Since a' is an NOC allocation at R', $a \notin \mathcal{C}(R')$. Thus, there exists $S \subseteq \mathcal{N}$ that blocks a at R' via $b \in \mathcal{A}^f$. Since the only difference between R and R' is i's preference, $i \in S$ and $b(i)\ P'_i\ a(i)$. Since $SUC(R_i, a(i)) = SUC(R'_i, a(i))$, $b(i)\ P_i\ a(i)$. Thus $a \notin \mathcal{C}(R)$, a contradiction. This completes the proof of Claim 3.

[11] $a(i)\ P'_i\ a'(i)$ and the construction of $\overline{R}'_i$ together imply $a(i)\ \overline{P}'_i\ a'(i)$. Note that $\omega(i)$ is the best object at $\overline{R}'_i$ among the objects worse than $a(i)$. Thus, $\omega(i)\ \overline{R}'_i\ a'(i)$.

Since $a'(i) \neq a(i)$, Claim 1,2 and 3 imply that the remaining possibility is $a'(i) = x$, i.e., $\varphi_i(R'_i, R_{-i}) = x$. Noting that R_i is the SU of R'_i at y, $\varphi_i(R_i, R_{-i}) \in \{x, y\}$ ($\because$ Lemma 2). This contradicts that $a(i) \notin \{x, y\}$. □

Corollary 1. *The student-proposing deferred-acceptance rule for college admission problems is strategy-proof for students.*

Proof. Theorem 2 in Gale and Shapley (1962) shows that the student-proposing DA algorithm hits the student-optimal "stable" matching. Since the set of stable matchings in college admission problems coincides with the core by Proposition 5.36 in Roth and Sotomayor (1990), Assumption 1 is satisfied. Gale and Sotomayor (1985) provide a brief proof of Assumption 2 in college admission problems. □

Corollary 2. *The top-trading cycles rule for housing markets is strategy-proof for $\mathcal{N}$.*

Proof. Roth and Postlewaite (1977) provide a brief proof for the uniqueness of the core allocation in housing markets. This fact implies that Assumption 1 and 2 are satisfied in housing markets. Note that Shapley and Scarf (1974) point out that Gale's TTC rule hits the unique core allocation for each housing market. □

4 Conclusion

In this paper, we characterize strategy-proofness for unit-demand agents with three invariance properties under preference transformation (Theorem 1). Utilizing it, we gave an elementary and transparent proof for strategy-proofness for N of N-optimal core rules for GIGAPs (Theorem 2). Consequently, both (i) strategy-proofness for students of the student-proposing DA rule for college admission problems and (ii) strategy-proofness of the TTC rule for housing markets are obtained. Before concluding the paper, we point out that the identical argument works to establish strategy-proofness of the worker-optimal stable rule, also known as the cumulative-offer process rule, for matching with contracts under weakened substitutes conditions. A detailed description for this topic is found in the working paper version (Anno and Takahashi, 2020).

References

Abdulkadiroğlu, A., and T. Sönmez (2003): "School Choice: A Mechanism Design Approach," *Americal Economic Review*, 93, 729–747.

Anno, H., and M. Kurino (2017): "Dual Organ Markets: Coexistence of Living and Deceased Donors," Department of Policy and Planning Sciences Discussion Paper Series No.1352, University of Tsukuba.

Anno, H., and S. Takahashi (2020): "A decomposition of strategy-proofness in discrete resource allocation problems," Department of Policy and Planning Sciences Discussion Paper Series No.1373, University of Tsukuba.

Dubins, L. E., and D. A. Freedman (1981): "Machiavelli and the Gale-Shapley algorithm," *American Mathematical Monthly*, 88, 485–494.

Gale, D., and L. S. Shapley (1962): "College Admissions and the Stability of Marriage," *American Mathematical Monthly*, 69, 9–15.

Gale, D., and M. A. O. Sotomayor (1985): "Some remarks on the stable matching problem," *Discrete Applied Mathematics*, 11, 223–232.

Hatfield, J., and P. Milgrom (2005): "Matching with Contracts," *American Economic Review*, 95, 913–935.

Hatfield, J. W., and F. Kojima (2010): "Substitutes and stability for matching with contracts," *Journal of Economic Theory*, 145, 1704–1723.

Maskin, E. (1999): "Nash Equilibrium and Welfare Optimality," *Review of Economic Studies*, 66, 23–38.

Roth, A. E. (1982a): "The Economics of Matching: Stability and Incentives," *Mathematics of Operations Research*, 7, 617–628.

——— (1982b): "Incentive Compatibility in a Market with Indivisibilities," *Economics Letters*, 9, 127–132.

Roth, A. E., and A. Postlewaite (1977): "Weak Versus Strong Domination in a Market with Indivisible Goods," *Journal of Mathematical Economics*, 4, 131–137.

Roth, A. E., and M. A. O. Sotomayor (1990): *Two-sided matching: a study in game-theoretic modeling and analysis*. Econometric Society monographs, Cambridge.

Shapley, L., and H. Scarf (1974): "On Cores and Indivisibility," *Journal of Mathematical Economics*, 1, 23–37.

Sönmez, T. (1999): "Strategy-Proofness and Essentially Single-Valued Cores," *Econometrica*, 67, 677–690.

Takamiya, K. (2003): "On strategy-proofness and essentially single-valued cores: A converse result," *Social Choice and Welfare*, 20, 77–83.