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Level-adjusted S-Gini index and its complementary index as a pair of sensitivity-adjustable inequality measures

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Abstract

The one-parameter family of S-Gini indices is the most representative of generalized Gini indices. This paper proposes a variant of the S-Gini index, called the level-adjusted S-Gini index (abbreviated as the aS-Gini index), together with its complementary one-parameter index, called the complementary level-adjusted S-Gini index (caS-Gini index). The relation of the new indices to the original index corresponds to that of the generalized entropy (GE) index to the Atkinson index, in a sense. The complementary index is introduced to overcome an issue arising from the failure of the aS-Gini index to satisfy some properties exhibited by the GE index. The combination of the aS-Gini and caS-Gini indices enables us to measure the extent of inequality in size distributions containing small portions of negative values, such as net wealth distributions, by different levels of sensitivity to higher values than to lower values. The caS-Gini index, as well as the S-Gini and aS-Gini indices, is also a generalization of the standard Gini index because the index is geometrically expressed as the area of a figure enclosed by a transformed egalitarian curve and a transformed Lorenz curve with a constant multiplier. For a specific parameter value, its expression coincides with the well-known geometrical expression of the standard Gini index.

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1. Desirable properties of sensitivity-adjustable relative inequality indices

The S-Gini index proposed by Kakwani (1980), Donaldson and Weymark (1980, 1983) and Yitzhaki (1983) is, like the Atkinson index, able to measure the extent of inequality under different inequality aversions by tuning a parameter value. However, there is also a need for measuring inequality with a different sensitivity to higher values than to lower values, i.e., a need for sensitivity-adjustable inequality indices, depending on users' interests in inequality in higher and lower classes; *c.f.*, Subramanian (2010, 2015, 2019). The generalized entropy (GE) index is used for this reason, despite its order equivalency to the Atkinson index. As the GE index is inapplicable to distributions containing negative values, it would be useful to investigate sensitivity-adjustable relative inequality indices for such distributions.

I consider that the following properties are desirable for a one-parameter family of relative inequality indices as a family of sensitivity-adjustable indices:

A: For a distribution with a heavier left/right tail relative to the right/left tail, the index value monotonically decreases/increases as the parameter value goes in a certain direction.

B: When the left/right tail becomes heavier, (the index exhibits an increase, and) the marginal increase rate monotonically falls/rises as the parameter value goes in the same direction as that in A, irrespective of the heaviness of both tails.

The fulfilment of A and B can be interpreted as that the relative sensitivity to higher/lower values weaken/strengthen as the parameter goes in the direction in A. The family of GE indices practically satisfies both properties at least for a certain class of distributions such as the Pareto and reciprocal Pareto distribution (or, equivalently, the power-function distribution); in contrast, the family of S-Gini indices fails to satisfy both properties, as shown in subsequent sections. To overcome this failure, this paper proposes alternative properties for a pair of one-parameter families of relative inequality indices with a common parameter: one (the main index) is sensitive to higher values, and the other (the complementary index) is sensitive to lower values.

C: For the base parameter value, both indices coincide.

D: For a distribution with a heavier left/right tail relative to the right/left tail, i) the main index is smaller/larger than the complementary index except for the base parameter value, and ii) the ratio of the main index to the complementary index monotonically decreases/increases as the parameter value moves far from the base parameter value.

E: When the left/right tail becomes heavier, (both indices exhibit increases,) i) the marginal increase rate of the former is lower/higher than that of the latter except for the base parameter value, and ii) the marginal increase rate of the former relative to that of the latter monotonically falls/rises as the parameter moves far from the base value, irrespective of the heaviness of both tails.

The fulfilment of D and E can be interpreted as that the sensitivity of the main index to lower/higher values weaken/strengthen relative to that of the complementary index as the parameter value moves far from the base value. Considering analytical tractability, this paper employs the Pareto/reciprocal Pareto distributions as the 'reference' distributions for A - E with heavier right/left tails relative to left/right tails to address issues in the domain of positive distributions and employs the reciprocal Pareto-negative uniform distribution instead of the reciprocal Pareto distribution for distributions containing small portions of negative values to show that the newly introduced pair of a variant of the S-Gini index and its complementary index practically satisfy properties C, D and E.

2. Level-adjusted S-Gini index

This paper assumes that the size distribution of a numerical variable X such as wealth has the positive finite mean μ, the probability density function (pdf) $f(x)$ except for a point mass at zero, the cumulative distribution function (cdf) $F(x)$, the (possibly negative) finite infimum $a = \inf X\ (= \inf F^{-1})$, and the (possibly infinite) supremum $b = \sup X\ (= \sup F^{-1})$. We frequently identify a distribution with its cdf. The S-Gini index with parameter $\nu(> 1)$ proposed by Kakwani (1980), Donaldson and Weymark (1980, 1983) and Yitzhaki (1983) is

$SG_\nu(F) = 1 - \nu(\nu - 1)\int_0^1 (1 - p)^{\nu-2} L_F(p)dp$ (Kakwani 1980)

$$= \nu(\nu - 1)\int_0^1 (1 - p)^{\nu-2}[p - L_F(p)]dp \quad (1)$$

$$= \nu\int_0^1 \left[\left(1 - L_F(p)\right)^{\nu-1} - (1 - p)^{\nu-1}\right] dL_F(p) \quad (2)$$

$$= 1 - \nu\int_a^b \frac{x}{\mu}\bar{F}(x)^{\nu-1} f(x)dx = \frac{1}{\mu}\int_a^b [\bar{F}(x) - \bar{F}(x)^\nu]dx \text{ (Yitzhaki 1983)}, \quad (3)$$

where $L_F(p) \coloneqq \frac{1}{\mu}\int_0^{F^{-1}(p)} x dF(x)$, $F^{-1}(p) = \inf_x\{F(x) \geq p\}$, and $\bar{F}(x) \coloneqq 1 - F(x)$. In (1), (2) and Kakwani's formula, $SG_\nu(F)$ is expressed using only the Lorenz curve of F. $SG_2(F)$ coincides with the standard Gini index $Gini(F)$. As mentioned by Donaldson and Weymark (1980, 1983) and Yitzhaki (1983), ν specifies the degree of inequality aversion, ranging from indifference to inequality ($SG_\nu(F) \to 0$ as $\nu \to 1$) to the relative maxmin rule ($SG_\nu(F) \to 1 - a/\mu$ as $\nu \to \infty$). In this regard, the S-Gini index resembles the Atkinson index (Atkinson 1970).

Here, we numerically compare the S-Gini and Atkinson indices using the Pareto distribution

$P_{c,\alpha}$, which has the pdf $\alpha c^{\alpha} x^{-\alpha-1}$ and Lorenz curve $L_{P_{c,\alpha}}(p) = 1-(1-p)^{(\alpha-1)/\alpha}$, $c \le x < \infty$, $\alpha > 1$; and the reciprocal Pareto distribution $RP_{c,\beta}$, which has the pdf $\beta x^{\beta-1}/c^{\beta}$ and Lorenz curve $L_{RP_{c,\beta}}(p) = p^{(\beta+1)/\beta}$, $0 < x \le c$, $\beta > 0$. $RP_{c,\beta}$ is generated by a random variable $c^2 X^{-1}$, $X \sim P_{c,\beta}$. As this paper addresses relative (or equivalently scale-invariant) indices, $P_{c,\alpha}$ and $RP_{c,\beta}$ are abbreviated as P_{α} and RP_{β}, respectively, hereafter. The S-Gini indices for P_{α} and RP_{β} are expressed as

$SG_{\nu}(P_{\alpha}) = \frac{\nu-1}{\alpha\nu-1}$ (Donaldson and Weymark 1983) and

$SG_{\nu}(RP_{\beta}) = 1 - \nu\frac{\beta+1}{\beta}B\left(\nu, \frac{\beta+1}{\beta}\right)$, where $B(p,q)$, $p, q > 0$, denotes the beta function.

The S-Gini and Atkinson indices for P_2, $P_{1.5}$, RP_1 and $RP_{0.5}$ are listed in Tables I and II. The formulas of the Atkinson index A_{ε} with parameter ε for P_{α} and RP_{β} are listed in Appendix 1.

When $\nu \to 1$ and $\varepsilon \to 0$, i.e., inequality aversion approaches inequality-neutrality, the indices decrease, and simultaneously, their sensitivity to higher values becomes stronger. The higher sensitivity (partly) causes higher ratios $SG_{\nu}(P_{1.5})/SG_{\nu}(P_2)$ and $A_{\varepsilon}(P_{1.5})/A_{\varepsilon}(P_2)$ for smaller ν and ε. The generalized entropy (GE) index GE_{δ} with parameter δ, order-equivalent to the Atkinson index $A_{1-\delta}$, more clearly exhibits sensitivity to higher values because when δ becomes higher (ε becomes lower), the index value $GE_{\delta}(P_{\alpha})$, $\alpha = 2$ and 2.5 (see Appendix 2) and the ratio $GE_{\delta}(P_{1.5})/GE_{\delta}(P_2)$ become higher in line with A and B. Furthermore, by extension to $\delta \ge 1$, GE_{δ} encompasses higher sensitivity to higher values than A_{ε}.

Table I. S- and aS-Gini indices for the Pareto and reciprocal Pareto distributions.

Index	Distribution model	Parameter ν							
		0.8	1	1.2	1.5	1.8	2	3	4
SG_{ν}	$P_{2.5}$	n.a.	n.a.	0.100	0.182	0.229	0.250	0.308	0.333
	P_2	n.a.	n.a.	0.143	0.250	0.308	0.333	0.400	0.429
	$P_2/P_{2.5}$	n.a.	n.a.	1.43	1.38	1.35	1.33	1.30	1.29
	$RP_{1.5}$	n.a.	n.a.	0.066	0.147	0.213	0.250	0.386	0.474
	RP_1	n.a.	n.a.	0.091	0.2000	0.286	0.333	0.500	0.600
	$RP_1/RP_{1.5}$	n.a.	n.a.	1.38	1.36	1.34	1.33	1.29	1.27
aSG_{ν}	$P_{2.5}$	1.000	0.667	0.500	0.364	0.286	0.250	0.154	0.111
	P_2	1.667	1.000	0.714	0.500	0.385	0.333	0.200	0.143
	$P_2/P_{2.5}$	1.67	1.50	1.43	1.38	1.35	1.33	1.30	1.29
	$RP_{1.5}$	0.394	0.343	0.330	0.294	0.266	0.250	0.193	0.158
	RP_1	0.556	0.480	0.455	0.400	0.357	0.333	0.250	0.200
	$RP_1/RP_{1.5}$	1.41	1.40	1.38	1.36	1.34	1.33	1.29	1.27

Table II. Atkinson and GE indices for the Pareto and reciprocal Pareto distributions.

Index	Distribution model	Parameter δ for the GE Index						
		-0.4	0	0.2	0.5	0.8	1	1.4
		Parameter ε for the Atkinson Index						
		1.4	1	0.8	0.5	0.2		
A_ε	$P_{2.5}$	0.130	0.105	0.090	0.063	0.028	n.a.	n.a.
	P_2	0.211	0.176	0.153	0.111	0.053	n.a.	n.a.
	$P_2/P_{2.5}$	1.62	1.67	1.71	1.78	1.88	n.a.	n.a.
	$RP_{1.5}$	0.232	0.144	0.109	0.063	0.023	n.a.	n.a.
	RP_1	0.442	0.264	0.196	0.111	0.041	n.a.	n.a.
	$RP_1/RP_{1.5}$	1.90	1.83	1.81	1.78	1.76	n.a.	n.a.
GE_δ	$P_{2.5}$	0.103	0.111	0.116	0.127	0.142	0.156	0.199
	P_2	0.178	0.193	0.205	0.229	0.267	0.307	0.470
	$P_2/P_{2.5}$	1.73	1.74	1.76	1.80	1.88	1.97	2.36
	$RP_{1.5}$	0.199	0.156	0.142	0.127	0.116	0.111	0.103
	RP_1	0.470	0.307	0.267	0.229	0.205	0.193	0.178
	$RP_1/RP_{1.5}$	2.36	1.97	1.88	1.80	1.76	1.74	1.73

We define the level-adjusted S-Gini (aS-Gini) index as $aSG_\nu(F) := SG_\nu(F)/(\nu-1)$; then, the smaller ν becomes, the higher $aSG_\nu(P_\alpha)$ becomes (see Table I). Furthermore, by extension to $0 < \nu \leq 1$ as in (4) and (5), aSG_ν encompasses higher sensitivity to higher values than SG_ν. aSG_ν is equivalent to $1/|\nu-1|$ times the Extended S-Gini index (Gisbert *et al.* 2010) except for $\nu = 1$. The lower limit of ν depends on the heaviness of the upper tail; e.g., $\nu > \alpha^{-1}$ for P_α.

$$aSG_\nu(F) = \begin{cases} \frac{1}{\nu-1} SG_\nu(F) = \nu \int_0^1 (1-p)^{\nu-2}[p - L_F(p)]dp\,, \ 0 < \nu \neq 1 & (4) \\ \int_0^1 \frac{p-L_F(p)}{1-p} dp = -\frac{1}{\mu}\int_a^b \bar{F}(x)\log \bar{F}(x)\,dx\,, \quad \nu = 1 & (5) \end{cases}$$

3. Complementary level-adjusted S-Gini index for positive distributions

However, Table I shows that the smaller ν becomes, the higher aSG_ν becomes for RP_β as well as for P_α, and the higher its marginal increase rate becomes for a change of RP_β to a heavier left tail as well as for a change of P_α to a heavier right tail; hence, aSG_ν violates A and B. In contrast, Table II indicates that GE_δ is in line with A and B. In fact, GE_δ, $0 \leq \delta \leq 1$, satisfies A and B except for particularly heavy-tailed distributions. For such heavy-tailed P_α and RP_β, GE_δ possibly violates both properties near $\delta = 0$ and near $\delta = 1$, respectively; e.g., $0 \leq \delta \leq 0.08$ for $P_{1.5}$, and $0.92 \leq \delta \leq 1$ for $RP_{0.5}$. Nevertheless, GE_δ can be regarded as practically satisfying A and B. This fact relates to the following property of GE_δ:

Consider Lorenz curves L_F and L_G of positive distributions F and G, respectively; L_F and L_G are mutually symmetric with respect to a diagonal other than the equality diagonal, as illustrated in Figure 1. Hereafter, I call this symmetric relation between F and G 'L-symmetry'. This symmetry is equivalent to the fulfillment of the simultaneous equations (eqs.) in (6). GE_δ exhibits property (7) regarding L-symmetry (see the appendix of Okamoto 2021):

$$L_F(p) = 1 - q \text{ and } p = 1 - L_G(q). \tag{6}$$

$$F \text{ and } G \text{ are mutually L-symmetric} \Longrightarrow GE_\delta(F) = GE_{1-\delta}(G). \tag{7}$$

From (7), for $\delta > \delta' > 0.5$, if $GE_\delta(F) > GE_{\delta'}(F) > GE_{1-\delta'}(F) > GE_{1-\delta}(F)$ and $\frac{GE_\delta(F_2)}{GE_\delta(F_1)} > \frac{GE_{\delta'}(F_2)}{GE_{\delta'}(F_1)} > \frac{GE_{1-\delta'}(F_2)}{GE_{1-\delta'}(F_1)} > \frac{GE_{1-\delta}(F_2)}{GE_{1-\delta}(F_1)}$, then, the inequalities hold in reverse order for G, G_1, and G_2. As P_α and $RP_{\alpha-1}$ have L-symmetry, property (7) relates to GE's fulfilment of A and B regarding both left and right tails.

Unlike GE_δ, aSG_ν does not satisfy (7). Then, to complement aSG_ν's failure to satisfy A and B, we create an index in the domain of positive distributions using an L-symmetric counterpart:

$$^c aSG_\nu(F) \coloneqq aSG_\nu(G). \tag{8}$$

Define the curve $L(q)$ implicitly for a given positive distribution F using eqs. (6) (after replacement of $L_G(q)$ with $L(q)$). As $L(q)$ is strictly convex and strictly monotonically increasing in addition to fulfilling conditions $L(0) = 0$ and $L(1) = 1$, an underlying positive distribution G for $L(q)$ exists uniquely except for scale-transformation (Thompson 1976). Hence, the relative index $^c aSG_\nu(F)$ is uniquely determined. Note that G also has its pdf (see Appendix 4). From the definition, the sensitivity of $^c aSG_\nu$ to the right/left tail is regarded as equivalent to that of aSG_ν to the left/right tail if we assume equivalence of the heaviness of the right tail of P_α to that of the left tail of $RP_{\alpha-1}$. From (2) and (6), $^c aSG_\nu(F)$ is expressed as

$$^c aSG_\nu(F) = \begin{cases} \frac{\nu}{\nu-1}\int_0^1 [p^{\nu-1} - L_F^{\nu-1}(p)]dp\,, & 0 < \nu \neq 1 \quad (9) \\ \int_0^1 [\log p - \log L_F(p)]dp = -1 - \int_0^1 \log L_F(p)\,dp\,, & \nu = 1 \quad . \quad (10) \end{cases}$$

The lower limit of ν depends on the heaviness of the left tail of the distribution; e.g., $\nu > (\beta+1)^{-1}$ for RP_β. $^c aSG_\nu$ is qualified as a relative inequality measure from (9) and (10). Clearly, $^c aSG_2(F) = aSG_2(F) = SG_2(F) = Gini(F)$.

Theorem 1: For a positive distribution F and its L-symmetric counterpart G, assume that L_F and L_G cross each other only once at their common Kolkata index p_K (Chatterjee *et al.* 2017), defined as p_K s.t. $p_K + L_H(p_K) = 1$ for $H = F, G$ (see Figure 1), and $L_F(p) \lesseqgtr L_G(p), p \lesseqgtr p_K$, i.e., L_F / L_G exhibits imbalance toward the lower/higher classes. [1] Then, $aSG_\nu(F) \lesseqgtr$

[1] Relations of a L-symmetric pair of distributions with the Kolkata, Gini (and some other symmetric) indices have been

${}^{c}aSG_{\nu}(F), \nu \lesseqgtr 2$. (The proof is given in Appendix 5.)

As a pair $RP_{\alpha-1}$ and P_{α} satisfy the prerequisites in Theorem 1 (see Appendix 6), Theorem 1 indicates that a pair aSG_{ν} and ${}^{c}aSG_{\nu}$ satisfy D-(i) for $\nu < 2$ and for $\nu > 2$ (if their roles are mutually interchange). As the index values of aSG_{ν} in the rows for $RP_{1.5}$ and RP_{1} in Table I coincide with those of ${}^{c}aSG_{\nu}$ for $P_{2.5}$ and P_{2}, respectively, the table can be regarded as illustrating the order relations between aSG_{ν} and ${}^{c}aSG_{\nu}$ in Theorem 1. However, the table also shows that the ratio $aSG_{\nu}(P_2)/aSG_{\nu}(P_{2.5})$ is higher than ${}^{c}aSG_{\nu}(P_2)/{}^{c}aSG_{\nu}(P_{2.5})$ even if $\nu > 2$, indicating that the pair fails to satisfy E for $\nu > 2$. Hence, we should be cautious about the uses of the pair with $\nu > 2$ as a pair of sensitivity-adjustable measures. Numerical analysis shows that a pair aSG_{ν} and ${}^{c}aSG_{\nu}$ satisfy D-(ii) for $\nu < 2$. This pair also satisfies E except for particularly heavy-tailed P_{α} with $\alpha \leq 1.67$ and RP_{β} with $\beta \leq 0.67$ for $\nu < 2$. In such exceptional cases, the pair can violate E near $\nu = 2$; e.g., $1.81 \leq \nu < 2$ for $P_{1.5}$ and $RP_{0.5}$ (see Figure 2). Nevertheless, the pair can be regarded as practically satisfying D and E. [2]

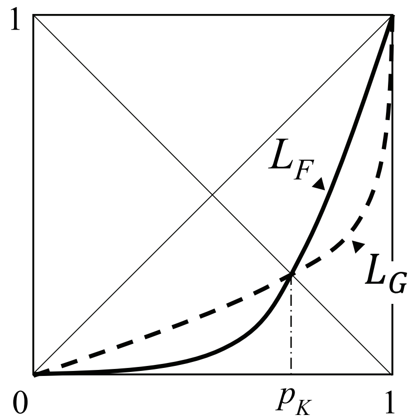

Fig. 1. Mutually symmetric Lorenz curves.

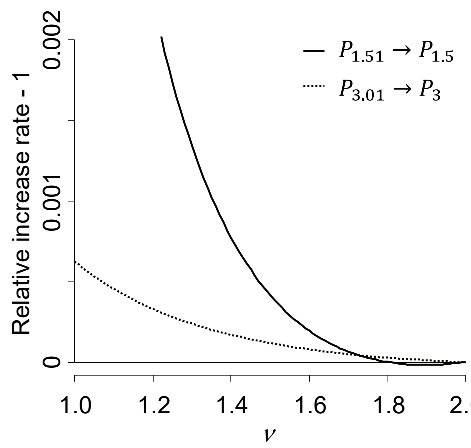

Fig. 2. Increase rate of aSG_{ν} relative to that of ${}^{c}aSG_{\nu}$ when $P_{\alpha+\Delta\alpha}$ changes to P_{α}, $\alpha = 1.5$ or 3, $\Delta\alpha = 0.01$.

Note. The relative increase rate is calculated as $\frac{aSG_{\nu}(P_{\alpha})/aSG_{\nu}(P_{\alpha+\Delta\alpha})}{{}^{c}aSG_{\nu}(P_{\alpha})/{}^{c}aSG_{\nu}(P_{\alpha+\Delta\alpha})}$.

4. Extension to distributions containing nonpositive values

The S- and aS-Gini indices are naturally extended to distributions containing negative values. If the infimums are finite, the Atkinson and GE indices also become applicable by adding an

essentially mentioned or illustrated in the literature frequently although most of them did not particularly pay attention to the Kolkata index. Banerjee *et al.* (2020) exceptionally focused on properties of this characteristic point of the Lorenz curve.

[2] The double-Pareto distribution $dP_{\alpha,\beta}$ (Reed 2003), or, equivalently, log-asymmetric Laplace distribution can also be used as the reference distribution instead of a pair P_{α} and RP_{β}. $dP_{\alpha,\beta}$ is generated by a product of two mutually independent random variables following P_{α} and RP_{β}. $dP_{\alpha,\beta}$ and $dP_{\beta+1,\alpha-1}$ have L-symmetry, and the pair satisfies the prerequisites in Theorem 1. The same conclusion is drawn for aSG_{ν} and ${}^{c}aSG_{\nu}$ from the use of $dP_{\alpha,\beta}$ as the reference distribution.

appropriate constant value to the variables; however, it is practically difficult to employ this simple method if the infimums vary depending on the distributions or a suitable common constant value is not definite. Here, we extend the ᶜaS-Gini index for $\nu > 1$ to distributions containing negative values as in (11). The rates of change of ${}^{c}aSG_{\nu}(F)$ in (11) by an increase in the constant value c depend on the distributions even if the means are the same; in contrast, $aSG_{\nu}(F)$ is reduced by $c/(\mu + c) \times 100\%$ for any distribution.

$$ {}^{c}aSG_{\nu}(F) = \frac{\nu}{\nu-1}\int_0^1 [p^{\nu-1} - \operatorname{sgn} L_F(p) \cdot |L_F|^{\nu-1}(p)]dp,\ \nu > 1, \quad (11) $$

[3]

where $\operatorname{sgn} q \coloneqq 1$ for $q > 0$, 0 for $q = 0$, and -1 for $q < 0$. Extension (11) retains the index properties required as a relative inequality measure and the equality ${}^{c}aSG_2(F) = aSG_2(F) = SG_2(F) = Gini(F)$. Note that, from (11), ${}^{c}aSG_{\nu}(F)$ can be geometrically expressed as $\nu/(\nu - 1)$ times the area of a figure enclosed by the transformed egalitarian curve $p^{\nu-1}$ and transformed Lorenz curve $\operatorname{sgn} L_F(p) \cdot |L_F|^{\nu-1}(p)$. The expression for $\nu = 2$ coincides with the well-known geometric expression of $Gini(F)$. ${}^{c}aSG_{\nu}(F) \to 0$ as $\nu \to \infty$. ${}^{c}aSG_{\nu}(F) \to \infty$ as $\nu \to 1$ if F has a point mass at zero or negative values. $(\nu - 1) \cdot {}^{c}aSG_{\nu}(F) \to L_F^{-1}(0)$ if F is nonnegative and $\to 2L_F^{-1}(0)$ otherwise as $\nu \to 1$, where $L_F^{-1}(0) \coloneqq \max_p\{L_F(p) \leq 0\}$. Hence, if both F and G contain negative values, ${}^{c}aSG_{\nu}(F)/\,{}^{c}aSG_{\nu}(G) \to L_F^{-1}(0)/L_G^{-1}(0)$ as $\nu \to 1$. The limiting cases indicate that 'zero' has a particular meaning as a threshold value for $(\nu - 1) \cdot {}^{c}aSG_{\nu}$ if ν is close to 1. The following theorem holds:

Theorem 2: Assume that F contains nonpositive values, L_F satisfies $1 - p \geq -L_F(p)$, $0 < p < 1$, and $L_{F^+}(p) \coloneqq \max\{0, L_F(p)\}$ crosses its L-symmetric counterpart only once; then, $aSG_{\nu}(F) \lesseqqgtr {}^{c}aSG_{\nu}(F)$ for $\nu \lesseqqgtr 2$. Furthermore, if F contains negative values, then $\left|aSG_{\nu}(F) - {}^{c}aSG_{\nu}(F)\right| > \left|aSG_{\nu}(F^+) - {}^{c}aSG_{\nu}(F^+)\right|$, where F^+ is an underlying distribution of L_{F^+}; i.e., $F^+(x) \coloneqq F(x)$ for $x \geq 0$ and 0 otherwise. (The proof is given in Appendix 7.)

Regarding distributions containing small portions of negative values such as household net wealth distributions, there exist neither standard models for size distributions nor established definitions of the heaviness of the left tails. Here, we adopt the three-parameter reciprocal Pareto-negative uniform distribution $RPNU_{\beta,p_0,x_0}$, as a reference distribution with an adjustable heaviness of the left tail, in addition to the Pareto distribution P_{α}, as one with an

[3] Some researchers have proposed transformations of net wealth to ease difficulties in processing data for statistical and analytical purposes such as graphical representations and measurements of inequality. The variant of the log-transformation of Biewen *et al*. (2021) has a form resembling that of the transformation of the Lorenz curve in (11). The concave log-like transformation of Ravallion (2017) is for inequality measurements. Unlike those transformations, ${}^{c}aSG_{\nu}$ employs a power function to transform the Lorenz curve (instead of the original variable) for inequality measurements.

adjustable heaviness of the right tail. $RPNU_{\beta,p_0,x_0}$ is defined as a distribution that has the pdf $f(x) = p_0/|x_0|$ for $x_0 \le x \le 0$ and $(1-p_0)\beta x^{\beta-1}$ for $0 < x \le 1$, where $\beta > 0$, $0 < p_0 \le 0.2$, and $-0.1 \le x_0 < 0$. We regard the changes $RPNU_{\beta,p_0,x_0} \to RPNU_{\beta,kp_0,x_0}$ and $RPNU_{\beta,p_0,x_0} \to RPNU_{\beta,kp_0,k'x_0}$, $1 < k' \le k$, as changes to heavier left tails. The formulas of aSG_ν and ${}^{c}aSG_\nu$ for $RPNU_{\beta,p_0,x_0}$ are given in Appendix 3. Table III presents numerical examples. Regarding the right-tail, a pair aSG_ν and ${}^{c}aSG_\nu$, $\nu < 2$, practically satisfy properties D and E, as explained in the previous section. The Lorenz curve of $RPNU_{\beta,p_0,x_0}$ is proven to satisfy the conditions in Theorem 2 in a similar manner as that of RP_β. Hence, a pair aSG_ν and ${}^{c}aSG_\nu$, $\nu < 2$, satisfy D-(i). Numerical calculation shows that the pair also satisfies D-(ii), as does E, if the condition $L^{-1}(0) < 0.5$ is imposed, unless positive values are intensively concentrated near zero as with $\beta < 0.4$ for E-(i) and unless $\nu < 1.2$ for E-(ii); e.g., the relative marginal increase rate of ${}^{c}aSG_\nu\left(RPNU_{\beta,p_0,x_0}\right)$ for $\nu \le 1.1$ obtained by a change to a heavier left tail is possibly slightly lower than the corresponding rate for $\nu = 1.2$ if $\beta > 4.8$ and p_0 is relatively large (near 0.2) (see Figure 3). Still, the indices are considered to practically satisfy D and E although further elaboration may be desirable for the relevant reference distributions.

Table III. aS- and ᶜaS-Gini indices for the reciprocal Pareto-negative uniform distribution.

Index	Distribution model	Parameter ν						
		1	1.2	1.5	1.8	2	3	4
aSG_ν	$RPNU_{1,0.1,-0.1}$	0.623	0.566	0.498	0.445	0.415	0.311	0.249
	$RPNU_{1,0.2,-0.1}$	0.766	0.693	0.607	0.539	0.501	0.369	0.291
	$RPNU_{1,0.2,-0.2}$	0.810	0.736	0.646	0.576	0.537	0.401	0.319
	$RPNU_{1,0.2,-0.1}/RPNU_{1,0.1,-0.1}$	1.23	1.22	1.22	1.21	1.21	1.19	1.17
	$RPNU_{1,0.2,-0.2}/RPNU_{1,0.1,-0.1}$	1.30	1.30	1.30	1.29	1.29	1.29	1.28
${}^{c}aSG_\nu$	$RPNU_{1,0.1,-0.1}$	n.a.	1.798	0.743	0.498	0.415	0.234	0.164
	$RPNU_{1,0.2,-0.1}$	n.a.	2.761	1.000	0.619	0.501	0.268	0.185
	$RPNU_{1,0.2,-0.2}$	n.a.	3.244	1.139	0.677	0.537	0.276	0.190
	$RPNU_{1,0.2,-0.1}/RPNU_{1,0.1,-0.1}$	1.68#	1.54	1.35	1.24	1.21	1.15	1.13
	$RPNU_{1,0.2,-0.2}/RPNU_{1,0.1,-0.1}$	1.94#	1.80	1.53	1.36	1.29	1.18	1.16

Ratios of $L^{-1}(0)$ for $RPNU_{1,0.2,-0.1}$ and $RPNU_{1,0.2,-0.2}$ to that for $RPNU_{1,0.1,-0.1}$.

5. Empirical example

As an empirical application, I apply the aS- and ᶜaS-Gini indices to measure inequalities in the 1995-2016 US per-adult net wealth distributions obtained from public-use microdata of the triennial Survey of Consumer Finances. The net wealth owned by a two-parent family is equally divided between the parents for the calculations. The index formulas for the sample survey data are given in Appendix 8. The estimates for 1995 and 2016 are listed in Table IV. The Lorenz curve for 2016 is lower than that for 1995 except for the lowest 0.1%; hence, the distribution in 1995 'almost' Lorenz dominates that in 2016.

The per-adult net wealth distributions in years after the 2007-2008 financial crisis satisfy the conditions in Theorem 2, whereas the earlier distributions fail to satisfy the conditions. Nevertheless, the order relation $aSG_\nu \lesseqgtr {}^{c}aSG_\nu$ for $\nu \lesseqgtr 2$ holds even before 2008. Although a substantial expansion of the share of the wealthiest 5% of adults draws attention (see Table V), the higher increase rate of ${}^{c}aSG_\nu$ relative to that of aSG_ν for $\nu < 2$ indicates that the distributional change on the lower tail contributed to the inequality rise more than that on the higher tail. Figure 4 illustrates the increases in $aSG_{1.5}$ and ${}^{c}aSG_{1.5}$ and the standard Gini index from 1995 for each survey year. A drastic increase in adults holding excess debt caused a surge in ${}^{c}aSG_{1.5}$ after the outbreak of the financial crisis (see also Table VI), whereas neither $aSG_{1.5}$ nor the standard Gini index shows the corresponding clearly visible indication.[4] Figure 4 also implies that we should care about differences in the level of sensitivity to distributional changes between the pair aSG_ν and ${}^{c}aSG_\nu$ and the standard Gini index.

Table IV. aS- and ^{c}aS-Gini indices for US per-adult net wealth distributions.

Index	Year	Parameter ν						
		1	1.2	1.5	1.8	2	3	4
aSG_ν	1995	2.847	1.900	1.239	0.913	0.777	0.445	0.313
	2016	3.217	2.127	1.373	1.002	0.846	0.473	0.327
	2016/1995	1.13	1.12	1.11	1.10	1.09	1.06	1.05
${}^{c}aSG_\nu$	1995	n.a.	2.625	1.288	0.917	0.777	0.445	0.311
	2016	n.a.	3.445	1.502	1.018	0.846	0.464	0.319
	2016/1995	1.53#	1.31	1.17	1.11	1.09	1.04	1.02

Ratio of $L^{-1}(0)$ for 2016 to that for 1995.

Table V. Share of net wealth by rank-group of per-adult net wealth in the US.

Year	Bottom10%	10-50%	50-90%	90-95%	Top5%
1995	-0.003	0.045	0.296	0.114	0.547
2016	-0.005	0.021	0.233	0.120	0.631
2016−1995	-0.002	-0.024	-0.064	0.006	0.084
2016/1995	1.68	0.46	0.79	1.05	1.15

4 The Zanardi index, a skewness measure for the Lorenz curve studied by Clementi *et al*. (2019a, 2019b), decreased from +0.003 for 1995 to -0.011 for 2016. This recent inclination to imbalance toward the lower classes also indicates that a drastic increase in adults holding excess debts contributed to the rising inequality in the US net wealth distribution more than a substantial expansion of the wealthiest' share. However, this example appears to contradict the views of Clementi *et al*. on the inequality measurement.

Table VI. Summary statistics for adults holding zero or negative net wealth in the US.

Year	Zero net wealth	Negative net wealth (excess debt)			$L^{-1}(0)$
	Population* share	Population* Share	Ratio to# ave. income	Ratio to## ave. wealth	
1995	0.019	0.062	-0.230	-0.048	0.228
2016	0.004	0.100	-0.340	-0.050	0.350

* Population shares in all adults. # Ratio of the average excess debt to the average disposable income of all adults. ## Ratio of the average excess debt to the average net wealth of all adults.

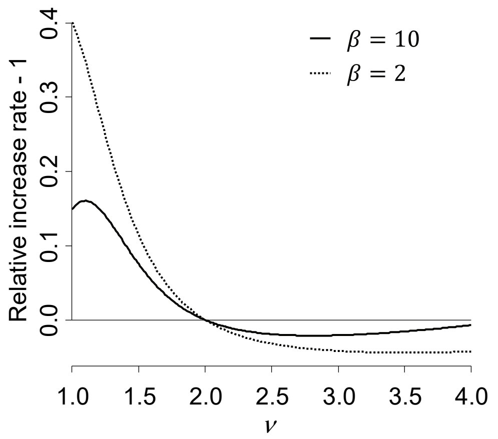

Fig. 3. Increase rate of ${}^{c}aSG_{\nu}$ relative to that of aSG_{ν} when $RPNU_{\beta,0.1,-0.1}$ changes to $RPNU_{\beta,0.2,-0.1}$, $\beta = 2$ or 10.

Note. The relative increase rate is calculated as $\frac{{}^{c}aSG_{\nu}(RPNU_{\beta,0.2,-0.1})/\,{}^{c}aSG_{\nu}(RPNU_{\beta,0.1,-0.1})}{aSG_{\nu}(RPNU_{\beta,0.2,-0.1})/aSG_{\nu}(RPNU_{\beta,0.1,-0.1})}$.

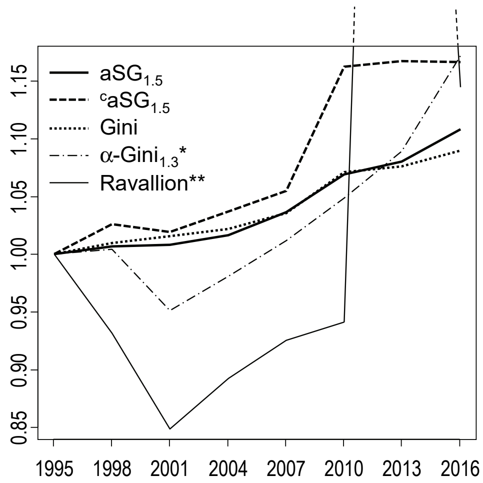

Fig. 4. Inequality in US per-adult net wealth distributions, 1995–2016 (1995 = 1.0).

* The α-Gini index (Chameni 2006) with α=1.3. This index is sensitive to distributional changes at low- and upper-ends.

** The GE_0-like index of Ravallion (2017) with the scale-adjustment para. set to a reciprocal of 10 times the average net wealth. This scale-dependent index is particularly sensitive to the existence of large negative values.

Note. Changes of the E-Gini index (Chakravrty 1988) with para.≤ 4 are close to the Gini index.

6. Concluding remarks

As mentioned above, for applications of a pair of the aS- and caS-Gini indices as sensitivity-adjustable indices, it would be better to set $\nu < 2$.[5] The empirical example illustrates a notable difference between the two indices when applied to distributions containing negative values.

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[5] If users place importance on specific properties such as the principle of positional transfer sensitivity (Mehran 1976, Kakwani 1980, Zoli 1999), it might be better to set $\nu > 2$.

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Appendices

Appendix 1. Atkinson indices for the Pareto and reciprocal Pareto distributions:

$$A_\varepsilon(P_\alpha) = \begin{cases} 1 - \alpha^{\varepsilon/(1-\varepsilon)} \frac{\alpha-1}{(\alpha-1+\varepsilon)^{1/(1-\varepsilon)}}, & 0 < \varepsilon \neq 1 \\ 1 - \frac{\alpha-1}{\alpha} e^{\frac{1}{\alpha}}, & \varepsilon = 1 \end{cases},$$

$$A_\varepsilon(RP_\beta) = \begin{cases} 1 - \beta^{\varepsilon/(1-\varepsilon)} \frac{\beta+1}{(\beta+1-\varepsilon)^{1/(1-\varepsilon)}}, & 0 < \varepsilon \neq 1 \\ 1 - \frac{\beta+1}{\beta} e^{-\frac{1}{\beta}}, & \varepsilon = 1 \end{cases}.$$

Appendix 2. GE indices for the Pareto and reciprocal Pareto distributions:

$$GE_\delta(P_\alpha) = \begin{cases} \frac{1}{\delta(\delta-1)}\left(\frac{\alpha^{1-\delta}(\alpha-1)^\delta}{\alpha-\delta} - 1\right), & \delta \neq 0, 1 \\ -\frac{1}{\alpha} + \log\frac{\alpha}{\alpha-1}, & \delta = 0 \\ \frac{1}{\alpha-1} - \log\frac{\alpha}{\alpha-1}, & \delta = 1 \end{cases},$$

$$GE_\delta(RP_\beta) = \begin{cases} \frac{1}{\delta(\delta-1)}\left(\frac{\beta^{1-\delta}(\beta+1)^\delta}{\beta+\delta} - 1\right), \delta \neq 0, 1 \\ \frac{1}{\beta} - \log\frac{\beta+1}{\beta}, & \delta = 0 \\ -\frac{1}{\beta+1} + \log\frac{\beta+1}{\beta}, & \delta = 1 \end{cases}.$$

Appendix 3. aS- and ᶜaS-Gini indices for the reciprocal Pareto-negative uniform distribution $RPNU_{\beta,p_0,x_0}$, $\beta > 0, 0 < p_0 < 1, -2\frac{1-p_0}{p_0}\frac{\beta}{\beta+1} < x_0 < 0$:

$$aSG_\nu(RPNU_{\beta,p_0,x_0}) = \begin{cases} \frac{1}{\nu-1}(1 + I_1 + I_2), & \nu \neq 1 \\ J_1 + J_2 + J_3, & \nu = 1, \end{cases}$$

where $I_1 = \nu(\nu-1)\frac{s_0}{p_0^2}\left[(2p_0 - 1)\frac{1-(1-p_0)^{\nu-1}}{\nu-1} + 2(1-p_0)\frac{1-(1-p_0)^\nu}{\nu} - \frac{1-(1-p_0)^{\nu+1}}{\nu+1}\right]$;

$I_2 = \nu s_0(1-p_0)^{\nu-1} - \nu(1+s_0)(1-p_0)^{\nu-1}\frac{\beta+1}{\beta}B\left(\nu, 1+\frac{1}{\beta}\right)$;

$J_1 = \left[-1 + \frac{s_0}{p_0^2}(1-2p_0)\right]\log(1-p_0) + \frac{s_0}{p_0}\left(1 - \frac{3}{2}p_0\right) - p_0$;

$J_2 = s_0(1-p_0)$, $J_3 = (1+s_0)\left(\psi\left(2+\frac{1}{\beta}\right) - \psi(1) - 1 + p_0\right)$;

$s_0 = -p_0\frac{x_0}{2}\Big/\left(p_0\frac{x_0}{2} + (1-p_0)\frac{\beta}{\beta+1}\right)$, i.e., the absolute share of negative values;

$\psi(x) = \Gamma'(x)/\Gamma(x)$, i.e., the digamma function,

$${}^c aSG_\nu(RPNU_{\beta,p_0,x_0}) = \frac{1}{\nu-1}(1 + K_1 + K_2 - K_3),$$

where $K_1 = \nu 2^{2\nu-1}s_0^{\nu-1}p_0 B_{1/2}(\nu,\nu)$; $K_2 = \nu(1-p_0)s_0^{\nu-1}\left(\frac{s_0}{1+s_0}\right)^{\beta/(\beta+1)}\frac{\beta}{\beta+1}B\left(\frac{\beta}{\beta+1},\nu\right)$;

$K_3 = \nu\frac{1-p_0}{1+s_0}\frac{\beta}{\beta+1}\sum_{i=0}^{\infty}\frac{(1/(\beta+1))_i}{i!}\frac{1}{(1+s_0)^i}B(\nu, i+1)$, $\left(\frac{1}{\beta+1}\right)_i = \frac{1}{\beta+1}\left(\frac{1}{\beta+1}+1\right)\cdots\left(\frac{1}{\beta+1} + i - 1\right)$ for $i = 1,2,\cdots$ or 1 for $i = 0$.

Appendix 4. *Proof* that if the positive distribution G is mutually L-symmetric with the positive distribution F that has pdf f, then G also has its pdf:

Proof: From (6), $L_F(F(x)) = 1 - G(y)$; $F(x) = 1 - L_G(G(y))$;

$\frac{x}{\mu_F}dF(x) = -dG(y)$; $dF(x) = -\frac{y}{\mu_G}dG(y)$; and hence, $y = \mu_F\mu_{F^c}/x$,

$$dG(y) = -\frac{x}{\mu_F}f(x)dx = \frac{\mu_F\mu_G^2}{y^3}f\left(\frac{\mu_F\mu_G}{y}\right)dy.$$

Thus, G has pdf $g(y) = \mu_F\mu_G^2/y^3 \cdot f(\mu_F\mu_G/y)$, identical to Taguchi's formula (1968). □

Appendix 5. *Proof* of Theorem 1:

Apply Theorem A1 (with a slight modification as explained below the theorem) to a pair F and G. Note that $aSG_\nu(G) = {}^c aSG_\nu(F)$ and $aSG_2(F) = {}^c aSG_2(F) = Gini(F)$, the Kolkata indices for F and G are identical, and L_F crosses L_G at least at the Kolkata index. □

Theorem A1 (Yitzhaki 1983): Assume that L_F crosses L_G once at $0 < p_0 < 1$, $L_F(p) \lesseqqgtr L_G(p)$ for $p \lesseqqgtr p_0$, and $\exists\nu_0$ s.t. $SG_{\nu_0}(F) = SG_{\nu_0}(G)$ then, $SG_\nu(F) \lesseqqgtr SG_\nu(G)$ for $\nu \lesseqqgtr \nu_0$.

Theorem A1 can be applied to $aSG_\nu, \nu > 0$, with a slight modification of the proof.

Appendix 6. *Proof* that a pair $RP_{\alpha-1}$ and $P_\alpha, \alpha > 1$ satisfy the prerequisite of Theorem 1:

First, we introduce the following lemma:

Lemma A2: Let the cdfs of positive distributions F and G with equal means be continuous and strictly increasing. If F and G (or equivalently, F^{-1} and G^{-1}) intersect twice and the sign sequence of $F(x) - G(x)$ is $+, -, +$ (that of $F^{-1}(p) - G^{-1}(p)$ is $-, +, -$), then L_F crosses L_G once at some p_0, and $L_F(p) \lesseqgtr L_G(p)$ for $p \lesseqgtr p_0$. [6]

Proof: Let $F^{-1}(p)$ and $G^{-1}(p)$ cross at a and b, $0 < a < b < 1$; then, $L_F(p) - L_G(p)$ decreases for $0 \le p \le a$ and $b \le p \le 1$ and increases for $a < p < b$ in the strict sense. As $L_F(p) - L_G(p) = 0$ at $p = 0,1$, the lemma must hold. □

Let F and G denote the cdfs of $RP_{\alpha-1}$ and P_α normalized to have means of one; then, $F^{-1}(p) = \frac{\alpha}{\alpha-1} p^{\frac{1}{\alpha-1}}$ and $G^{-1}(p) = \frac{\alpha-1}{\alpha}(1-p)^{-\frac{1}{\alpha}}$. Consider the ratio $\mathcal{R}(p) = F^{-1}(p)/G^{-1}(p) = Kp^{\frac{1}{\alpha-1}}(1-p)^{\frac{1}{\alpha}}, K = \left(\frac{\alpha}{\alpha-1}\right)^2$. $\mathcal{R}(p)$ has an inverted U-shape in the sense that $\mathcal{R}(p)$ attains its maximum at $p_m = \alpha/(2\alpha-1)$; $\mathcal{R}(0), \mathcal{R}(1) = 0$; and $\mathcal{R}(p)$ increases for $0 \le p \le p_m$ and decreases for $p_m < p \le 1$ in the strict sense. As $\mathcal{R}(p)$ ranges from zero to a maximum larger than unity, it coincides with unity at two points. This means that F^{-1} and G^{-1} cross twice with the sign sequence $-, +, -$. Thus, from Lemma A2, L_F crosses L_G once (at the Kolkata index common to $RP_{\alpha-1}$ and P_α).

A pair of double-Pareto distributions $dP_{\alpha,\beta}$ and $dP_{\beta+1,\alpha-1}$, $\alpha > \beta + 1, \beta > 0$ (see footnote 2) are also proved to satisfy the prerequisite in Theorem 1 in a similar manner. □

Appendix 7. *Proof* of Theorem 2:

First, assume that F is a nonnegative distribution with a point mass at zero and, at the same time, F satisfies the conditions in the theorem. In this case, $-L_F(p) \le 0 \le 1 - p$ for $0 < p < 1$, and $F^+ = F, L_{F^+} = L_F$. Let L_G and p_K denote the curve symmetric to L_F with respect to a diagonal other than the equality diagonal and the Kolkata index for F, respectively. Define a Lorenz curve $L^{(k)}$ as follows:

$$L^{(k)}(p) \coloneqq \max\left\{L_F(p), \tfrac{1}{k} L_G(p)\right\},\ k = 2, 3, \cdots.$$

Let $F^{(k)}$ denote $L^{(k)}$'s underlying distribution. As $L^{(k)}(p) \searrow L_F(p)$ when $k \to \infty$,

[6] If F and G intersect once, and the sign sequence of $F(x) - G(x)$ is $-, +$, then L_F does not intersect with L_G, and $L_F > L_G$ (Theorem 3.A.44 of Shaked and Shanthikumar 2006).

$aSG_\nu(F^{(k)}) \nearrow aSG_\nu(F)$ and ${}^c aSG_\nu(F^{(k)}) \nearrow {}^c aSG_\nu(F)$ when $k \to \infty$. From (4), for $k < l$,

$$aSG_\nu(F^{(l)}) - aSG_\nu(F^{(k)}) = \nu \int_0^1 (1-p)^{\nu-2}[L^{(k)}(p) - L^{(l)}(p)]dp = (*). \tag{A3}$$

As $1 - p > L^{(k)}(p)$ and $L^{(k)}(p) \geq L^{(l)}(p)$ for $p < p_K$, $L^{(k)}(p) > L^{(l)}(p)$ near $p = 0$, and $L^{(k)}(p) = L^{(l)}(p)$ for $p \geq p_K$, for $1 < \nu < 2$,

$$\begin{aligned}(*) &< \nu \int_0^1 L^{(k)}(p)^{\nu-2}[L^{(k)}(p) - L^{(l)}(p)]dp \\ &< \frac{\nu}{\nu-1}\int_0^1 [L^{(k)}(p)^{\nu-1} - L^{(l)}(p)^{\nu-1}]dp = {}^c aSG_\nu(F^{(l)}) - {}^c aSG_\nu(F^{(k)}).\end{aligned} \tag{A4}$$

The right-most equation is derived from (9). Analogously, for $2 < \nu$,

$$\begin{aligned}(*) &> \nu \int_0^1 L^{(k)}(p)^{\nu-2}[L^{(k)}(p) - L^{(l)}(p)]dp \\ &> \frac{\nu}{\nu-1}\int_0^1 [L^{(k)}(p)^{\nu-1} - L^{(l)}(p)^{\nu-1}]dp = {}^c aSG_\nu(F^{(l)}) - {}^c aSG_\nu(F^{(k)}).\end{aligned} \tag{A5}$$

As a pair of $F^{(k)}$ and its L-symmetric counterpart $G^{(k)}$ satisfy the conditions in Theorem 1, $aSG_\nu(F^{(k)}) < {}^c aSG_\nu(F^{(k)})$ for $1 < \nu < 2$. Hence, from (A4),

$$\begin{aligned}aSG_\nu(F) &= \lim_{l\to\infty}[aSG_\nu(F^{(l)}) - aSG_\nu(F^{(k)})] + aSG_\nu(F^{(k)}) \\ &< \lim_{l\to\infty}[{}^c aSG_\nu(F^{(l)}) - {}^c aSG_\nu(F^{(k)})] + {}^c aSG_\nu(F^{(k)}) = {}^c aSG_\nu(F).\end{aligned} \tag{A6}$$

Analogously, for $\nu > 2$, from (A5),

$$aSG_\nu(F) > {}^c aSG_\nu(F). \tag{A7}$$

Next, assume that F is a distribution containing negative values; then, (A6) and (A7) hold for F^+, an underlying distribution of $L_{F^+}(p) \coloneqq \max\{0, L_F(p)\}$. Then, we evaluate the differences between the indices for F and F^+ in a similar way as (A3) - (A5), as follows:

$$aSG_\nu(F) - aSG_\nu(F^+) = \nu \int_0^1 (1-p)^{\nu-2}[L_{F^+}(p) - L_F(p)]dp = (**). \tag{A8}$$

From the assumption, $L_{F^+}(p) = 0$ and $1 - p \geq |L_F(p)|$ for p s.t. $L_F(p) < 0$, and $L_{F^+}(p) = L_F(p)$ for p s.t. $L_F(p) \geq 0$. Hence, for $1 < \nu < 2$,

$$\begin{aligned}(**) &< \nu \int_0^1 |L_F(p)|^{\nu-2}[L_{F^+}(p) - L_F(p)]dp \\ &< \frac{\nu}{\nu-1}\int_0^1 [L_{F^+}(p)^{\nu-1} - \operatorname{sgn} L_F(p) \cdot |L_F(p)|^{\nu-1}]dp = {}^c aSG_\nu(F) - {}^c aSG_\nu(F^+),\end{aligned} \tag{A9}$$

and for $\nu > 2$,

$$\begin{aligned}(**) &> \nu \int_0^1 |L_F(p)|^{\nu-2}[L_{F^+}(p) - L_F(p)]dp \\ &> \frac{\nu}{\nu-1}\int_0^1 [L_{F^+}(p)^{\nu-1} - \operatorname{sgn} L_F(p) \cdot |L_F(p)|^{\nu-1}]dp = {}^c aSG_\nu(F) - {}^c aSG_\nu(F^+).\end{aligned} \tag{A10}$$

As $aSG_\nu(F^+) < {}^c aSG_\nu(F^+)$ for $1 < \nu < 2$ from (A6) (after replacing F with F^+),

$$\begin{aligned}&aSG_\nu(F) - {}^c aSG_\nu(F) = [aSG_\nu(F) - aSG_\nu(F^+)] + aSG_\nu(F^+) \\ &-[{}^c aSG_\nu(F) - {}^c aSG_\nu(F^+)] - {}^c aSG_\nu(F^+) < aSG_\nu(F^+) - {}^c aSG_\nu(F^+) < 0.\end{aligned} \tag{A11}$$

Analogously, for $\nu > 2$,

$$aSG_\nu(F) - {}^c aSG_\nu(F) > aSG_\nu(F^+) - {}^c aSG_\nu(F^+) > 0. \quad \text{(A12)}$$

Note that condition $-L_F(p) \le 1 - p$ holds if both $-\inf F^{-1} < \mu_F$ and $L_F(0.5) \ge 0$ are satisfied because $-L_F(p) = -\int_0^p \frac{F^{-1}(q)}{\mu_F} dq \le p \le 1 - p$ for $0 < p \le 0.5$. □

Appendix 8. Formulas for aSG_ν and ${}^c aSG_\nu$ for calculation from sample survey data.

Formulas (A13) - (A16) correspond to discretizations of (3), (5), (11) and (10), respectively, and are applicable to weighted discrete data $D = \{x_i, w_i\}$, $x_1 < \cdots < x_n$, $w_1, \cdots, w_n > 0$. Before applying the formulas, we collapse records with an identical value into a single record if duplicates exist in the array $\{x_i\}_i$. For example, if $x_2 < x_3 = x_4 < x_5$, then we replace the weight for the 3rd record w_3 with $w_3 + w_4$ and remove the 4th record.

$$aSG_\nu(D) = \frac{1}{\nu-1}\left[1 - \frac{1}{\hat{\mu}}\sum_{i=2}^{n}(1 - c_{i-1})^\nu (x_i - x_{i-1})\right]; \quad \text{(A13)}$$

$$aSG_1(D) = -\frac{1}{\hat{\mu}}\sum_{i=2}^{n}(1 - c_{i-1})\log(1 - c_{i-1}) \cdot (x_i - x_{i-1}); \quad \text{(A14)}$$

$${}^c aSG_\nu(D) = \frac{1}{\nu-1}\left(1 - \sum_i \Delta_i p_i\right), \Delta_i = \nu \cdot \mathrm{sgn}(l_i) \cdot |l_i|^{\nu-1} \text{ for } l_{i-1} = l_i,$$
$$|l_i|^{\nu-1} - |l_{i-1}|^{\nu-1} \text{ for } l_{i-1} < 0 < l_i, \text{ and } \frac{|l_i|^\nu - |l_{i-1}|^\nu}{l_i - l_{i-1}} \text{ otherwise}; \quad \text{(A15)}$$

$${}^c aSG_1(D) = -\sum_{i=1}^{n} \frac{l_i \log l_i - l_{i-1}\log l_{i-1}}{l_i - l_{i-1}} p_i; \quad \text{(A16)}$$

where $\nu \ne 1$ in (A13) and (A15); $\hat{\mu} = \sum_{j=1}^n w_j x_j / \sum_{j=1}^n w_j$; $c_i = \sum_{j=1}^i w_j / \sum_{j=1}^n w_j$, $l_i = \sum_{j=1}^i w_j x_j / \sum_{j=1}^n w_j x_j$, and $p_i = w_i / \sum_{j=1}^n w_j$, $i = 1,2,\cdots,n$; $l_0 = 0$; $l_0 \log l_0 = 0$. (A13) with multiplication by $\nu - 1$ is equivalent to the formula of Chotikapanich and Griffiths (2001) for the S-Gini index.