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Differential multiplier effect in the Leontief-Keynes model

Louis de Mesnard
CREGO, University of Burgundy

Abstract

“Closing” input-output models, i.e., taking into account the fact that households spend the income they receive from the productive sectors, is an important issue for the development of realistic macroeconomic models. Classical Type II multipliers fail to do that. This is why we consider here what we term the Leontief-Keynes (LK) model by introducing the Keynesian circuit into the Leontief model. We find that LK multipliers are deduced from the Leontief multipliers by adding to them a constant equal to the product of the marginal propensity to consume, the Keynesian multiplier, and the average ordinary output multiplier. With respect to the ordinary Leontief multipliers, the smallest LK multipliers are multiplied by a value greater than k, the greatest LK multipliers are multiplied by a value lower than k, such that the average LK multiplier is exactly multiplied by k. Thus, the macroeconomic Keynesian effect is not applied uniformly among sectors in relative terms: the better placed a sector is, with respect to its Leontief multiplier, the less it benefits from the Keynesian multiplier, and conversely. A numerical example shows that Type II multipliers are nevertheless a particular case of LK multipliers, both coinciding when the “pivot value” for the marginal propensity to consume is chosen.

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Contact: Louis de Mesnard - louis.de-mesnard@u-bourgogne.fr.
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1. Introduction

The Leontief (1986) input-output model remains the basis for national accounts and most economic models at the macro, international, interregional, regional and urban levels and also for environmental economics (Leontief 1970; Kagawa 2012). The output multipliers play a central role: they indicate what the increase in the total output in all industries of the economy is when the demand increases of $1 (Miller & Blair 2022). However, the Leontief model is *opened*: the macroeconomic aggregate "households" receives the value-added of the sectors but does not spend this income. If we want to build a realistic macro-economic model, it is important to *close* the model, that is, to examine how households spend the income they receive from the sectors. Even in its 1928 paper (Leontief 1991), yet entitled "The economy as a circular flow", Leontief has never considered closing his model. Even though they allow to take into account the households, Miyazawa (1960)'s and the classical Type II multipliers approaches[1] fail to *close* the Leontief model. Miller & Blair (2022, p. 36) clearly explain that households are treated as a supplementary sector and the final demand of households in each commodity is endogenous with respect to the output of this commodity: household behavior is treated in the same way as producer behavior (Miller & Blair 2022). Moreover, there is no propensity to consume.

An alternate approach consist into combining the Leontief model and the Keynes (1936) model as done by Miyazawa and Masegi (1963) to give which we term the Leontief-Keynes (hereafter LK) model.[2] This idea will conduct us to introduce a true circuit —industrial sectors / households / industrial sectors—, by making the household consumption endogenous with respect to the income, and by introducing an exogenous marginal propensity to consume, the income being the sum of the sectors' value added. This will allows us to demonstrate some original and interesting properties about the way the Keynesian multiplier is applied among sectors. We will focus here on output multipliers.

2. The LK model

2.1. Reminder: The traditional approach

Consider the following accounting equation $\mathbf{Z}\mathbf{s}_n + \mathbf{f} = \mathbf{x}$, where $\mathbf{Z} \geq \mathbf{0}$ is the matrix of deliveries from sector i in row to sector j in column, $\mathbf{x} \geq \mathbf{0}$ the output vector, $\mathbf{f} \geq \mathbf{0}$ the final demand (or net output) vector, all in currency units, and $\mathbf{s}_n$ indicates that the sum vector $\mathbf{s}$ is of dimension n. The Leontief (1985) model writes as $\mathbf{A}\mathbf{x} + \mathbf{f} = \mathbf{x}$, where $\mathbf{A} = \mathbf{Z}[\hat{\mathbf{x}}]^{-1}$ is the technical coefficients matrix of dimensions (n, n), the hat denoting a diagonal matrix made from a vector. Its solution is given by $\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{f}$. Following the Rouché-Capelli-Kronecker-Fontené-Frobenius theorem, the solution is unique because $n = rank(\mathbf{I} - \mathbf{A})$ if and only if $\det(\mathbf{I} - \mathbf{A}) \neq 0$. Thus, as $\mathbf{A}$ is productive by construction ($\mathbf{A}\mathbf{x} \leq \mathbf{x}$), which implies that the maximal eigenvalue in absolute value is less than 1 (Nikaido 1972, chapter 3; Murata 1977, chapter 4), the solution is nonnegative from the moment that $\mathbf{f}$ is nonnegative. From there, Leontief multipliers are classically defined. Consider a change in the final demand of sector 1: $\Delta f_1 = 1$. This generates an increase in the output of all sectors

[1] See: Moore and Peterson (1955), Hirsch (1959), Sandoval (1967), Bradley and Gander (1967), Katz (1980), ten Raa and Chakraborty (1983), Batey 1985, Miller & Blair (2022).

[2] Synthetized in Miyazawa (1976). Notice that this model has nothing to do with what Morishima and Nosse (1972, pp. 76-95) call a "mixed Leontief-Keynes model", an extension of Miyazawa (1960)'s model.

given by the first column of $(\mathbf{I}-\mathbf{A})^{-1}$. This is why Leontief multipliers are defined as the sum of the terms of the inverse matrix $(\mathbf{I}-\mathbf{A})^{-1}$: $\mathbf{m}' = \mathbf{s}_n' (\mathbf{I}-\mathbf{A})^{-1}$, the prime denoting the transposition operator.

In the approach of the so-called *Type I* and *Type II* multipliers, households are treated as a new sector. Consider the following accounting equations where the first n rows and columns concern the sectors and the $n+1$ row and column concern the households:

$$\begin{bmatrix} \mathbf{Z} & \mathbf{z}_h^D \\ \mathbf{z}_h^{L\prime} & 0 \end{bmatrix} + \begin{bmatrix} \mathbf{f}^* \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{x} \\ x_h \end{bmatrix} \tag{1}$$

where $\mathbf{z}_h^D$ is the household demand vector (in this simplified version, we exclude the intra-household consumption, for instance, housekeeping services, private lessons, etc.), $\mathbf{z}_h^L$ is the vector of purchases of labor by sectors, $z_{h,h}$ is the; x_h is the output of the household sector (a very questionable idea), $\mathbf{f}^*$ is the remaining final demand (without those of households captured in $\mathbf{z}_h^D$). In terms of coefficients, $\mathbf{h}^D$ is the vector of household consumption coefficients: $\widehat{\mathbf{h}^D} = \widehat{\mathbf{z}_h^D} \begin{bmatrix} \hat{\mathbf{x}} & 0 \\ 0 & x_h \end{bmatrix}^{-1}$, $\mathbf{h}^L$ the vector of labor input coefficients: $\widehat{\mathbf{h}^L} = \widehat{\mathbf{z}_h^L} \begin{bmatrix} \hat{\mathbf{x}} & 0 \\ 0 & x_h \end{bmatrix}^{-1}$, and $h = z_{h,h}/x_h$ the intra-household consumption coefficient. We define also a matrix of coefficients of dimensions $(n+1, n+1)$: $\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{h}^D \\ \mathbf{h}^{L\prime} & 0 \end{bmatrix}$. This gives the following model

$$\begin{bmatrix} \mathbf{A} & \mathbf{h}^D \\ \mathbf{h}^{L\prime} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ x_h \end{bmatrix} + \begin{bmatrix} \mathbf{f}^* \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{x} \\ x_h \end{bmatrix}$$

It is not really closed in the sense of the economic theory of the circuit: we have no circuit here because, $\mathbf{z}_h^D$ remains exogenous with no propensity to consume.

2.2. Combining the Leontief model with the Keynesian circuit

One of the main characteristics of the Keynesian circuit is that households take their income from the productive sector and spend it with the productive sector. We define the vector of coefficients (of dimension n) that indicate how an income is spent on each commodity

$$\mathbf{d} = \mathbf{f} f^{-1} \tag{2}$$

where $f = \mathbf{s}_n'\mathbf{f}$ is the total final demand or total consumption in currency units. We notice that (2) is a consumption function of fixed coefficients. Obviously, it holds that $\mathbf{s}_n'\mathbf{d} = \mathbf{d}\mathbf{s}_n = 1$. This allows us to define the average Leontief output multiplier as the mean of sectors' Leontief output multipliers weighted by the demand coefficients:

$$m = \mathbf{m}'\mathbf{d} \tag{3}$$

We define the vector of labor input coefficients $\mathbf{w}$: $\hat{\mathbf{w}} = \hat{\mathbf{v}} [\hat{\mathbf{x}}]^{-1}$ where $\mathbf{v}$ is the vector (of dimension n) of quantities of labor in currency units, i.e., the vector of sectors income. We have:

$$\mathbf{w}' = \mathbf{s}_n' (\mathbf{I}-\mathbf{A}) \Leftrightarrow \mathbf{w}' (\mathbf{I}-\mathbf{A})^{-1} = \mathbf{s}_n' \tag{4}$$

The Keynesian circuit is as follows. Consider a change $\Delta\mathbf{x}$ in the output vector. This generates a change $\Delta\mathbf{v} = \hat{\mathbf{w}}\Delta\mathbf{x}$ in the sectors income. The change of the total income is $\Delta R = \mathbf{s}_n'\Delta\mathbf{v} = \mathbf{w}'\Delta\mathbf{x}$. It generates an increase of the total final demand $\Delta F = c\Delta R = c\mathbf{w}'\Delta\mathbf{x}$, where c is the propensity to consume (note that c denotes $z_{h,h}$ in Katz (1980)), calculated as the ratio between consumption and national income and $1-c$ is the savings leakage ratio, both well known in the Keynesian model. This itself generates an increase of final demands in each commodity following (2): $\Delta\mathbf{f}_h = \Delta F\mathbf{d} = (\mathbf{w}'\Delta\mathbf{x})\,c\mathbf{d}$. We denote by $k = 1/(1-c)$ the Keynesian multiplier.

To combine Leontief and Keynes we follow Miyazawa and Masegi (1963). We add a new vertex to the graph: the households, that is, to add a row $\mathbf{w}'$ and a column $c\mathbf{d}$ to the matrix $\mathbf{A}$. So, we have a circuit, as in the Keynes model. See Figure 1.

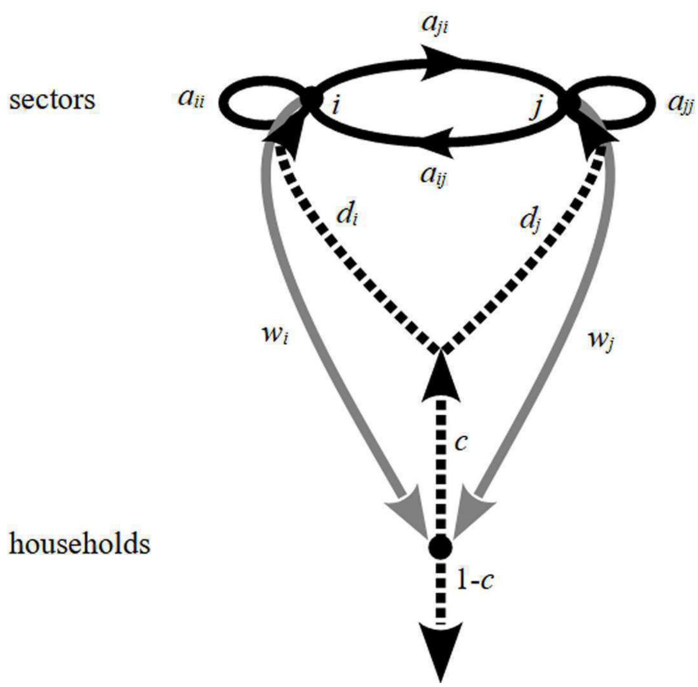

Figure 1: Graph of the Leontief-Keynes circuit for two sectors i and j. a_{ij} is the term $\{i,j\}$ of matrix $\mathbf{A}$, d_i is the term $\{i\}$ of vector $\mathbf{d}$, w_i is the term $\{i\}$ of vector $\mathbf{w}$, and c is the propensity to consume.

This leads us to define a matrix

$$\mathbf{E} = \begin{bmatrix} \mathbf{A} & c\mathbf{d} \\ \mathbf{w}' & 0 \end{bmatrix}$$

of dimensions $(n+1, n+1)$, where the term $(n+1, n+1)$ is null because there is no vertex households/households in a Keynesian circuit. In matrix $\mathbf{E}$, the n^{th} row and column correspond to the households. The cell $\{n+1, n+1\}$ is set to zero, i.e., there is no intra-consumption from households to households, because, unlike in the Type-I and Type-II approach or Miyazawa (1960), *households are not treated as a productive sector*. Now, we can write the LK model as follows: $\mathbf{Ez} + \mathbf{g} = \mathbf{z}$, where R is the total income, $\mathbf{g} \equiv \begin{bmatrix} \mathbf{f} \\ f_h \end{bmatrix}$ and $\mathbf{z} \equiv \begin{bmatrix} \mathbf{x} \\ y \end{bmatrix}$. Its solution is given by

$$\mathbf{z} = (\mathbf{I} - \mathbf{E})^{-1}\,\mathbf{g} \tag{5}$$

2.3. Existence of the solution of the LK model

Proposition 1. *It holds that* $\det(\mathbf{I}-\mathbf{A})/\det(\mathbf{I}-\mathbf{E}) = k$ *(Sandoval 1967).*

Proof. $\det(\mathbf{I}-\mathbf{E}) = \det\left(\begin{bmatrix} \mathbf{I}-\mathbf{A} & -c\mathbf{d} \\ -\mathbf{w}' & 1 \end{bmatrix}\right) = \det(\mathbf{I}-\mathbf{A})\det\left(1-\mathbf{w}'(\mathbf{I}-\mathbf{A})^{-1}c\mathbf{d}\right)$ by applying Liebnitz' formula. Thus, $\det(\mathbf{I}-\mathbf{E}) = \det(\mathbf{I}-\mathbf{A})\det(1-c\mathbf{s}'\mathbf{d}) = k^{-1}\det(\mathbf{I}-\mathbf{A})$. □

It is handy to define also a direct matrix in a compact form, denoted $\tilde{\mathbf{E}}$, of dimensions (n,n):

$$\tilde{\mathbf{E}} = \mathbf{A} + c\mathbf{d}\mathbf{w}'$$

where $\tilde{e}_{ij} = a_{ij} + cd_i w_j$. A term $\tilde{e}_{ij}$ indicates the payments of sector j to sector i for delivering the input i (a_{ij}) *plus* the monetary effect on sector i of the income paid by sector j to households. We have $\mathbf{s}_n'\left(\mathbf{I}-\tilde{\mathbf{E}}\right) = (1-c)\mathbf{w}'$ because $\mathbf{s}_n'\left(\mathbf{I}-\tilde{\mathbf{E}}\right) = \mathbf{s}_n'(\mathbf{I}-\mathbf{A}) - c\mathbf{s}_n'\mathbf{d}\mathbf{w}' = \mathbf{w}' - c\mathbf{w}'$.

Proposition 2. $\left(\mathbf{I}-\tilde{\mathbf{E}}\right)^{-1} = (\mathbf{I}-\mathbf{A})^{-1}(\mathbf{I}+ck\mathbf{d}\mathbf{s}_n')$.

Proof. $\mathbf{I}-\tilde{\mathbf{E}} = \mathbf{I}-\mathbf{A}-c\mathbf{d}\mathbf{w}'$. We have $\det\left(\mathbf{I}-\tilde{\mathbf{E}}\right) > 0$ because $\det(\mathbf{I}-\mathbf{A}) > 0$. Assume that proposition 2 holds. Therefore, $\left(\mathbf{I}-\tilde{\mathbf{E}}\right)\left(\mathbf{I}-\tilde{\mathbf{E}}\right)^{-1} = (\mathbf{I}-\mathbf{A}-c\mathbf{d}\mathbf{w}')\left((\mathbf{I}-\mathbf{A})^{-1} + \frac{c}{1-c}(\mathbf{I}-\mathbf{A})^{-1}\mathbf{d}\mathbf{s}'\right)$
$= \mathbf{I} + \frac{c}{1-c}\mathbf{d}\mathbf{s}' - c\mathbf{d}\left(\mathbf{w}'(\mathbf{I}-\mathbf{A})^{-1}\right) - \frac{c^2}{1-c}\mathbf{d}\left(\mathbf{w}'(\mathbf{I}-\mathbf{A})^{-1}\right)\mathbf{d}\mathbf{s}'$
$= \mathbf{I} + c\mathbf{d}\mathbf{s}'\left(\frac{1}{1-c} - 1 - \frac{c}{1-c}\right) = \mathbf{I}$ because $\mathbf{w}'(\mathbf{I}-\mathbf{A})^{-1} = \mathbf{s}'$ because of (4) and $\mathbf{s}'\mathbf{d} = 1$.
The same holds for $\left(\mathbf{I}-\tilde{\mathbf{E}}\right)^{-1}\left(\mathbf{I}-\tilde{\mathbf{E}}\right)$. So, proposition 2 is true. □

Proposition 3.

$$(\mathbf{I}-\mathbf{E})^{-1} = \left[\begin{array}{c|c} \left(\mathbf{I}-\tilde{\mathbf{E}}\right)^{-1} & ck(\mathbf{I}-\mathbf{A})^{-1}\mathbf{d} \\ \hline k\mathbf{s}_n' & k \end{array}\right] \tag{6}$$

Proof. $\mathbf{I}-\mathbf{E} = \left[\begin{array}{c|c} \mathbf{I}-\mathbf{A} & -c\mathbf{d} \\ \hline -\mathbf{w}' & 1 \end{array}\right]$. Assume that (6) is true. Then, $(\mathbf{I}-\mathbf{E})(\mathbf{I}-\mathbf{E})^{-1}$

$$= \left[\begin{array}{c|c} (\mathbf{I}-\mathbf{A})\left(\left(\mathbf{I}-\tilde{\mathbf{E}}\right)^{-1} - \frac{c}{1-c}\mathbf{d}\mathbf{s}'\right) & (\mathbf{I}-\mathbf{A})\frac{c}{1-c}(\mathbf{I}-\mathbf{A})^{-1}\mathbf{d} - \frac{c}{1-c}\mathbf{d} \\ \hline -\mathbf{w}'\left(\mathbf{I}-\tilde{\mathbf{E}}\right)^{-1} + \frac{1}{1-c}\mathbf{s}' & -\mathbf{w}'\frac{c}{1-c}(\mathbf{I}-\mathbf{A})^{-1}\mathbf{d} + \frac{1}{1-c} \end{array}\right]$$

$$= \left[\begin{array}{c|c} \mathbf{I} + \frac{c}{1-c}\mathbf{d}\mathbf{s}' - \frac{c}{1-c}\mathbf{d}\mathbf{s}' & \frac{c}{1-c}\mathbf{d} - \frac{c}{1-c}\mathbf{d} \\ \hline -\left(\mathbf{s}' + \frac{c}{1-c}\mathbf{s}'\mathbf{d}\mathbf{s}'\right) + \frac{1}{1-c}\mathbf{s}' & -\frac{c}{1-c}\mathbf{s}'\mathbf{d} + \frac{1}{1-c} \end{array}\right] = \left[\begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right].$$

The same holds for $(\mathbf{I}-\mathbf{E})^{-1}(\mathbf{I}-\mathbf{E})$. So, (6) is true. □

We can now explicitly compute the solution (5) (notice that the labor (vector $\mathbf{w}'$) does not explicitly appear in $(\mathbf{I}-\mathbf{E})^{-1}$ even if it is included in matrix $\mathbf{E}$). A familiar result is that the solution of the Leontief model exists only if $\det(\mathbf{I}-\mathbf{A}) \neq 0$. The question is: assuming that the Leontief model has a solution, does the closed model have one also?

Proposition 4. *It holds that* $\det\left(\mathbf{I}-\tilde{\mathbf{E}}\right) = \det(\mathbf{A}+c\mathbf{d}\mathbf{w}')$.

Proof. $\det\left(\mathbf{I}-\tilde{\mathbf{E}}\right)=\det\left(\mathbf{I}-\mathbf{A}\right)-c\det\left(\mathbf{dw}'\right)$. As $\det\left(\mathbf{dw}'\right)=0$, $\det\left(\mathbf{I}-\tilde{\mathbf{E}}\right)=\det\left(\mathbf{I}-\mathbf{A}\right)$. □

Proposition 5. If the Leontief model has no solution, the LK model has no solution.

Proof. We have to prove that $\det\left(\mathbf{I}-\mathbf{A}\right)=0\Leftrightarrow\det\left(\mathbf{I}-\tilde{\mathbf{E}}\right)=0$ and $\det\left(\mathbf{I}-\mathbf{A}\right)=0\Leftrightarrow\det\left(\mathbf{I}-\mathbf{E}\right)=0$. The proof is obvious from lemmas 4 and 1, respectively. □

3 The output multipliers in the multi-sector LK model

The inverse matrix (6) writes as

$$\left(\mathbf{I}-\mathbf{E}\right)^{-1}=\left[\begin{array}{c|c}\left(\mathbf{I}-\mathbf{A}\right)^{-1}\left(\mathbf{I}+ck\mathbf{ds}_n'\right) & ck\left(\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{d}\\ \hline k\mathbf{s}_n' & k\end{array}\right]\tag{7}$$

The value of LK multipliers is found by summing the terms of the columns of $\left(\mathbf{I}-\mathbf{E}\right)^{-1}$ (see table 1).

LK multipliers	n sectors	households
n sectors	$\mu'=\mathbf{s}_n'\left(\mathbf{I}-\mathbf{A}\right)^{-1}\left(\mathbf{I}+ck\mathbf{ds}_n'\right)$	$\mu^{H,S}=ck\mathbf{s}_n'\left(\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{d}=c\mu$
households	$\mu^{S,H\prime}=k\mathbf{s}_n'$	$\mu^H=k$

Table 1: LK multipliers in the multi-sector LK model: notations.

Theorem 1. *When we switch from the Leontief model to the LK model, all output LK multipliers are increased by the same amount, ckm:* $\mu_j=m_j+ckm$ for any $j=1$ to n.

Proof. From (7), $\mu'=\mathbf{s}_n'\left(\mathbf{I}-\mathbf{A}\right)^{-1}\left(\mathbf{I}+ck\mathbf{ds}_n'\right)=\mathbf{m}'+ck\mathbf{m}'\mathbf{ds}_n'=\mathbf{m}'+ckm\mathbf{s}_n'$. □

Corollary 1. *The order of output multipliers is preserved when we switch from the Leontief model to the LK model. This order does not depend on the propensity to consume, c.*

Proof. We deduce from theorem 1 that $m_{j_1}>m_{j_2}\Leftrightarrow\mu_{j_1}>\mu_{j_2}$ for any $j_1=1..n$ and $j_2=1..n$, $j_1\neq j_2$. c does not play any role. □

Corollary 2. *In relative terms of the ratio of Leontief output multipliers to LK output multipliers, the lower the Leontief output multipliers are, the larger the Keynesian effect is.*

Proof. $\mu_j/m_j=\left(m_j+mkc\right)/m_j=1+m_j^{-1}mkc$. Thus, the differential Keynesian effect in percentage, measured by the ratio μ_j/m_j, is inversely proportional to the Leontief output multiplier m_j. □

Proposition 6. *The average output LK multiplier, μ, is equal to the average Leontief output multiplier m times the Keynesian multiplier k: $\mu=mk$.*

Proof. The average LK output multipliers is $\mu=\mu'\mathbf{d}$. From equation (3), we deduce that $\mu=\mu'\mathbf{d}=\mathbf{m}'\mathbf{d}+ckm\mathbf{s}_n'\mathbf{d}=m+ckm=mk$. □

Proposition 7. $m_j=m\Rightarrow\mu_j=\mu$; $m_j>m\Rightarrow\mu_j>\mu$; $m_j<m\Rightarrow\mu_j<\mu$.

Proof. We use proposition 6: $\mu_j/\mu = \left(m_j + mkc\right)/mk = c + k^{-1}m_j/m$. □

Proposition 8. *The value of the LK households/sector multiplier* $\mu^{H,S}$ is equal to $c\mu$.

Proof. From (6), we have $\mu^{H,S} = ck\mathbf{m}'\mathbf{d} = ckm = c\mu$. □

Proposition 9. $\mu_j = m_j + c\mu = m_j + \mu^{H,S}$ for any $j = 1$ to n.

Proof. The proof is immediate by combining theorem 1, propositions 6 and 8. □

4. Numerical example

We develop here a classroom numerical example to illustrate what LK output multipliers are from (1) with the data of table 2.

	Sector 1	Sector 2	*Household demand*	*Other demand*	Total demand	Total income
Sector 1	10	20	*25*	*15*	40	70
Sector 2	30	40	*5*	*5*	10	80
Labor	30	20				50
Total cost	70	80	*30*	*20*	50	

Table 2: Numerical example: data

For the Leontief output multipliers We have $\mathbf{A} = \begin{bmatrix} 1/7 & 1/4 \\ 3/7 & 1/2 \end{bmatrix}$, $(\mathbf{I}-\mathbf{A})^{-1} = \begin{bmatrix} 1.556 & .778 \\ 1.333 & 2.667 \end{bmatrix}$ and the output multipliers are $\mathbf{s}'(\mathbf{I}-\mathbf{A})^{-1} = \begin{bmatrix} 2.889 & 3.444 \end{bmatrix}$. For Type II output multipliers, we have $\bar{\mathbf{A}} = \begin{bmatrix} 1/7 & 1/4 & 1/2 \\ 3/7 & 1/2 & 1/10 \\ 3/7 & 1/4 & 1/5 \end{bmatrix}$, $\left(\mathbf{I}-\bar{\mathbf{A}}\right)^{-1} = \begin{bmatrix} 35/6 & 91/18 & 77/18 \\ 6 & 22/3 & 14/3 \\ 5 & 5 & 5 \end{bmatrix}$ and the output multipliers are given by $\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}\left(\mathbf{I}-\bar{\mathbf{A}}\right)^{-1} = \begin{bmatrix} 71/6 & 223/18 \end{bmatrix}$.

To find $\mathbf{E}$, we need to calculate vector $\mathbf{d}$: it is more logical to take the total consumption $\mathbf{f} = \begin{bmatrix} 40 \\ 10 \end{bmatrix}$, entirely paid by the value added. We have $\mathbf{s}'\mathbf{f} = 50$ and $\mathbf{d} = \begin{bmatrix} 4/5 \\ 1/5 \end{bmatrix}$; thus $m = 3$. And we must consider the propensity to consume $c < 1$. We take it as a parameter. This gives $c\mathbf{d} = \begin{bmatrix} 4c/5 \\ c/5 \end{bmatrix}$. We have $\mathbf{w}' = \begin{bmatrix} 1 & 1 \end{bmatrix}(\mathbf{I}-\mathbf{A}) = \begin{bmatrix} 3/7 & 1/4 \end{bmatrix}$. Thus $\mathbf{E} = \begin{bmatrix} 1/7 & 1/4 & 4c/5 \\ 3/7 & 1/2 & c/5 \\ 3/7 & 1/4 & 0 \end{bmatrix}$,

$$(\mathbf{I}-\mathbf{E})^{-1} = \begin{bmatrix} \frac{7}{45}k(10-c) & \frac{7}{45}k(5+4c) & \frac{7}{5}c \\ \frac{4}{15}k(5+c) & \frac{8}{15}(5-2c) & \frac{8}{5}kc \\ k & k & k \end{bmatrix}$$

with LK output multipliers $\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}(\mathbf{I}-\mathbf{E})^{-1} = \begin{bmatrix} \frac{1}{9}k(26+c) & \frac{1}{9}k(-31+4c) \end{bmatrix}$.

We verify that the difference between LK output multipliers and the corresponding Leontief output multipliers is equal to $ckm = \frac{3c}{1-c}$. For sector 1, for which the Leontief output multiplier is below m, the LK output multiplier is more increased in percentage by respect to the Leontief output multiplier than for sector 2 for which the Leontief output multiplier is above: sector 1 benefits of a differential Keynesian effect by respect to sector 2. Moreover, we find a ratio of 1.22 between sectors' Leontief output multipliers and $\frac{26+c}{31-4c} < 1$ between sectors' LK output multipliers: the range is reduced. When $c = \frac{161}{296} = .544$ (the "pivot value", found by solving $\frac{1}{9}\frac{26+c}{1-c} = \frac{97}{15}$ and $\frac{1}{9}\frac{31-4c}{1-c} = \frac{97}{15}$, two equations that have the same solution) LK output multipliers are equal to Type II output multipliers and when $c > \frac{161}{296}$, they are above. The figure in appendix draws the LK output multipliers by respect to c: they do not evolve linearly by respect to c even if the model is globally linear and the relative difference, $\frac{156+5c}{186-25c}$, between both LK output multipliers decreases when c grows.

5. Conclusion

In the Leontief input-output model, Type II output multipliers are the classical approach to take into account the consumption effect on the sectors but do not really succeed in *closing* the economy. To the contrary, the Leontief-Keynes (LK) model really makes endogenous consumption by introducing the Keynesian circuit. LK output multipliers μ_j are deduced from Leontief output multipliers m_j by adding to them a constant equal to the product of the marginal propensity to consume c, the Keynesian multiplier k and the average Leontief output multiplier m. Relatively to the Leontief output multipliers, the smallest LK output multipliers are multiplied by a value greater than k, the greatest LK output multipliers are multiplied by a value lower than k, such that the average LK output multiplier is exactly multiplied by k.

Thus, we shown a relative differential macroeconomic Keynesian effect which is not applied uniformly among sectors in relative terms, even if the hierarchy of output multipliers remains unchanged (the better placed a sector is, with respect to its Leontief output multiplier, the less it benefits from the Keynesian effect, and conversely): the interest of some sectors, for which the Leontief output multiplier is below the mean, is underestimated with Leontief output multipliers. With LK output multipliers, the selection of sectors for economic stimulus policy is less important because their range of variation is smaller. Type II multipliers are nevertheless a particular case of LK multipliers (both coincide when the "pivot value" for the propensity to consume is chosen), LK multipliers should always been used.

The data required for using them are not different to those of Type II output multipliers: they are standard national accounts. LK output multipliers are not more difficult to compute than Type II output multipliers, except that c must be determined: their applicability is the same. These results could be generalized, for instance by splitting consumption into parts as in Kim et al. (1985).

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