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Financial transaction taxes and expert advice

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Abstract

This note uses strategic communication to study the impact of a financial transaction tax on expert advice to traders. The central finding is that the tax worsens expert advice by strengthening experts' incentives to misreport information. Such negative tax impact is moderated by expert commissions and exacerbated by uncertainty. Expert advice thus emerges as a new channel through which the tax makes traders less informed. This result advances the debate on tax suitability beyond the conventional arguments.

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1. Introduction

In the wake of the financial crisis, several countries decided to tax financial transactions. France introduced a 0.2 percent tax on security transactions in 2012, Italy a 0.1 percent tax in 2013, and Spain a 0.2 percent tax in 2021. The European Commission has advanced a proposal to introduce a pan-European financial transaction tax that is currently discussed. Similar proposals have been made in the United States and in many other countries.

The idea of taxing financial transactions has sparkled a longstanding debate. According to proponents like Tobin (1978), the tax should improve markets' informational efficiency by curbing the proportion of trading unrelated to information (noise trading). Opponents like Edwards (1993) contend that the tax reduces the proportion of informed trading and hence prevents markets from incorporating all available information.[1]

Yet, informed traders act on information from experts. Institutional investors trade following advice of sell-side analysts. Retail investors trade following recommendations of financial advisors. Such experts are remunerated through trading commissions. Commission gains shape experts' incentives to report information (Jackson, 2005). By discouraging trading, the tax hence influences experts' incentives. The strategic communication model developed in this note studies the tax impact on expert advice to traders.

Our model complements a number of other theoretical studies. Pallay (1999) shows that the tax raises informational efficiency when the proportion of noise relative to informed traders is high. Song and Zhang (2005) also consider the negative tax impact on liquidity. Sørensen (2017) studies the tax impact on adverse selection between noise and informed traders. Cipriani *et al.* (forthcoming) investigate the tax impact in a sequential trading model with price-elastic noise and informed traders. Davila (forthcoming) derives optimal taxes as a function of noise and informed trading proportions. Our focus is different. We aim at shedding light on how the tax affects traders' information through expert advice.

2. Model

To model trading on expert advice with financial transaction taxes, we consider a framework of strategic communication between an expert (he) and an informed trader (she). The expert privately observes the realization of a state $x \in [-s, s]$, with $0 < s < \infty$. The state captures optimal trading of a security in absence of commissions and taxes.[2] When $x < 0$ it is optimal to sell x of the security, when $x = 0$ it is optimal not to trade, and when $x > 0$ it is optimal to buy x of the security. It is common knowledge that x is uniformly distributed on $[-s, s]$ with zero mean and variance σ^2.

The expert advises the trader on optimal trading of the security. Specifically, after observing x he reports a costless, unverifiable message $m \in [-s, s]$ to the trader.[3] Since the message and state space are the same, the message sent can be thought of as a literal statement or a submitted report about the state of the world. He chooses message m to maximize his profit

$$\pi_E = \Pi_E - (y(m) - x)^2 + \kappa y^2(m), \tag{1}$$

[1] As the tax hampers trading, it also decreases liquidity thereby slowing the correction of mispricing.

[2] In practice financial advisors and sell-side analysts determine optimal trading of a security for a client through research on the security and knowledge of the client's (financial) features and objectives. Hence the assumption that the expert privately observes the realization of the state.

[3] Since the expert's message is costless, our model is a cheap talk game. This model thus differs from signaling models in which a sender can send a costly message to a receiver.

where $0 < \Pi_E < \infty$ is the maximum profit that the expert can realize and $y(m) \in [-s, s]$ is the trading decision of the trader given the expert's message m: if $y(m) < 0$ the trader sells $y(m)$ of the security, if $y(m) = 0$ she does not trade, and if $y(m) > 0$ she buys $y(m)$ of the security.[4] The expert has two objectives. First, he wants the trader to make a suitable trading decision. An unsuitable decision may cause a reputation loss or a punishment by a regulatory authority. The $-(y(m) - x)^2$ term captures the expert's concern for suitability.[5]

Second, the expert wants the trader to trade because she pays him a commission on the transaction. The $\kappa y^2(m)$ term represents the expert's gain from commission, where $\kappa \in (0, 1)$ is the expert commission on the amount traded.[6] Such gain is quadratic in the amount traded. More trading hence increases the expert's commission gain more than if it was linear in the amount traded. The reason is that in addition to raising the commission amount, more trading fosters the expert's career thus increasing his future gains. Indeed, financial advisors and sell-side analysts that generate more trading and thereby contribute the most to their firm's performance often get promoted (Hoechle *et al.*, 2018).

After receiving the expert's message m, the trader makes trading decision $y(m) \in [-s, s]$. Her trading decision maximizes her expected profit conditional on expert's message m

$$E(\pi_T|m) = E\left[\Pi_T - (y - x)^2 - (\kappa + \tau)y^2|m\right], \tag{2}$$

where $0 < \Pi_T < \infty$ is the maximum profit that the trader can realize. The trader wants her trading decision to match optimal trading but pays an expert commission $\kappa \in (0, 1)$ and a tax $\tau \in (0, 1 - \kappa]$ on the transaction. The $-(y - x)^2$ term represents the trader's loss from not matching optimal trading of the security while the $-(\kappa + \tau)y^2$ term represents the transaction cost. The transaction cost is convex in the amount traded to capture the fact that the larger is the amount traded, the larger is the price impact.

The timing is as follows. First, the expert privately observes optimal trading of the security. Second, the expert advises the trader. Third, the trader makes the trading decision. Finally, the trader pays the commission and the tax. All aspects of the model except optimal trading are common knowledge. Without loss of generality, we set $\Pi_E = \Pi_T = 0$.

3. Equilibrium

Following the definition of perfect Bayesian equilibria in cheap talk games by Crawford and Sobel (1982), in our cheap talk model we let the expert's communication strategy specify the probability of sending message $m \in [-s, s]$ conditional on observing state x, and we denote it by $q(m|x)$. The trader's decision strategy maps message $m \in [-s, s]$ into trading decision, and we denote it $y^*(m)$. Finally, the belief function of the trader is denoted by $\mu(x|m)$ and states the probability of state x conditional on observing message m. Perfect Bayesian equilibria require that (a) the expert's communication strategy is

[4] The expert maximizes his profit rather than his expected profit because the expert knows state x, i.e., the expert observes the state rather than a signal about the state.

[5] In addition to regulations that encourage sell-side analysts and financial advisors to correctly report information, sell-side analysts are frequently ranked according to the accuracy of their past recommendations. See Jackson (2005).

[6] Commissions of sell-side analysts and financial advisors usually consist of a share of the amount traded by clients.

optimal for the expert given the decision strategy of the trader, (b) the trader's decision strategy is optimal for the trader given the belief function, (c) the belief function is derived from the expert's communication strategy using Bayes's rule whenever possible.

To derive perfect Bayesian equilibria, we start from the trader's maximization problem. After receiving message m from the expert, the trader chooses to trade y solving

$$\begin{aligned} \max_y E\left(\pi_T|m\right) &= E\left[-\left(y-x\right)^2-(\kappa+\tau)y^2|m\right] \\ &= -\left[y-E(x|m)\right]^2-Var(x|m)-(\kappa+\tau)y^2, \end{aligned} \quad (3)$$

where $E(x|m)$ and $Var(x|m)$ are the expectation and the variance of optimal trading x conditional on message m, respectively. The first-order condition for the trader's best response conditional on message m yields trading decision

$$y^\star(m) = \frac{E(x|m)}{1+\kappa+\tau}. \quad (4)$$

We see from (4) that the trader's decision is her posterior expectation of optimal trading after expert advice discounted by the transaction cost. From (1) and (4) the expert's profit given the trader's decision is then

$$\pi_E = -\left(\frac{E(x|m)}{1+\kappa+\tau}-x\right)^2+\kappa\left(\frac{E(x|m)}{1+\kappa+\tau}\right)^2. \quad (5)$$

Before deriving the expert's best response, it is useful to consider his desired trading decision y_E. Taking the first-order condition of (1) with respect to trading decision y we obtain

$$y_E = \frac{x}{1-\kappa}. \quad (6)$$

Suppose that the trader knows state x. From (4) her desired trading decision y_T is

$$y_T = \frac{x}{1+\kappa+\tau}. \quad (7)$$

We see from (6) and (7) that $y_E > y_T$ for $x > 0$, $y_E = y_T$ for $x = 0$, and $y_E < y_T$ for $x < 0$. As the expert benefits from trading, if he correctly communicates optimal trading x, the purchase (sale) y_T that he would expect the trader to make is too small from his perspective. Trading is more attractive for the expert than for the trader for two reasons. First, the trader pays the commission to the expert. Second, the tax makes trading less attractive to the trader. The expert has hence an incentive to exaggerate optimal trading: when it is optimal to trade, he wants to convince the trader to trade more than what would be optimal to. The expert's incentive to exaggerate strengthens with the tax, the commission, and optimal trading volume $|x|$.

Crawford and Sobel (1982) have demonstrated that when a sender's desired decision is larger than that of the receiver for all states x, then every perfect Bayesian equilibrium of the cheap talk game is partitional: the state space is partitioned into finite intervals and every sender's message in an interval induces the same decision. Gordon (2010) provides a general characterization of partition equilibria when, as in our model, there is a state x for which sender's and receiver's desired decisions agree ($x = 0$ in our model). Following Crawford and Sobel (1982) and Gordon (2010), we can characterize equilibria using an indifference condition. This condition says that an equilibrium partition is determined in such a manner that an expert observing state x corresponding to a partition point is indifferent between the decision induced by communicating that the state is in the interval on the left-hand side of the partition point and the decision induced by communicating that the state is in the interval on the right-hand side of it.

To derive this indifference condition, let $\underline{a}$ and $\bar{a}$ be two points in $[-s, s]$ such that $\underline{a} < \bar{a}$. Suppose that the expert reports $m = \{x \in [\underline{a}, \bar{a})\}$. Since x is uniformly distributed, the trader's posterior expectation of optimal trading x given message $m = \{x \in [\underline{a}, \bar{a})\}$ is $E\left[x|m = \{x \in [\underline{a}, \bar{a})\}\right] = (\underline{a} + \bar{a})/2$ and from (4) her trading decision is

$$y^{*}\left(m = \{x \in [\underline{a}, \bar{a})\}\right) = \frac{\underline{a} + \bar{a}}{2(1 + \kappa + \tau)}. \tag{8}$$

In an equilibrium partition with N intervals, the indifference condition says that each partition point a_i must be such that the expert observing $x = a_i$ is indifferent between inducing $y^{\star}\left(m = \{x \in (a_{i-1}, a_i]\}\right)$ and $y^{\star}\left(m = \{x \in [a_i, a_{i+1})\}\right)$, that is,

$$\begin{aligned} &-\left(y^{\star}\left(m = \{x \in (a_{i-1}, a_i]\}\right) - x\right)^2 + \kappa y^{\star 2}\left(m = \{x \in (a_{i-1}, a_i]\}\right) = \\ &-\left(y^{\star}\left(m = \{x \in [a_i, a_{i+1})\}\right) - x\right)^2 + \kappa y^{\star 2}\left(m = \{x \in [a_i, a_{i+1})\}\right). \end{aligned} \tag{9}$$

Using (8), (9) can be written as

$$a_{i+1} - a_i = a_i - a_{i-1} + \frac{4(2\kappa + \tau)}{1 - \kappa} a_i. \tag{10}$$

This second-order difference equation determines the equilibrium partition for given κ, τ, and N, where $a_0 = -s$, $a_N = s$ and $a_i < a_{i+1}$ for $i = 0, 1, 2, .., N-1$. Our first proposition summarizes these results and characterizes perfect Bayesian equilibria.

Proposition 1. *For every positive integer N, there exists at least one equilibrium $(q(m|x), y^*(m), \mu(x|m))$, where*

(i) $q(m|x)$ is uniform, supported on $[a_{i-1}, a_i]$ if $x \in (a_{i-1}, a_i)$;

(ii) $\mu(x|m)$ is uniform, supported on $[a_{i-1}, a_i]$ if $m \in (a_{i-1}, a_i)$;

(iii) $a_{i+1} - a_i = a_i - a_{i-1} + \frac{4(2\kappa+\tau)}{1-\kappa} a_i$ for $i \in \{1, 2, .., N-1\}$ with $a_0 = -s$ and $a_N = s$;

(iv) $y^*(m) = \frac{a_{i-1}+a_i}{2(1+\kappa+\tau)}$ for all $m \in (a_{i-1}, a_i)$.

All other equilibria are economically equivalent.

Proof. See the Appendix. □

Figure 1 illustrates equilibria. In each equilibrium, the expert communicates in which interval is optimal trading.[7] The second-order difference equation in (iii) defines the size of intervals. The size of an interval $(a_{i+1} - a_i)$ is the size of the preceding interval $(a_i - a_{i-1})$ plus $\frac{4(2\kappa+\tau)}{1-\kappa} a_i$. The expert's incentive to exaggerate optimal trading determines how information is distorted. Since the expert's incentive to exaggerate increases in the tax, the commission and optimal trading volume, so does the size of intervals. Thus, expert advice becomes less accurate, the larger are the tax, the commission, and optimal trading volume.

[7] As an example, suppose that the expert observes that it is optimal for the trader to buy 10k USD of stock A (before commission and tax). He then advises the trader to buy between 9k and 12k USD of stock A.

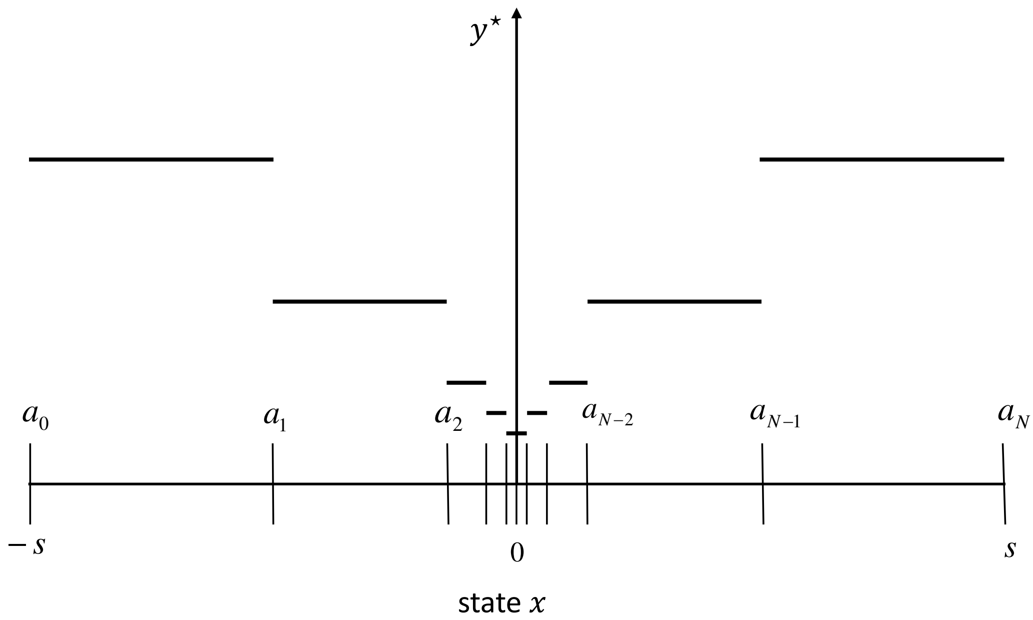

Figure 1: Equilibria

Intervals are symmetric around zero because the expert's incentive to exaggerate is symmetric for sales and purchases. For any state x in a given interval, the expert sends the same message m - he reports that x is in such interval - and the trader makes the same trading decision $y^\star(m)$. Trading decision $y^\star$ is hence a step function of x, symmetric around zero. The equilibrium number of intervals can be infinite since the expert has an incentive to exaggerate except when it is optimal not to trade (see Gordon, 2010). In this equilibrium intervals are infinitesimally small close to zero and become larger moving away from it. The expert communicates more information than in any other equilibrium. Since both expert and trader are best off in this equilibrium, we plausibly assume that they manage to coordinate on it and we focus our analysis on such equilibrium.

4. Tax Impact on Expert Advice

We measure quality of expert advice as the residual variance faced by the trader after expert advice $RV = E\left[(x - E(x|m))^2\right]$. The higher is residual variance, the lower is quality of expert advice.

Lemma 1. *In the equilibrium with $N \to \infty$, the residual variance is*

$$RV = \frac{2\kappa + \tau}{3 + 5\kappa + 4\tau}\sigma^2. \tag{11}$$

Proof. See the Appendix. □

Intuitively, quality of expert advice depends on the tax, the commission, and the variance of optimal trading.

Proposition 2. *The tax reduces quality of expert advice. This negative tax impact is mitigated by the commission and exacerbated by uncertainty.*

Proof. From (11) we have $\partial RV/\partial\tau > 0$, $\partial^2 RV/\partial\tau\partial\kappa < 0$, and $\partial^2 RV/\partial\tau\partial\sigma^2 > 0$. □

Proposition 2 shows that the tax worsens expert advice. The tax strengthens the expert's incentive to exaggerate so expert advice becomes less accurate and less information is communicated in equilibrium. Thus, quality of expert advice declines.

The negative tax impact on expert advice is moderated by the commission. The reason is that the stronger is the expert's incentive to exaggerate optimal trading because of the commission, the weaker is the effect of the tax on such incentive. A higher commission hence reduces the negative impact of the tax on quality of expert advice. Such impact is instead exacerbated by uncertainty as a higher variance of optimal trading amplifies the tax distortion of advice accuracy.

This novel mechanism through which the tax affects traders' information advances the tax debate beyond its focus on the proportion of noise versus informed trading and on liquidity. In particular, expert advice emerges as a new channel through which the tax makes traders less informed. The negative tax impact on expert advice can hence help to explain the observed decline in informational efficiency after the tax introduction in France (Colliard and Hoffmann, 2017).

Besides, our results point to a relationship between regulation of expert commissions and financial transaction taxes. The former is debated in the context of investor protection while the latter are discussed in the context of financial stability. Yet, Proposition 2 suggests that commission regulations aimed at improving expert advice -such as the introduction of caps that reduce maximum commissions- could backfire since lower commissions strengthen the negative tax impact on expert advice. Thus, commission regulations and financial transaction taxes should be jointly discussed.

5. Discussion

In this section we briefly discuss our modeling choice, alternative models to study the tax impact on expert advice as well as limits and extensions of our analysis.

Our model focuses on advice given by financial advisors to retail investors and by sell-side analysts to institutional investors. In this setting experts usually give advice without paying a cost and for this reason the expert's message is costless in our model. Such model is hence a cheap talk game. A feature of cheap talk is that unless the expert's and the trader's interests are aligned, the expert does not fully reveal information to the trader in the sense that the expert truthfully reveals in which interval is the state but his message is coarse - intervals are not singletons, they contain several possible states. As a consequence, perfect Bayesian equilibria are not separating. Since the expert's message is costless, one cannot recast the game as a standard signaling game in which the expert's message is costly.

However, as an alternative, one could consider a different model of financial advice in which the expert pays a cost to send a message to the trader. For instance, Kartik et al. (2007) build a model in which an expert can lie to investors, some of whom are sophisticated while others are naive. There are lying costs that are increasing and convex in the magnitude of the lie. In this framework, there is a separating equilibrium in which the expert sends a message higher than the actual state. The expert's message fully reveals her private information in the sense that a sophisticated investor correctly infers the state while a naive investor blindly believes the expert. In this equilibrium, the expert can induce investors to take a more favorable action on average by deviating to a higher message, but this gain can be offset by the marginal increase in the lying cost. In this setting, a tax on financial transaction would reduce quality of expert advice in the sense that the tax would increase the conflict of interests with investors and thereby motivate the expert to lie more in spite of lying costs, that is, to send a higher message. In contrast, in our cheap talk game the tax increases the conflict of interests between the expert and the trader thereby leading the expert to send coarser messages.

Since our analysis is the first to study the tax impact on expert advice, we have focused on the simplest possible model of expert advice. Arguably, adding more details to the model could generate more accurate predictions on the tax impact. For instance, we assume that the expert observes the state. One could add a step in the model in which the expert exerts costly effort to search for information about the state. Moreover, while our results indicate that the tax worsens expert advice, they are not conclusive about the overall tax impact on market informational efficiency. For this reason, embedding the model developed in this note in a more general framework with expert-advised and noise traders would generate better predictions about the impact of financial transaction taxes on market informational efficiency. Finally, the relationship between financial transaction taxes and expert commissions identified in this note deserves further investigation.

6. Conclusion

The longstanding debate on the taxation of financial transactions neglects the fact that many traders act on experts' recommendations. We develop a model of trading on expert advice to study the impact of a financial transaction tax on expert advice to traders. We find that the tax worsens expert advice and thereby makes traders less informed. Our analysis hence contributes to the debate on tax suitability by shedding light on a new channel through which the tax affects traders' information.

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Appendix

Proof of Proposition 1. We first prove the existence of perfect Bayesian equilibria and then add further details on their characterization relative to Section 3. Let x_A be the state such that the expert's and the trader's desired decisions coincide, that is $y_E(x_A) = y_I(x_A)$. From (6) and (7) we have $x_A = 0$. (6) and (7) also imply that the expert's desired decision given the state is larger than that of the trader in the sense that $y_E(x) > y_I(x)$ for $x > 0$ and $y_E(x) < y_I(x)$ for $x < 0$. Furthermore, (6) and (7) imply that the expert's preferences are continuous in y and the trader's desired decision is continuous in x. We can therefore apply Theorem 4 of Gordon (2010) to determine that there exists an equilibrium with an infinite number of intervals. Existence then follows from Theorem 2 of Gordon (2010) which shows that if there is an infinite equilibrium, then there is an equilibrium with N intervals for every positive integer N. We can further characterize perfect Bayesian equilibria described in Section 3 solving the second-order difference equation (10) using boundary conditions $a_0 = -s$ and $a_N = s$. Specifically, the solution of the second-order difference equation is

$$a_i = \frac{s}{\left(z_1^N - z_2^N\right)} \left[\left(1 + z_2^N\right) z_1^i - \left(1 + z_1^N\right) z_2^i\right] \quad \text{for all } i \in \{0, 1, 2, .., N\} \tag{12}$$

where the distinct roots of the second-order difference equation are

$$z_1 = \frac{1}{1-\kappa}\left[1 + 3\kappa + 2\tau + \sqrt{(1 + 3\kappa + 2\tau)^2 - (1-\kappa)^2}\right] \tag{13}$$

$$z_2 = \frac{1}{1-\kappa}\left[1 + 3\kappa + 2\tau - \sqrt{(1 + 3\kappa + 2\tau)^2 - (1-\kappa)^2}\right] \tag{14}$$

and satisfy $z_1 z_2 = 1$ with $z_1 > 1$. It is readily seen from (12) that for any integer $0 \leq K \leq N$, $a_K = -a_{N-K}$, i.e., intervals are symmetrically distributed around zero. Since x is uniformly distributed over $[-s, s]$, if $x \in (a_{i-1}, a_i)$, the expert's communication rule $q(m|x)$ specifying probability of sending message m conditional on observing state x is uniform, supported on $[a_{i-1}, a_i]$. It then follows that if $m \in (a_{i-1}, a_i)$, the trader's belief function $\mu(x|m)$ stating the probability of state x conditional on observing message m is uniform, supported on $[a_{i-1}, a_i]$. $\square$

Proof of Lemma 1. The proof of Lemma 1 follows those of Lemma A2 and Lemma 1 in Alonso et al. (2008). The proof consists of four steps. Let $\bar{m} = E(x|m)$. (i) First, we prove that $E(\bar{m}x) = E(\bar{m}^2)$. Applying the law of iterated expectations we have that

$$E(\bar{m}x) = E\left[E(\bar{m}x|x \in (a_{i-1}, a_i))\right] = E\left[\bar{m}E(x|x \in (a_{i-1}, a_i))\right] = E\left(\bar{m}^2\right). \tag{15}$$

(ii) Second, we show that

$$E\left(\bar{m}^2\right) = \frac{s^2}{4}\left[\frac{\left(z_1^{3N}-1\right)(z_1-1)^2}{\left(z_1^N-1\right)^3\left(z_1^2+z_1+1\right)} - \frac{z_1^N(z_1+1)^2}{z_1\left(z_1^N-1\right)^2}\right]. \tag{16}$$

For this purpose, using (12) we have

$$\begin{aligned}
E\left(\bar{m}^2\right) = {} & \frac{1}{2s}\sum_{i=1}^N \int_{a_{i-1}}^{a_i}\left(\frac{a_i+a_{i-1}}{2}\right)^2 dx = \frac{1}{8s}\sum_{i=1}^N (a_i-a_{i-1})(a_i+a_{i-1})^2 \\
= {} & \frac{s^2}{8\left(z_1^N-z_2^N\right)^3}\sum_{i=1}^N\Bigg[\left(1+z_2^N\right)^3 z_1^{3(i-1)}(z_1+1)^2(z_1-1) \\
& +\left(1+z_1^N\right)^3 z_2^{3(i-1)}(z_2+1)^2(1-z_2) + \left(1+z_1^N\right)^3 z_2^{3(i-1)}(z_2+1)^2(1-z_2) \\
& +\left(1+z_2^N\right)\left(1+z_1^N\right)^2 z_2^{i-1}\left(z_2^2-1\right)(z_1+1) \\
& -\left(1+z_2^N\right)^2\left(1+z_1^N\right) z_1^{i-1}\left(z_1^2-1\right)(z_2+1)\Bigg].
\end{aligned} \tag{17}$$

Carrying out the summation in (17) and using the property that $z_1 z_2 = 1$ yields

$$\begin{aligned}
E\left(\bar{m}^2\right) &= \frac{s^2}{4}\left[\frac{\left(z_1^{3N}-1\right)(z_1-1)^2}{\left(z_1^N-1\right)^3\left(z_1^2+z_1+1\right)} - \frac{\left(z_1^N+1\right)^2(z_2+1)(z_1+1)}{z_1^N\left(z_1^N-z_2^N\right)^2}\right] \\
&= \frac{s^2}{4}\left[\frac{\left(z_1^{3N}-1\right)(z_1-1)^2}{\left(z_1^N-1\right)^3\left(z_1^2+z_1+1\right)} - \frac{z_1^N(z_1+1)^2}{z_1\left(z_1^N-1\right)^2}\right].
\end{aligned} \tag{18}$$

(iii) Third, we prove that

$$\lim_{N\to\infty} E\left(\bar{m}^2\right) = (1-\lambda)\sigma^2 \tag{19}$$

where $\lambda = (2\kappa+\tau)/(3+5\kappa+4\tau)$. Taking the limit for $N \to \infty$ of each term in (16) we get

$$\lim_{N\to\infty}\frac{\left(z_1^{3N}-1\right)(z_1-1)^2}{\left(z_1^N-1\right)^3\left(z_1^2+z_1+1\right)} = \frac{z_1-1}{z_1^2+z_1+1} \tag{20}$$

and

$$\lim_{N\to\infty}\frac{z_1^N(z_1+1)^2}{z_1\left(z_1^N-1\right)^2} = 0. \tag{21}$$

It then follows from (16), (20), and (21) that

$$\lim_{N\to\infty} E\left(\bar{m}^2\right) = \frac{s^2}{4}\frac{(z_1+1)^2}{z_1^2+z_1+1} \tag{22}$$

where, making use of (13) we have

$$\begin{aligned}(z_1+1)^2 &= z_1^2+1+2z_1 \\ &= \frac{4(1+\kappa+\tau)}{1-\kappa}\frac{1}{1-\kappa}\left[1+3\kappa+2\tau+\sqrt{(1+3\kappa+2\tau)^2-(1-\kappa)^2}\right] \\ &= \frac{4(1+\kappa+\tau)}{1-\kappa}z_1 \qquad (23)\end{aligned}$$

and

$$\begin{aligned}z_1^2+z_1+1 &= \frac{3+5\kappa+4\tau}{1-\kappa}\frac{1}{1-\kappa}\left[1+3\kappa+2\tau+\sqrt{(1+3\kappa+2\tau)^2-(1-\kappa)^2}\right] \\ &= \frac{3+5\kappa+4\tau}{1-\kappa}z_1. \qquad (24)\end{aligned}$$

Using (22), (23), (24), and $\sigma^2 = s^2/3$ yields

$$\lim_{N\to\infty} E\left(\bar{m}^2\right) = \frac{s^2}{4}\frac{(z_1+1)^2}{z_1^2+z_1+1} = s^2\frac{1+\kappa+\tau}{3+5\kappa+4\tau} = \frac{3(1+\kappa+\tau)}{3+5\kappa+4\tau}\sigma^2 = (1-\lambda)\sigma^2 \qquad (25)$$

where $\lambda = (2\kappa+\tau)/(3+5\kappa+4\tau)$.

(iv) Fourth, we show that $RV = (2\kappa+\tau)\sigma^2/(3+5\kappa+4\tau)$. Applying the Lebesgue dominated convergence theorem to (25) we have

$$\lim_{N\to\infty} E\left(\bar{m}^2\right) = E\left(\bar{m}^2\right). \qquad (26)$$

Using (i), (iii), (25), and (26) we then obtain

$$RV = E\left[(x-E(x|m))^2\right] = \sigma^2 - E\left(\bar{m}^2\right) = \lambda\sigma^2 = \frac{2\kappa+\tau}{3+5\kappa+4\tau}\sigma^2. \qquad (27)$$

□