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Using the zeta function to explain 'downside' and 'upside' inequality aversion

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Abstract

This paper presents a single-parameter generalization of the Gini coefficient of inequality. The generalization yields a unique sequence of measures parametrized by the integer k which runs from minus infinity to plus infinity, and is based on the zeta function (defined on the set of integers). Using suitably normalized income weights, one can generate a family of welfare functions and associated inequality measures. For k belonging to {…,-3,-2,-1}, one has a family of decreasingly 'upside inequality aversion' measures; when k is zero, one has the familiar 'transfer-neutral' Gini coefficient; and for k belonging to {1,2,3,…}, one has a family of increasingly 'downside inequality aversion' measures. As k tends to minus infinity, the underlying social welfare function mimics a utilitarian rule, and as k tends to plus infinity, the Rawlsian rule. When k is 1, the corresponding inequality measure turns out to be the Bonferroni coefficient.

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1. Introduction

This paper is concerned with presenting a class of inequality measures which is an addition to the stock of parametric families available in the literature. It has three principal features of interest: (a) it may have some pedagogical or expository value in rehearsing and clarifying certain issues relating to the transfer-sensitivity property of inequality indices; (b) it has the practical utility of advancing a specific class of measures which are differentiated from each other by the nature and degree of transfer-sensitivity they display; and (c) the means to the end has been the zeta function of number theory—a very familiar mathematical entity which can be handily exploited to present a set of inequality measures under a unified framework of analysis that describes a spectrum from Bentham to Rawls via Gini and Bonferroni.

Of key relevance to this paper is Kolm's (1969) 'principle of diminishing transfers'. This principle, stated loosely, requires that the reduction in inequality following on a progressive transfer between two individuals should be larger at the lower (poorer) end of an income distribution than at the upper (richer) end of the distribution. This requirement is also called 'transfer-sensitivity' (Shorrocks and Foster, 1987). There are a number of ways in which the transfer-sensitivity property can be stated. One weak version would require that a given progressive transfer of income between two individuals who are both a fixed income and a fixed number of ranks apart should reduce inequality by more the poorer the pair of individuals is.

Transfer-sensitivity, along the lines described above, has also been referred to as 'down-side aversion to inequality' (Davies and Hoy, 1985). In contrast, 'up-side aversion to inequality', or 'reverse transfer-sensitivity', would require that a given progressive transfer of income between two individuals who are both a fixed income and a fixed number of ranks apart should reduce inequality by more the *richer* the pair of individuals is. At first glance, this may appear to be a somewhat perverse inversion of what has routinely come to be accepted as *the* 'right' response to judging between transfers at the lower and upper ends of a distribution. A small numerical example may help to illuminate why this perspective is not necessarily completely compelling.

Consider two ordered three-person income distributions given, respectively, by $\mathbf{b} = (15,15,30)$ and $\mathbf{c} = (10,25,25)$, both of which have the same mean income of 20. Which of **b** and **c** is the less unequal distribution? Notice that the relatively rich population stands apart more starkly in distribution **b** than in distribution **c**, in the sense that this population in **b** (constituted by person 3 with an income of 30) is in a minority, while this population in **c** (constituted by persons 2 and 3 with an income of 25 each) is in a majority. Notice also that each of the distributions **b** and **c** can be seen to have been derived from the distribution $\mathbf{a} = (10,20,30)$: **b** is derived from **a** by taxing person 2 and transferring a part of her income to 1, while **c** is derived from **a** by taxing person 3 and transferring a part of his income to 2. Is it fairer to leave the richest person untaxed, as in **b**, or to tax him, as in **c**? If the answer is in the negative, then one has a case—as in this instance—in favour of 'up-side' over 'down-side' transfer-sensitivity. Again, a 'Rawlsian' 'maximin' perspective, which ranks the inequality of income distributions solely in terms of the status of the worst-off person would pronounce distribution **b** to be less unequal than distribution **c**; however, inequality judgements can also plausibly be informed by a 'minimax' criterion, by which the inequality of income distributions is ranked solely in terms of the status of the best-off person—a rule that aims to reward the curbing of extreme wealth, so that, in line with this perspective, distribution **c** would be pronounced less unequal than distribution **b**. Briefly, a uniform and blanket endorsement of transfer-sensitivity, and opposition to reverse transfer-sensitivity, may not always be in conformity with what intuition dictates across a range of situations and income distributions.

In view of the above, one could advance a case for a generalized family of inequality indices such that its members are differentiated from each other not only in the *degree* of transfer-sensitivity, but in the *kind* of transfer-sensitivity. This would pave the way for a sequence of measures, some of whose members display the 'usual' transfer-sensitivity property in different degrees, and some a reverse transfer-sensitivity property, again in different degrees. The family of measures dealt with in this paper is based on the zeta function of number theory, and includes, as special cases, two well-known inequality indices—the Gini and the Bonferroni coefficients. (For deeper and more thoroughly investigated treatments of related issues, the reader is referred to, among others, Aaberge et al (2021).)

The components of the preceding introductory discussion are elaborated on in the rest of the paper.

2. Welfare Functions and Inequality Measures

An income distribution is a non-decreasingly ordered *n*-vector $\mathbf{x} = (x_1,...,x_i,...,x_n)$, where x_i is the income of the *i*th poorest person in a society of *n* individuals, *n* being a positive integer. $\mathbf{X}_n$ is the set of all such *n*-vectors of income, and $\mathbf{X}$ is the set $\bigcup_{n=1}^{\infty}\mathbf{X}_n$. The set of real numbers is denoted by $\mathbb{R}$.

A *social welfare function* is a mapping $W:\mathbf{X}\to\mathbb{R}$ such that, for every $\mathbf{x}\in\mathbf{X}$, $W(\mathbf{x})$ is a real number signifying the extent of welfare associated with the income distribution $\mathbf{x}$.

An *inequality measure* is a mapping $I:\mathbf{X}\to\mathbb{R}$ such that, for every $\mathbf{x}\in\mathbf{X}$, $I(\mathbf{x})$ is a real number signifying the extent of inequality associated with the income distribution $\mathbf{x}$.

For all $\mathbf{x}\in\mathbf{X}$, $N(\mathbf{x})$ is the set of individuals whose incomes are represented in **x**, $n(\mathbf{x})(\equiv|N(\mathbf{x})|)$ is the dimensionality of **x**, $\mu(\mathbf{x})(\equiv\frac{1}{n(\mathbf{x})}\sum_{i=1}^{n(\mathbf{x})}x_i)$ is the mean of the distribution **x**, and, for all $i=1,...,n(\mathbf{x})$, $r_i(\mathbf{x})\equiv n(\mathbf{x})+1-i$ is the (inverse) rank of the *i*th poorest person in **x**. In what follows, the argument **x** in $N(\mathbf{x}), n(\mathbf{x}), \mu(\mathbf{x})$ and $r_i(\mathbf{x})$ will be suppressed wherever there is no ambiguity.

In the analysis of the welfare basis of income-inequality comparisons, it is customary to adopt one of two broad approaches to the specification of a social welfare function. In one approach, popularised by Atkinson (1970), social welfare is expressed as a sum of individual valuation functions defined on income, where the function in question can—but need not—be interpreted as a utility function: in this approach, given an *n*-person distribution of incomes **x**, social welfare $W(\mathbf{x})$ is identified with the sum $\sum_{i=1}^{n}\varphi(x_i)$, where, typically, $\varphi(.)$ is taken to be a symmetric, increasing and concave function. An alternative approach is one in which social welfare is taken to be a weighted sum of incomes, with the weights being positive and declining in income: a widely-employed weighting scheme is the (inverse) rank-order weighting system, derived from the so-called 'Borda Rule' (or some appropriate variant of it) which has found application in the aggregation of preferences in voting problems. It is with the latter of these two broad approaches that this paper is concerned.

Let v_i be a weight attached to the *i*th poorest person's income, and let v be the sum of the weights over the incomes in the reference population: $v \equiv \sum_{j=1}^{n} v_j$. Then, for all $i = 1,...,n$, $w_i \equiv v_i / v$ is a normalized weight attached to the *i*th poorest person's income: it may be interpreted as the social value placed on her income. Notice that, since w_i is normalized, $\sum_{j=1}^{n} w_j = 1$. A typical member of the class of social welfare functions considered in this paper is a weighted sum of individual incomes, that is, for all $\mathbf{x} \in \mathbf{X}$:

(1) $W(\mathbf{x}) = \sum_{i=1}^{n} w_i x_i$.

The concern in this paper, as stated earlier, will be with the class of welfare functions in (1) which are further restricted by the requirements that for all *i*, (i) the weights w_i are specific positive transformations of the inverse rank-orders r_i; and (ii) w_i is a declining function of *i*. The fact that the w_i are positive ensures that the welfare function is increasing in individual incomes; and the fact that the weights decline as income increases ensures that the welfare function is 'equity-sensitive'. Further, since individuals are indexed by (some positive transformation of) their rank-orders, it is clear that income-valuations are independent of the identities of who owns what income. Briefly, the class of welfare functions represented in (1) is a class of rank-dependent, increasing, equity-sensitive and symmetric/anonymous welfare functions. It is this class of welfare functions, and the inequality measures derived from them and discussed below, that are of relevance to this paper.

Given the class of welfare functions just described, define, for any $\mathbf{x} \in \mathbf{X}$, Atkinson's (1970) *equally distributed equivalent income* x_{ede} as that level of income such that, if it is assigned to every individual, then the resulting level of welfare will be the same as that which obtains for the distribution **x** under consideration; that is, x_{ede} is obtained from the equation

$W(x_{ede}, x_{ede},..., x_{ede}) = W(\mathbf{x})$ whence, given (1), we have:

(2) $x_{ede} = \sum_{i=1}^{n} w_i x_i$.

For every $\mathbf{x} \in \mathbf{X}$, define $I(\mathbf{x})$ --see Atkinson (1970)—as the inequality measure obtained by taking the proportionate deviation of the equally distributed equivalent income from the mean income of the distribution:

(3) $I(\mathbf{x}) = 1 - x_{ede} / \mu = 1 - \frac{1}{\mu} \sum_{i=1}^{n} w_i x_i$.

(1), (2) and (3), and the accompanying discussion, together define the general class of welfare functions and inequality measures with which this paper is concerned. Before narrowing this class down to a specific family, it is useful to consider some 'equity-related' properties of an inequality measure.

3. Transfer and Differential Transfer-Sensitivity

For all $\mathbf{x},\mathbf{y}\in\mathbf{X}$, $\mathbf{y}$ is said to be derived from $\mathbf{x}$ through *a permissible progressive transfer* if $y_i = x_i$ for all $i=\{1,...,n\}-\{j,k\}$ for some j,k such that $x_j < x_k, y_j = x_j + \delta$ and $y_k = x_k - \delta$, where $0<\delta \le (x_k - x_j)/2$. The Pigou-Dalton 'Transfer' axiom rewards progressive transfers, other things being equal:

Transfer (*Axiom T*): An inequality index I will be said to satisfy Axiom T whenever, for all $\mathbf{x},\mathbf{y}\in\mathbf{X}$, if $\mathbf{y}$ has been derived from $\mathbf{x}$ through a permissible progressive transfer, then $I(\mathbf{y}) < I(\mathbf{x})$.

Inequality measures that satisfy the transfer axiom can be restricted further by constraints on how they should behave in the presence of progressive transfers at the lower and upper ends of an income distribution. This requires comparing the downward impact on inequality of a given progressive income transfer between two pairs of persons of which one pair is poorer than the other, when the individuals in each pair are a *fixed number of ranks* apart, or when they are a *fixed income* apart (see Foster, 1985). Formulations of transfer-sensitivity (or insensitivity) based on individuals a fixed number of ranks apart are '*positional*' formulations (Mehran, 1976; Zoli, 1999), and those based on individuals a fixed income apart are related to what Kolm (1969) called 'the principle of diminishing [increasing] transfers'. In the definitions considered here, resort is had to a formulation that is weaker than (that is, is implied by) each of the two types of formulation just described, and requires comparisons of the magnitude of inequality-reduction for progressive transfers between pairs of poorer and richer individuals such that each pair is both a fixed number of ranks, *and* a fixed income, apart. This leads to the following definitions of transfer-sensitivity, transfer-neutrality, and reverse transfer-sensitivity:

Transfer-Sensitivity (*Axiom TS*): An inequality index I will be said to satisfy Axiom TS whenever, for all $\mathbf{x},\mathbf{y},\mathbf{z}\in\mathbf{X}$, if $\mathbf{y}$ has been derived from $\mathbf{x}$ through a given permissible progressive transfer of income δ from individual *h* to individual *g*, and $\mathbf{z}$ has been derived from $\mathbf{x}$ through the same permissible progressive transfer of income δ from individual *q* to individual *p*, where $h-g=q-p\equiv u>0$ and $x_h - x_g = x_q - x_p > 0$, then $I(\mathbf{y}) < I(\mathbf{z}) < I(\mathbf{x})$.

Transfer-Neutrality (*Axiom TN*). An inequality index I will be said to satisfy Axiom TN whenever, for all $\mathbf{x},\mathbf{y},\mathbf{z}\in\mathbf{X}$, if the antecedents in the statement of Axiom TS are satisfied, then $I(\mathbf{x}) = I(\mathbf{y}) = I(\mathbf{z})$.

Reverse Transfer-Sensitivity (*Axiom RTS*). An inequality index I will be said to satisfy Axiom RTS whenever, for all $\mathbf{x},\mathbf{y},\mathbf{z}\in\mathbf{X}$, if the antecedents in the statement of Axiom TS are satisfied, then $I(\mathbf{z}) < I(\mathbf{y}) < I(\mathbf{x})$.

As noted earlier, the class of inequality indices in (3) is equity-sensitive, in the sense that each member of the class satisfies the Transfer Axiom, which follows from the fact that for all j,k such that $x_k > x_j$, it is the case that $w_k < w_j$. When individuals are indexed in ascending order of income, the Transfer principle requires that w_i be a declining function of *i* for all *i*. Further, Transfer-Sensitivity is satisfied whenever the difference between the weights on a pair of successive individuals declines as one moves up the income ladder, that is, when $w_i - w_{i+1}$ declines as *i* increases, for all $i=1,...,n-1$. Transfer-Neutrality would require $w_i - w_{i+1}$ to remain constant with an increase in *i*, while Reverse Transfer-Sensitivity would require $w_i - w_{i+1}$ to increase with an increase in *i*. Equivalently, Transfer-Neutrality requires the weighting function to be declining and *linear* in the rank-order of an individual; and Transfer-Sensitivity (respectively, Reverse Transfer-Sensitivity) requires the weighting function to be declining and strictly *convex* (respectively, strictly *concave*) in the rank-order.

These properties are examined, in the following section, in the context of a particular class of rank-dependent, equity-sensitive, symmetric inequality measures yielded by an appropriate parametrization based on the well-known *zeta function* of number theory.

4. A Class of Inequality Measures Based on the Zeta Function

Let $\mathbb{Z} \equiv \{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$ be the set of integers. Then, the zeta function, restricted to the set of integers, is given by

(4) $\zeta(k) = 1 + \frac{1}{2^k} + \frac{1}{3^k} + \ldots;\ k \in \mathbb{Z}.$

Perhaps the best known zeta function is the one realized for $k = 1$, which yields the well-known *harmonic series* (and plays an important part in the composition of the Bonferroni index of inequality, on which more later):

(5) $\zeta(1) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$

The *partial sum* of the first *i* terms of the zeta function is given by

(6) $H_i(k) \equiv 1 + \frac{1}{2^k} + \frac{1}{3^k} + \ldots + \frac{1}{i^k}\ ,\ i = 1,2,\ldots;\ k \in \mathbb{Z}.$

It is useful, here, to define $H_0(k) \equiv 0, \forall k \in \mathbb{Z}$.

Consider now a class of welfare functions parametrized by the integer *k* which, for every *n*-vector of incomes $\mathbf{x} \in \mathbf{X}$, is given by

(7) $W_k(\mathbf{x}) = \sum_{i=1}^{n} w_i(k) x_i\ ,\ k \in \mathbb{Z},$

where $w_i(k) = \dfrac{H_n(k) - H_{i-1}(k)}{\sum_{j=1}^{n}[H_n(k) - H_{j-1}(k)]},\ i = 1,\ldots,n.$

The corresponding class of inequality measures (see (3)) is given, for every $\mathbf{x} \in \mathbf{X}$, by

(8) $I_k(\mathbf{x}) = 1 - \frac{1}{\mu}\sum_{i=1}^{n} w_i(k) x_i\ ,\ k \in \mathbb{Z},$

where $w_i(k)$ is as defined in (7) for all $i = 1,\ldots,n$.

Equation (8) can be simplified. Taking note of the definition of the partial sum $H_i(k)$ ($i = 0,1,\ldots,n$ and $k \in \mathbb{Z}$), (8) can be rewritten to read: for every $\mathbf{x} \in \mathbf{X}$,

(9) $I_k(\mathbf{x}) = 1 - \frac{1}{S_n(k)\mu}\sum_{i=1}^{n}(\sum_{j=i}^{n}\frac{1}{j^k})x_i\ ,$

where $S_n(k) \equiv \sum_{i=1}^{n} (\sum_{j=i}^{n} \frac{1}{j^k})$.

A few straightforward points can be quickly noted. It is clear that the class of inequality measures described by (9) is a subset of the class of inequality measures described by (3), which are derived from the class of increasing, rank-dependent, anonymous and equity-sensitive social welfare functions described by (1). Specifically, for each of the inequality measures in (9), the weights are positive, rank-related, and declining in income (the weight on the *i*th lowest income being the sum of the last $n+1-i$ terms of the partial sum H_n of the zeta function (which of course declines as i increases)). Thus, while all of the inequality measures described by (9) satisfy the Transfer Axiom, they do not all satisfy the same degree, nor even kind, of transfer-sensitivity. A small diagrammatic example should illustrate the last point.

Figures 1a and 1b portray the behaviour of the weighting function in the class of inequality measures I_k as k varies. For illustrative purposes, the dimensionality n of the income distribution is restricted to the small number of five. Figure 1a shows how the weight changes as income-rank increases, for $k = 0$ and for two negative values of *k*, -1 and -3. Figure 1b does the same for $k = 0$ and for two positive values of *k*, 1 and 3. As can be seen from the figures, the weighting function is downward sloping: all members of the I_k class of inequality measures satisfy the Transfer Axiom.

However, for $k = -1$ and $k = -3$, the weighting functions are downward sloping and strictly concave—more concave for $k = -1$ than for $k = -3$: in the latter case, the curve is already beginning to flatten out (Figure 1a). Indeed, in the limit, as *k* tends to $-\infty$, the weighting structure converges on one in which each income has the same weight, of $1/n$, so that the underlying social welfare function mimics the Utilitarian rule of counting all persons' interests equally, and is identical with the mean income, whose size is all that matters, with no concern for its inter-personal distribution. For negative values of *k*, the increasing concavity of the weighting function as *k* becomes larger indicates that the measure I_k becomes progressively less reverse-transfer-sensitive as *k* increases in value.

Both Figures 1a and 1b reveal that, for *k* = 0, the weighting function is downwad-sloping and linear, suggesting that I_0 is a transfer-neutral inequality measure.

Figure 1b shows that for *k* = 1 and *k* = 3, the weighting functions are declining and strictly convex—more convex for *k* = 3 than for *k* = 1; in the former case, the curve is already one in which the bulk of the weight is on the smallest income. Indeed, as *k* goes to ∞, the weighting structure converges on one in which the weight on the lowest-ranked income is unity, with the other ranks receiving zero weight: this is in conformity with the Rawlsian maximin rule. For positive values of *k*, the increasing convexity of the weighting function as *k* becomes larger indicates that the measure I_k becomes progressively more transfer-sensitive as *k* increases in value.

Figure 1a: 'Zeta Function' Weights for a 5-person Distribution, for k=0, k = (-)1, k = (-)3

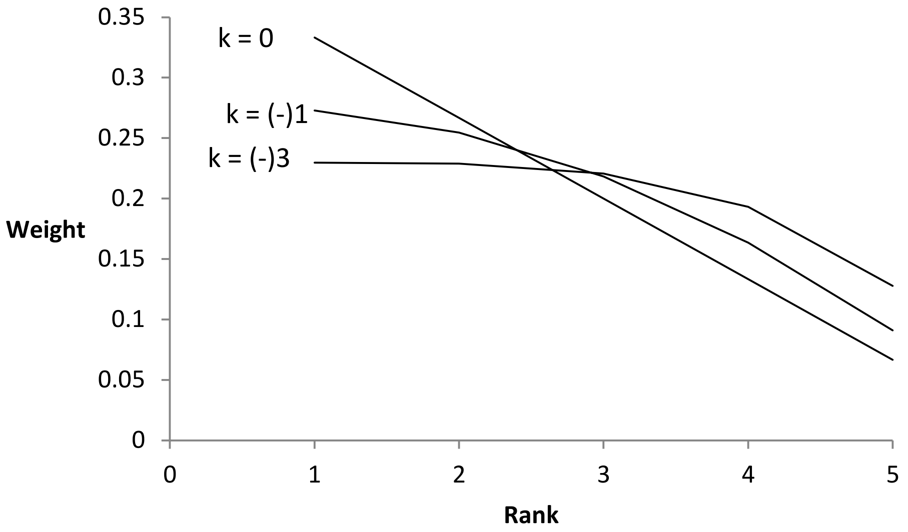
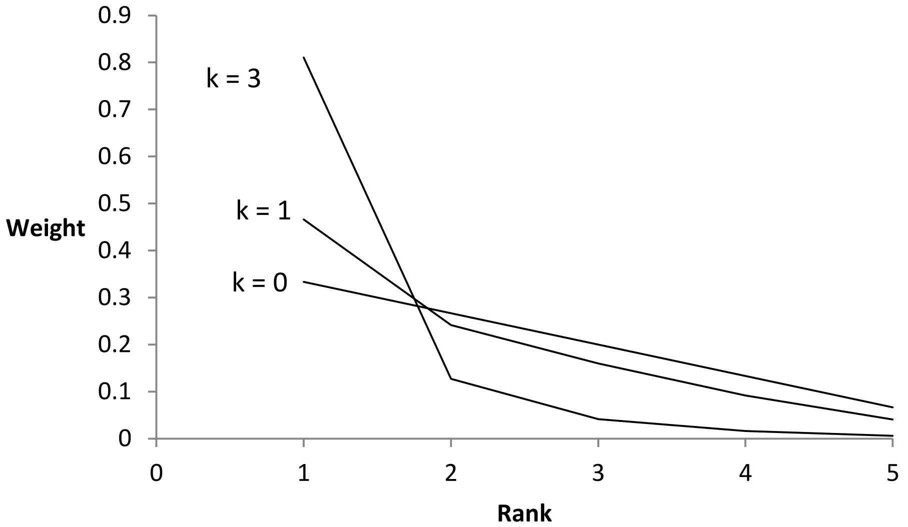


Figure 1b: 'Zeta Function' Weights for a 5-person Distribution, for k=0, k= 1, k= 3



The transfer-sensitivity properties of the I_k class of inequality measures can be more directly seen from the following. Let **x**, **y** and **z** be three n-person distributions satisfying the antecedents in the statement of Axiom TS, and let $\mathbb{Z}_{--}$ (respectively, $\mathbb{Z}_{++}$) be the set of all strictly negative (respectively, strictly positive) integers. Then, given (9), it can be verified (after some algebraic manipulation) that, for all $k \in \mathbb{Z}$:

(10) $$\Delta_k(\mathbf{y},\mathbf{z}) \equiv I_k(\mathbf{y}) - I_k(\mathbf{z}) = \frac{\delta}{S_n(k)\mu}[\sum_{i=p}^{p+u-1}\frac{1}{i^k} - \sum_{i=g}^{g+u-1}\frac{1}{i^k}].$$

It is clear that for k = 0, $\Delta_k(\mathbf{y},\mathbf{z}) = 0$: the measure I_0 is transfer-neutral. Further, since $p > g$, it is the case that for all $k \in \mathbb{Z}_{--}$, $\Delta_k(\mathbf{y},\mathbf{z}) > 0$ and the quantity $\Delta_k(\mathbf{y},\mathbf{z})$ declines as the absolute value of k declines: that is, for all $k \in \mathbb{Z}_{--}$, I_k is a diminishingly reverse-transfer-sensitive index. Similar reasoning shows that for all $k \in \mathbb{Z}_{++}$, I_k is an increasingly transfer-sensitive index. (It may be noted that that the family of indices $\{I_k\}_{k\in\mathbb{Z}_{++}}$ has already been considered and discussed in Subramanian, 1987.)

Finally, two distinguished members of the $\{I_k\}$ family of indices are realized for k = 0 and k = 1. It can be easily verified, given (9), that for all $\mathbf{x} \in \mathbf{X}$:

(11) $$I_0(\mathbf{x}) = 1 - \frac{2}{n(n+1)\mu}\sum_{i=1}^{n}(n+1-i)x_i \text{ ; and}$$

(12) $$I_I(\mathbf{x}) = 1 - \frac{1}{n\mu}\sum_{i=1}^{n}(\sum_{j=i}^{n}\frac{1}{j})x_i \ .$$

As it happens, I_0 and I_1 are, respectively, just the familiar Gini and Bonferroni coefficients of inequality. (On generalizations which include the Gini and Bonferroni indices as special cases, see Chakravarty, 2007.) As is well-known, Gini is transfer-neutral, and Bonferroni is transfer-sensitive. To complete the picture, one may add the reverse-transfer-sensitive index I_{-1} which is 'isomorphic' with I_1, and which, it can be checked, is given, for all $\mathbf{x} \in \mathbf{X}$, by:

(13) $$I_{-1}(\mathbf{x}) = 1 - \frac{6}{n(n+1)(2n+1)\mu}\sum_{i=1}^{n}(\sum_{j=i}^{n} j)x_i \quad .$$

Finally, a simple numerical example which illustrates the concerns of this paper might be helpful. Returning to the three-person ordered income vectors **a** = (10,20,30), **b** = (15,15,30) and **c** = (10,25,25), Table 1 sets out the values of the inequality measure I_k for parametric variation in k: $k \to -\infty, k = -3, k = -1, k = 0, k = +1, k = +3, k \to +\infty$. As one might expect, the value of the inequality measure for each distribution rises with k (reading the Table column-wise); inequality is zero as k tends to $-\infty$; the inequality-value is the ratio of the poorest person's income to mean income as $k \to +\infty$; for finite values of k, inequality for **a** is greater than for each of **b** and **c** because the measure satisfies the Pigou-Dalton axiom of Transfers; and for finite negative values of k, inequality for **b** is greater than for **c** because of reverse-transfer sensitivity, while for positive values, inequality for **c** is greater than for **b**, because of transfer-sensitivity. The table bears out an observation of Sen's on 'ethical' measures of inequality, to the effect that they lend themselves to the consideration of two types of variation: '(I) variations of the income distribution vector given the degree of [inequality aversion]'; and '(II) variations of the degree of [inequality aversion] given the income distribution vector' (Sen, 1982 [1978]; p.419). The two variations can be seen from a row-wise (respectively, column-wise) reading of Table 1.

Table 1: Values of the Inequality Measure I_k for Parametric Variation in *k*, for Each of Three Distributions: a = (10,20,30), b = (15,15,30) and c = (10,25,25)

Distribution → / $k \downarrow$	**a** = (10,20,30)	**b** = (15,15,30)	**c** = (10,25,25)	Rank in descending order of inequality
$k \to -\infty$	0	0	0	$\mathbf{a} \approx \mathbf{b} \approx \mathbf{c}$
$k = -3$	.0459	.0434	.0255	$\mathbf{a} \succ \mathbf{b} \succ \mathbf{c}$
$k = -1$	.1071	.0893	.0714	$\mathbf{a} \succ \mathbf{b} \succ \mathbf{c}$
$k = 0$	.1667	.1250	.1250	$\mathbf{a} \succ \mathbf{b} \approx \mathbf{c}$
$k = +1$	.2500	.1667	.2084	$\mathbf{a} \succ \mathbf{c} \succ \mathbf{b}$
$k = +3$	.4133	.2297	.3904	$\mathbf{a} \succ \mathbf{c} \succ \mathbf{b}$
$k \to +\infty$	.5000	.2500	.5000	$\mathbf{a} \approx \mathbf{c} \succ \mathbf{b}$

5. Concluding Note

A number of parametric generalizations of the Gini coefficient of inequality are available in the literature on inequality measurement—see, for example, Donaldson and Weymark (1980), Kakwani (1980), Yitzhaki (1983), Chakravarty (1988), Chameni (2006), Subramanian (2021), and Okamoto (2022). The present paper, which exploits the well-known zeta function of number theory to present a family of Ginis differentiated by the nature and degree of their 'transfer-sensitivity', supplies an addition to the available set of generalizations.

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