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Braess paradox with mixed strategies

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Abstract

The Braess paradox persists if drivers play mixed strategies. In equilibria in mixed strategies, traffic flows are almost the same as in equilibria in pure strategies.

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1. Introduction

In principle, additional road connections can reduce traffic flows and travel times if drivers follow specific instructions which route to take. However, an additional road may increase or not all affect travel times if it is left to the drivers what route to take. This intriguing possibility was demonstrated by Braess (1968)[1] and became known as Braess paradox. Once it was pointed out, the phenomenon has gained ample attention among experts in traffic planning, operations research, economics and game theory as well as by computer scientists concerned with the flow and routing of information in computer networks. It was also reported in the popular press after some actual occurrences.

The present note deals with the case of random choices and addresses the question whether such choices would undo the Braess paradox. The basic model is like Braess's original example, but with 6000 rather than 6 drivers. It turns out that the Braess paradox persists if drivers play mixed strategies. Moreover, in each equilibrium in mixed strategies, traffic flows are almost the same as in the corresponding equilibrium in pure strategies.

2. Basic Model

Every hour, $N = 6000$ car drivers go from A to D. A car driver wants to get from A to D as fast as possible. Travel time is measured in minutes. A driver's utility is given by $u(t) = -t$ if it takes the driver t minutes to go from A to D. Therefore, the driver's objective is to minimize t. A driver has the choice between several routes. The driver's travel time depends on the chosen route and the choices of all others. The drivers play a strategic game where each driver chooses a route from A to D and driver i's payoff is $-t_i$ where t_i is i's travel time.

2.1. Originally, a driver has the choice between two routes, ABD and ACD, as depicted in Figure 1, with the following travel times:

- It takes $t_{AB} = \frac{1}{100} \cdot x$ minutes to go from A to B if x cars per hour use road AB.
- It takes $t_{BD} = 50 + \frac{1}{1000} \cdot x$ minutes to go from B to D if x cars per hour use road BD.
- It takes $t_{AC} = 50 + \frac{1}{1000} \cdot x$ minutes to go from A to C if x cars per hour use road AC.
- It takes $t_{CD} = \frac{1}{100} \cdot x$ minutes to go from C to D if x cars per hour use road CD.

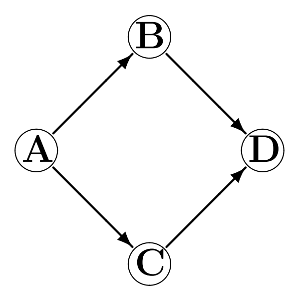

Figure 1

Let x_{ABD} be the number of drivers who choose route ABD each hour and x_{ACD} the number of drivers who choose route ACD each hour. Note that for each of the two routes, the travel time $\frac{1}{100} \cdot x + 50 + \frac{1}{1000} \cdot x$ is increasing in x.

[1] Braess et al. (2005) is an English translation of the original paper.

Suppose that $x_{ABD} \neq x_{ACD}$, w.l.o.g. $x_{ABD} > x_{ACD}$. Then $x_{ABD} > 3000 > x_{ACD}$, hence $x_{ABD} > x_{ACD} + 2$. If a driver switches from ABD to ACD, then $x_{ACD} + 1$ cars use the latter route and the driver is better off than before the switch. Therefore, (x_{ABD}, x_{ACD}) with $x_{ABD} > x_{ACD}$ cannot be an equilibrium outcome. Similarly for $x_{ABD} < x_{ACD}$. We have shown that a Nash equilibrium in pure strategies requires $x_{ABD} = x_{ACD} = 3000$.

$x_{ABD} = x_{ACD} = 3000$ is a Nash equilibrium outcome, indeed. For suppose that one driver deviates, say switches from ABD to ACD. Then there are 3001 cars on route ACD and the driver is worse off. The equilibrium travel time is $\frac{1}{100} \cdot 3000 + 50 + \frac{1}{1000} \cdot 3000 = 30 + 50 + 3 = 83$.

2.2. Suppose that a new one-way road from B to C opens as depicted in Figure 2, so that now a driver can choose between three routes to go from A to D: ABD, ACD and ABCD.

- It takes $t_{BC} = 10 + \frac{1}{1000} \cdot x$ minutes to go from B to C if x cars per hour use road BC.

Let x_{ABCD} be the number of drivers who choose route ABCD each hour.

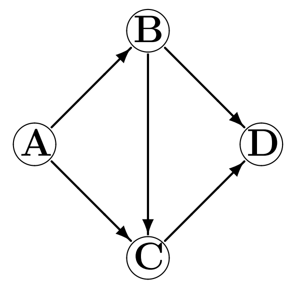

Figure 2

First, there exists a Nash equilibrium in pure strategies with $x_{ABD} = x_{ACD} = x_{ABCD} = 2000$. Namely, if $x_{ABD} = x_{ACD} = x_{ABCD} = 2000$, then

4000 cars use road AB,
4000 cars use road CD,
2000 cars use road BD,
2000 cars use road AC,
2000 cars use road BC.

The travel times are
$t_{ABD} = t_{ACD} = \frac{1}{100} \cdot 4000 + 50 + \frac{1}{1000} \cdot 2000 = 40 + 50 + 2 = 92$ and
$t_{ABCD} = \frac{1}{100} \cdot 4000 + 10 + \frac{1}{1000} \cdot 2000 + \frac{1}{100} \cdot 4000 = 40 + 10 + 2 + 40 = 92$.

By the argument given in 2.1, nobody wants to switch between ABD and ACD. If a driver switches from ABD to ABCD, then the traffic on AB remains the same. But there is one more car using BC and CD. If a driver switches from ACD to ABCD, the effect is analogous. Similarly for reverse switches. This shows the claim.

Second, it follows immediately from mutually best responses, that in equilibrium, travel times for two routes differ at most by $2/100 + 1/1000$.

2.3. Suppose that the one-way road from B to C is replaced by a two-way road as in Figure 3, so that travel both from B to C and from C to B is possible. Then a driver can choose between four routes to go from A to D: ABD, ACD, ABCD, and ACBD.

- Like before, it takes $t_{BC} = 10 + \frac{1}{1000} \cdot x$ minutes to go from B to C if x cars per hour go from B to C.
- It takes $t_{CB} = \frac{1}{1000} \cdot x$ minutes to go from C to B if x cars per hour go from C to B.

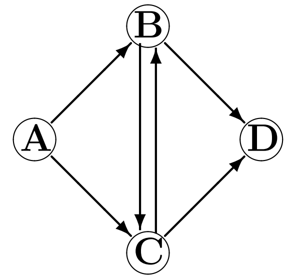

Figure 3

Let x_{ACBD} be the number of drivers who choose route ACBD each hour. It turns out that there exists a Nash equilibrium with $x_{ABD} = x_{ACD} = x_{ABCD} = 2000$, $x_{ACBD} = 0$. Namely, we know from 2.2 that there are no beneficial switches between routes ABD, ACD and ABCD if $x_{ABD} = x_{ACD} = x_{ABCD} = 2000$. It remains to be shown that nobody wants to switch to route ACBD. A switching driver has to take roads AC and BD, with total travel time exceeding 100. The driver would be worse off and, therefore, does not want to switch.

3. Mixed Strategies

In this section, drivers may choose routes randomly, that is play mixed strategies. Each driver's objective is to minimize her expected travel time.

3.1. Let us start with the original road network. Recall that for each of the two routes ABD and ACD, the travel time is $\frac{1}{100} \cdot x + 50 + \frac{1}{1000} \cdot x$ when x cars use the route. Hence we obtain the expected travel time, if x is replaced by the expected number of cars. We find that each driver randomizing 50:50 constitutes a Nash equilibrium in mixed strategies, with expected travel time 83.

For suppose that $N - 1$ drivers play the 50:50 strategy. Without the remaining player, the expected number of cars on each of the routes ABD and ACD is $\frac{1}{2}(N - 1)$. If the remaining player chooses one of the routes, then the expected number of cars on that route is $\frac{1}{2}(N - 1) + 1$. Hence this driver is indifferent between the two routes and randomizing 50:50 is a best response. In that equilibrium, the expected number of cars for each route is $\frac{1}{2}N = 3000$ and, consequently, the expected travel time is 83.

Next we are going to show that there is a unique completely mixed Nash equilibrium. Let $p_i \in (0, 1)$ be the probability that driver i chooses route ABD. Consider two drivers i and j. Let E_i denote the expected number of cars taking route ABD, not counting i, and E_j denote the expected number of cars taking route ABD, not counting j. Moreover, let E_{ij} denote the expected number of cars taking route ABD, not counting i and j.

Since $p_i \in (0,1)$ is a best response for i, driver i is indifferent between the two routes, which is the case if and only if $E_i = \frac{1}{2} \cdot (N-1)$. Similarly, $E_j = \frac{1}{2} \cdot (N-1)$ has to hold. But $E_i = p_j + E_{ij}$ and $E_j = p_i + E_{ij}$. Hence $E_i = E_j$ implies $p_i = p_j$. Since i and j were chosen arbitrarily, all drivers randomize the same way. That is, there exists $p \in (0,1)$ such that each driver chooses route ABD with probability p. Then for any driver i, $E_i = p \cdot (N-1)$ holds and $E_i = \frac{1}{2} \cdot (N-1)$ implies $p = 1/2$.

There also exist partially mixed equilibria where the expected travel volume on each route is 3000. Furthermore, there exist Nash equilibria where the expected travel volume is not the same for both routes. Let again p_i denote the probability that driver i chooses route ABD. Let E_i denote again the expected number of cars taking route ABD, not counting i. Suppose that 151 drivers choose $p_i = p = \frac{1}{100}$, 2998 drivers choose $p_i = 1$ and 2851 drivers choose $p_i = 0$.
For a randomizing player, $E_i = 150p + 2998 = 2999.5 = \frac{1}{2} \cdot (N-1)$.
For a player choosing ABD, $E_i = 151p + 2997 = 2998.51 < \frac{1}{2} \cdot (N-1)$.
For a player choosing ACD, $E_i = 151p + 2998 = 2999.51 > \frac{1}{2} \cdot (N-1)$.
Hence each driver plays a best response against the joint strategy of the others. The expected number of cars taking route ABD is 2999.51 whereas the expected number of cars taking route ACD is $2851 + 151 \cdot (1-p) = 2851 + 151 \cdot 0.99 = 3000.49$.

This example shows that in some Nash equilibrium, $\overline{x}_{ABD}$, the expected travel volume on route ABD can be different from $\overline{x}_{ACD}$, the expected travel volume on route ACD. However, the difference is rather small: $|\overline{x}_{ABD} - \overline{x}_{ACD}| \leq 2$ in any Nash equilibrium. This follows from the fact that $\overline{x}_{ABD} \in [2999, 3001]$ and $\overline{x}_{ACD} \in [2999, 3001]$ in any Nash equilibrium. For instance, suppose that $\overline{x}_{ABD} > 3001$. Then for every driver i, $E_i > 3000$ and $p_i = 0$ is i's unique best response against the joint strategy of the other players. Thus a joint strategy yielding $\overline{x}_{ABD} > 3001$ is not a fixed point of the best response correspondence. Hence it is not a Nash equilibrium.

3.2. Next let us reconsider the road system after road BC has been added. We are going to show that at every Nash equilibrium, $\overline{x}_{ABD}, \overline{x}_{ACD} \in (1998, 2003)$ and $\overline{x}_{ABCD} \in (1996, 2002)$. To begin with, it follows by the previous argument that in any Nash equilibrium,

$$\overline{x}_{ABD}, \overline{x}_{ACD} \in [(6000 - \overline{x}_{ABCD})/2 - 1, (6000 - \overline{x}_{ABCD})/2 + 1] \tag{1}$$

when routes ABD, ACD and ABCD are available. Further, the inequalities

$10 + \frac{1}{1000}(\overline{x}_{ABCD} + 1) + \frac{1}{100}(\overline{x}_{ABCD} + \overline{x}_{ACD} + 1) \geq 50 + \frac{1}{1000}\overline{x}_{ABD}$,
$10 + \frac{1}{1000}\overline{x}_{ABCD} + \frac{1}{100}(\overline{x}_{ABCD} + \overline{x}_{ACD}) \leq 50 + \frac{1}{1000}(\overline{x}_{ABD} + 1)$

have to hold in equilibrium, to rule out beneficial switches from ABD to ABCD or vice versa. It follows that

$10 + \frac{1}{1000}\overline{x}_{ABCD} + \frac{1}{100}\overline{x}_{ABCD} + \frac{1}{100}\overline{x}_{ACD} \in [50 + \frac{1}{1000}\overline{x}_{ABD} - \frac{1}{1000} - \frac{1}{100}, 50 + \frac{1}{1000}\overline{x}_{ABD} + \frac{1}{1000}]$
or $\frac{1}{1000}\overline{x}_{ABCD} + \frac{1}{100}\overline{x}_{ABCD} + \frac{1}{100}\overline{x}_{ACD} \in [40 + \frac{1}{1000}\overline{x}_{ABD} - \frac{1}{1000} - \frac{1}{100}, 40 + \frac{1}{1000}\overline{x}_{ABD} + \frac{1}{1000}]$ or

$$11 \cdot \overline{x}_{ABCD} + 10 \cdot \overline{x}_{ACD} \in [40000 + \overline{x}_{ABD} - 11, 40000 + \overline{x}_{ABD} + 1]. \tag{2}$$

First, (1) and (2) imply

$11 \cdot \overline{x}_{ABCD} + 10 \cdot [(6000 - \overline{x}_{ABCD})/2 - 1] \leq 40000 + [(6000 - \overline{x}_{ABCD})/2 + 1] + 1$ or

$6.5 \cdot \overline{x}_{ABCD} + 30000 - 10 \leq 43000 + 2$ or $6.5 \cdot \overline{x}_{ABCD} \leq 13000 + 12$, hence $\overline{x}_{ABCD} < 2000 + 2$.

Second, (1) and (2) imply

$11 \cdot \overline{x}_{ABCD} + 10 \cdot [(6000 - \overline{x}_{ABCD})/2 + 1] \geq 40000 + [(6000 - \overline{x}_{ABCD})/2 - 1] - 11$ or

$6.5 \cdot \overline{x}_{ABCD} + 30000 + 10 \geq 43000 - 12$ or $6.5 \cdot \overline{x}_{ABCD} \geq 13000 - 22$, hence $\overline{x}_{ABCD} > 2000 - 4$. To summarize,

$$\overline{x}_{ABCD} \in (1996, 2002) \tag{3}$$

has to hold in every Nash equilibrium. (1) and (3) imply

$$\overline{x}_{ABD}, \overline{x}_{ACD} \in (1998, 2003). \tag{4}$$

3.3. Finally, suppose that all roads AB, AC, BD, CD, BC and CB exist and $\overline{t}_{CB} \geq 0$ where $\overline{t}$ stands for expected travel time. Then in every Nash equilibrium, $\overline{x}_{ACBD} = 0$; that is, road CB is not used. For suppose $\overline{x}_{ACBD} > 0$. Then $\overline{t}_{ACBD} \geq \overline{t}_{AC} + \overline{t}_{BD}$, $\overline{t}_{ABD} = \overline{t}_{AB} + \overline{t}_{BD}$ and $\overline{t}_{ACD} = \overline{t}_{AC} + \overline{t}_{CD}$. A driver chooses route ACBD only if switching to route ABD or route ACD does not shorten the expected travel time. A switch to ABD increases the expected travel time on route ABD by 1/100. If the switch is not beneficial, then $\overline{t}_{ABD} + 1/100 \geq \overline{t}_{ACBD}$. Hence $\overline{t}_{AB} + \overline{t}_{BD} + 1/100 \geq \overline{t}_{AC} + \overline{t}_{BD}$ and $\overline{t}_{AB} > \overline{t}_{AC} - 1/100 > 49$. It follows that $\overline{x}_{AB} > 4900$. By the same token, $\overline{x}_{CD} > 4900$. Consequently, $\overline{x}_{AC} < 1100$ and $\overline{x}_{BD} < 1100$. It follows that $\overline{x}_{BC} > 3800$ and $\overline{t}_{CB} < 1100$. Therefore, $\overline{t}_{ABCD} = \overline{t}_{AB} + \overline{t}_{BC} + \overline{t}_{CD} \geq 49 + 13.8 + 49 = 111.8$ whereas $\overline{t}_{ACBD} = \overline{t}_{AC} + \overline{t}_{CB} + \overline{t}_{BD} < 51.1 + 1.1 + 51.1 = 103.3$. This implies that taking route ABCD with positive probability cannot be part of a best response for any player — which contradicts the fact more than 3800 players make such a choice so that $\overline{x}_{BC} > 3800$ obtains.

4. Conclusion

Introduction of mixed strategies causes only slight changes in expected equilibrium travel times so that, by and large, the Braess paradox remains intact. In any Nash equilibrium (in pure or mixed strategies), the expected travel times are close to 83 minutes when the original road system is in place. After addition of road BC, the expected equilibrium travel times are close to 92 minutes. Further addition of the fast road CB has no effect: That road is never used.

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