Economics Bulletin

Volume 44, Issue 4

The joint dynamics of the saving rate and factor income shares

Jaime Alonso-Carrera
University of Vigo, Spain

María-Jesús Freire-Serén
University of Vigo, Spain

Abstract

We assess the empirical plausibility of the economic conditions underlying the well-known result on the non-monotonic dynamics of the saving rate in the exogenous growth model with aggregate CES production function. It is well known that a hump-shaped saving rate can emerge along the transition dynamics when capital and labor are complementary. We show that under this condition, the marginal productivity of capital, which drives the substitution effect of capital accumulation, declines faster than the average productivity, which drives the income effect. We also show that some key features of the model are empirically implausible. In particular, the data do not support the declining path of the capital income share that the model shows in reproducing the hump-shaped saving rate.

Financial support from the Spanish Government and FEDER trough grant PID2021-124015NB-I00 is gratefully acknowledged.
Citation: Jaime Alonso-Carrera and María-Jesús Freire-Serén, (2024) "The joint dynamics of the saving rate and factor income shares", *Economics Bulletin*, Volume 44, Issue 4, pages 1469-1480
Contact: Jaime Alonso-Carrera - jalonso@uvigo.es, María-Jesús Freire-Serén - mjfreire@uvigo.es.
Submitted: September 17, 2024. **Published:** December 30, 2024.

1 Introduction

This paper conducts an economic and empirical evaluation of the conditions necessary for the neoclassical growth model to reproduce the observed hump-shaped path of the saving/investment rate during the development process. The literature has largely documented this empirical phenomenon using different data sources and time periods. For example, Alvarez-Cuadrado (2008), Alonso-Carrera et al. (2021) and Alonso-Carrera et al. (2024) analyze the postwar economic dynamics of some European countries, while Hayashi (1986, 1989) or Christiano (1989) study the Japanese postwar reconstruction process. Another branch of the aforementioned literature shows that these hump-shaped dynamics of the saving rate can be identified by cross-country analysis. Antras (2001) provides convincing evidence of this pattern for 24 OECD countries. Loayza et al. (2000) also finds non-monotonic saving trends in developing countries.

The literature has also demonstrated that a hump-shaped saving rate may arise in the one-sector neoclassical growth model when capital and labor are complementary (see, e.g., Antras, 2001; Smetters, 2003; Gómez, 2008; Litina and Palivos, 2010). These studies follow a phase diagram analysis to derive the necessary conditions for this non-monotonic dynamics of the saving rate. However, they do not provide economically intuitive conditions and, therefore, do not reveal the economic mechanisms and intuition underlying this result, which prevents us from assessing the extent to which it is empirically plausible.[1] In this paper, we contribute to this debate by unraveling the mechanism that drive the dynamic behavior of the saving rate in the standard neoclassical growth model.

We show that the key for the dynamics of the saving rate is the evolution of the ratio between the marginal and the average product of capital. The economic intuition of this result follows directly from the fact that the marginal product determines the substitution effect of changes in the capital stock, while the average product determines the income effect. The relationship between these two technological features and the willingness of individual to intertemporally substitute consumption, which is determined by the constant elasticity of intertemporal substitution (IES), determines whether the substitution effect or the income effect dominates. In the standard model with a Cobb-Douglas production function, the ratio of the marginal to average product of capital is constant for each level of capital. This implies that the balance between the income and the substitution effects of capital accumulation also remains constant along the equilibrium dynamics, which explains the well-known model's prediction of a monotonic behavior of the saving rate.

Based on the previous result, we conclude that a non-monotonic behavior of the saving rate in a neoclassical growth model requires a decoupling of the marginal from the average product of capital. If the ratio between these two technological features depends on the level of the capital stock, then the ranking between the income and the substitution effects of capital accumulation can be reversed along the transition. More specifically, we prove that a hump-shaped path of the saving rate can arise when the aforementioned ratio between the marginal and average product of capital decreases. In this case, the

[1] Antras (2001) explains that the hump-shaped behavior of the saving/investment rate may arise when capital and labor are complementary, because in this case the marginal product of capital is finite as capital tends to zero, which weakens the substitution effect when capital is low. However, as this paper shows, this does not seem to be the real reason because in this case the average product of capital is also finite.

income effect is likely to dominate for small values of capital, while the substitution effect may become the dominant effect when capital is above a certain threshold.

This necessary property of technology is satisfied when the elasticity of substitution between capital and labor is less than one. This can be used to empirically test the technological explanation of the observed hump-shaped dynamics of the saving rate. On the one hand, there is ample evidence that capital and labor are complementary at the aggregate level in the U.S. economy (see, e.g., Antràs, 2004; Klump et al., 2007; León-Ledesma et al., 2010). On the other hand, we test the empirical plausibility of our theoretical explanation of the saving rate dynamics by examining the behavior of the capital income share of GDP along the development process. Note that this share is equal to the ratio between the marginal and the average product of capital. We show that the European economies involved in the World War II exhibited hump-shaped saving rates but a declining labor income shares in the years following the conflict.

The paper is organized as follows. Section 2 presents the model. Section 3 characterizes the equilibrium dynamics. Section 4 provides the economic interpretation of the conditions that guarantee the hump-shaped saving rate. Section 5 calibrates and simulates the model to empirically test its dynamic properties. Section 6 concludes.

2 The model

We consider a one-sector model of economic growth in continuous time. There is a single good Y that can be used for consumption and investment in capital. This good is produced by means of the CES production function

$$Y = \left[\alpha K^{\frac{\pi-1}{\pi}} + (1-\alpha)(AL)^{\frac{\pi-1}{\pi}}\right]^{\frac{\pi}{\pi-1}}, \tag{1}$$

with $\alpha \in (0,1)$, and where K is the aggregate capital stock; L is the total amount of labor; $\pi \geq 0$ is the elasticity of substitution between K and L; and A measures the efficient units of labor. We consider a Harrod-neutral technological change, so that A grows at the constant rate η. Capital depreciates at the constant rate δ. Finally, firms operate under perfect competition so that the rental rate of capital, r, and labor, w, are equal to their respective marginal products.

The economy is populated by identical and infinitely lived individuals. The population grows at a constant rate n. In each period, the representative agent is endowed with one unit of time, which he supplies inelastically on the labor market. This individual then faces the following budget constraint:

$$\dot{k} = w + rk - c - (n+\delta)k, \tag{2}$$

where c and k are the per capita levels of consumption and capital stock, respectively.

Preferences are given by the following utility function:

$$U = \int_0^{\infty} \mathrm{e}^{(n-\rho)t}\left[\frac{c^{1-\sigma}-1}{1-\sigma}\right]\mathrm{d}t, \tag{3}$$

where ρ is the constant subjective discount rate of time preferences, and $\sigma > 0$ is the constant coefficient of relative risk aversion, which is also the inverse of IES. We assume that $\rho > n + (1-\sigma)\eta$, which guarantees that the representative's utility (3) is finite.

Thus, the representative consumer faces the problem of choosing the time paths c and k that maximize (3) subject to (2) and the non-negative constraints on c and k.

Given the initial values of the capital stock, a *competitive equilibrium* of this economy consists of a path of prices $\{w, r\}$ and a path of allocations $\{k, c\}$ that are consistent with consumer and firm optimization. The equilibrium path is thus fully characterized by the dynamic system of equations in k and c given by the aggregate resource constraint and the standard Euler equation determining the growth rate of consumption. Furthermore, a *balanced growth path (BGP) equilibrium* of this economy is a path along which the allocations of aggregate output, consumption and capital grow at the constant rate $n+\eta$.

3 Equilibrium dynamics

Given the properties of BGP, we fully characterize the equilibrium path by means of the following dynamic system in the capital stock per efficient units of labor, $z = K/AL$, and the ratio of consumption to output, $x = C/Y$:

$$\dot{z} = f(z)(1-x) - (\delta + n + \eta)z, \tag{4}$$

and

$$\frac{\dot{x}}{x} = f'(z)\left[x + \frac{(\delta + n + \eta)\, z}{f(z)} + \frac{1-\sigma}{\sigma}\right] - \frac{\delta + \rho + \sigma\eta}{\sigma}, \tag{5}$$

where $f(z)$ is the output in efficient units of labor, i.e.,

$$y = f(z) \equiv \left[\alpha z^{\frac{\pi-1}{\pi}} + 1 - \alpha\right]^{\frac{\pi}{\pi-1}}. \tag{6}$$

At this point, we note that the capital income share, rz/y, is given by the following function of z:

$$m(z) = \frac{1}{\left[1 + \left(\frac{1-\alpha}{\alpha}\right) z^{\frac{1-\pi}{\pi}}\right]} \in [0, 1]. \tag{7}$$

By differentiating this function, we see that the capital income share decreases in z when capital and labor are complementary (i.e., $\pi < 1$), while it increases in z when these two factors are substitutes (i.e., $\pi > 1$). Finally, this share is constant and equal to α in a Cobb-Douglas production function as $\pi = 1$.

We now characterize the equilibrium dynamics by following a phase diagram analysis of the dynamic system composed of Equations (4) and (5). We define the z-locus and the x-locus in the space (z, x) as the combinations of these two variables for which $\dot{z} = 0$ and $\dot{x} = 0$, respectively. We see from (4) that the z-locus is given by the following equation:

$$x = H(z) \equiv 1 - \frac{(\delta + n + \eta)z}{f(z)}. \tag{8}$$

This locus is then given by a decreasing function of x on z. The equilibrium is only defined for $z < \bar{z}^1$, where $\bar{z}^1$ is the level of z that satisfies $H(\bar{z}^1) = 0$. We also get that $\partial\dot{z}/\partial x < 0$ at any point in the z-locus.

We also see from (5) that the x-locus is given by the following function:

$$x = G(z) \equiv H(z) + \left(\frac{1}{\sigma}\right)\left[\frac{\delta+\rho+\sigma\eta}{f'(z)} - 1\right]. \tag{9}$$

By differentiating this function, we obtain that $G'(z) > 0$ if and only if the following condition holds:

$$m(z) < \frac{\phi}{\sigma\pi}, \tag{10}$$

with

$$\phi = \frac{\delta+\rho+\sigma\eta}{\delta+n+\eta}.$$

Since the capital income share $m(z)$ is a monotonic function, we then conclude that the locus $G(z)$ can exhibit one of the following alternative shapes. First, if $\pi = 1$, then $m(z) = \alpha$ and $G(z)$ is monotonic with: (a) $G'(z) > 0$ if $\alpha\sigma < \phi$; and, (b) $G'(z) < 0$ if $\alpha\sigma > \phi$. Second, if $\pi < 1$, then $m'(z) < 0$, $m(0) = 1$ and with: (a) $G'(z) > 0$ for all z when $\sigma\pi < \phi$; and, (b) when $\sigma\pi > \phi$, there exists a value of z given by

$$\hat{z} = \left[\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{\sigma\pi}{\phi} - 1\right)\right]^{\frac{\pi}{1-\pi}}, \tag{11}$$

so that $G'(z) < 0$ for $z < \hat{z}$, while $G'(z) > 0$ for $z > \hat{z}$. Third, if $\pi > 1$, then $m'(z) > 0$, $\lim_{z\to 0} m(z) = 0$, and $G'(z) > 0$ when $z < \hat{z}$, while $G'(z) < 0$ when $z > \hat{z}$. Finally, we also get that $\partial\dot{x}/\partial x > 0$ at any point in the x-locus in all of these three cases.

Along the BGP, the values of z and x remain constant. By solving the system of equations (8) and (9), we obtain that these stationary values are given by $f'(z^\star) = \delta + \rho + \sigma n$ and $x^\star = 1 - m(z^\star)/\phi$. In the appendix, we prove that there is at most one unique BGP. Furthermore, we also prove that this BGP is locally saddle-path stable.

The main conclusion from the previous analysis is that the shape of the policy function relating the consumption-output ratio with the capital stock in efficient units of labor depends on the dynamic behavior of the capital income share, i.e., how the ratio between the marginal and the average product of capital evolves along the transition dynamics. In the next section, we will focus on the case with $\pi \in (\phi/\sigma, 1)$, as this is the unique case where a hump-shaped saving rate can arise. Figure 1 plots the phase diagram that can occur in this benchmark case. The figure plots the U-shape of the policy function $x = \Phi(z)$ that arises under certain conditions. Note that these dynamics require the x-locus to be U-shaped. Next, we state the conditions that ensure this.

4 A hump-shaped saving rate

From the previous section, we conclude that the saving rate, $1 - x$, can have a hump shape only if $\pi \in (\phi/\sigma, 1)$. As noted earlier in the literature, capital and labor must then complement each other to reproduce the hump-shaped path of the saving rate observed in the data. We contribute by providing a clear economic intuition of the mechanism behind this result.

Consider the case where the initial stock of capital in efficient units of labor is below its stationary value, i.e., $z_0 < z^\star$. The dynamic behavior of the saving rate depends on the balance between the substitution and income effects of capital accumulation caused by this initial imbalance. The increase of z reduces both the marginal and the average

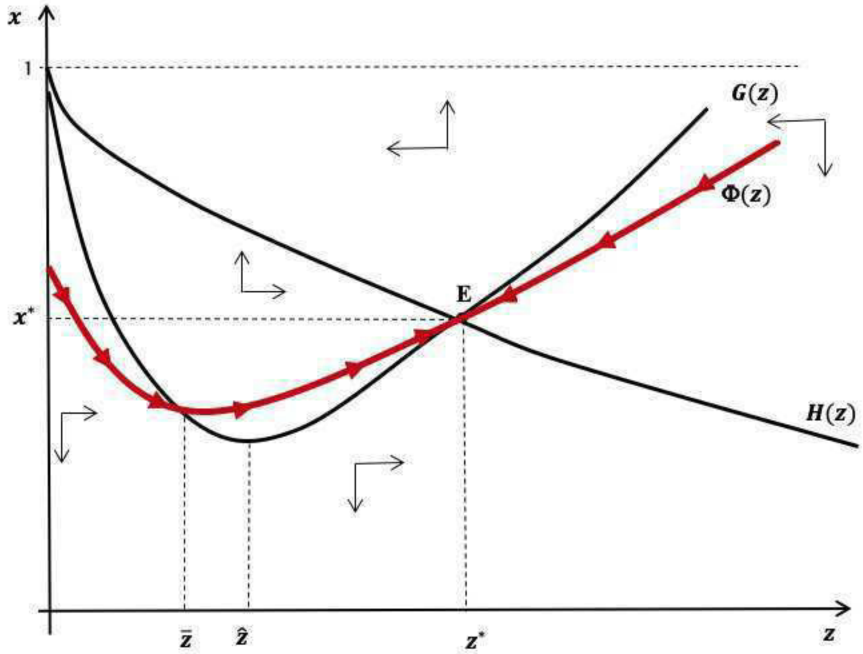


Figure 1: Phase diagram for benchmark dynamics

productivity of capital. The decrease in the marginal productivity reduces the rate of return on savings, leading to an intertemporal substitution of consumption towards the present. On the contrary, the decrease in the average productivity implies a smaller impact of capital accumulation in reducing the distance of current income in efficient units of labor with respect to its stationary value. This stimulates savings to accelerate the reduction in this income gap. The balance between these two effects depends on the IES since it determines the consumer's desire to smooth the path of consumption. The substitution effect dominates for sufficiently large values of IES (i.e., for small values of σ).

Thus, in order to have a non-monotonic path of the saving rate, the balance between substitution and income effects must change along the transition dynamics leading the economy to the BGP. Since the IES in our model is constant, and equal to $1/\sigma$, any change in the dominance of the two aforementioned effects of capital accumulation can only occur if there is a variation in their relative size. This requires decoupling the dynamics of the marginal and average productivity of capital. Next, we provide the conditions that are necessary for this result.

The first necessary condition for a hump-shaped saving rate is that capital and labor complements each other in production, i.e., $\pi < 1$. The relevance of this condition is that it implies that the capital income share should be a decreasing function of z, i.e., $m'(z) < 0$. In this case, the marginal productivity decreases faster than average productivity as z increases. This can change the balance between the income and substitution effects, even when the IES is constant in our model. Other conditions are then necessary to ensure that the reversal in the dominance of the effects actually occurs and it is in the right direction as the economy approaches to the BGP.

On the one hand, a hump-shaped saving rate requires that the income effect of capital accumulation must dominate to the substitution effect for small values of z. This depends on the relative size of these effects, which is given by the capital income share $m(z)$ (i.e., the ratio between marginal and average productivity), and on the value of the IES, which determines the balance between them. In particular, from the previous section, we conclude that the income effect dominates when the condition $m(z) > \frac{\phi}{\sigma\pi}$ holds, which implies that the income effect is not relatively small with respect to the substitution effect. Noting that $m(z)$ converges to one as z tends to zero when $\pi < 1$, the necessary condition for the income effect to dominate for any sufficiently small value of z is that $\sigma\pi > \phi$. This defines a lower bound for σ given by

$$\underline{\sigma} = \frac{\delta + \rho}{\pi(\delta + n + \eta) - \eta}.$$

Given a ratio of marginal to average productivity, the income effect dominates when σ is sufficiently high, i.e., when the consumers' willingness to intertemporally substitute consumption is small.[2] In this case, the saving rate then increases with z.

On the other hand, a hump-shaped saving rate also requires that the condition $m(z) > \frac{\phi}{\sigma\pi}$ be reversed as z approaches to the steady state. If this were to occur, then the substitution effect of capital accumulation would begin to dominate the income effect, and thus the saving rate would thereafter decline with z. We proved above that this reversal of the order of dominance between effects occurs at $z = \hat{z}$. Obviously, we need that this threshold of z to be less than its stationary value, i.e., $\hat{z} < z^{\star}$. This happens when

$$\frac{1}{\sigma} > \pi\left(\delta + n + \eta\right)\left(\frac{\alpha}{\delta + \rho + \sigma\eta}\right)^{\pi}. \tag{12}$$

Condition (12) implicitly imposes an upper bound on the value of σ, which we denote by $\overline{\sigma}$. Given the uneven evolution of the marginal and average products of capital, the reversal of the dominance of the income effect over the substitution effect at some point in the transition dynamics also requires that consumers' willingness to smooth consumption over their lifetime is sufficiently high.

Note that the above set of conditions given is necessary but not sufficient for having a hump-shaped savings rate. In particular, the range of values of σ under which the saving rate follows a hump-shaped path may be smaller than the interval $(\underline{\sigma}, \overline{\sigma})$. The overlap between these two intervals will be smaller, the larger the distance of the initial value of z from its stationary value.

5 Empirical evaluation

We now provide an empirical test of the theory presented above. To this end, we first calibrate the parameters of the model to reproduce some long-run features of the U.S. economy. Table I provides the benchmark values of the parameters. We define a period as one year. We set the population growth rate to $n = 0.01$ and the rate of technical progress to $\eta = 0.02$. We choose $\alpha = 0.3479$ so that the steady state of the model replicates a labor income share of 0.67 (see Valentinyi and Herrendorf, 2008). We assign a value of 0.84 to the capital-labor elasticity of substitution π, which is the value estimated by Herrendorf et al. (2015), and it is in line with other estimates in the literature (see, e.g.,

[2] We must assume that $\pi(\delta + n + \eta) > \eta$. Otherwise, the condition $\sigma\pi > \phi$ cannot hold, since it would require a negative IES and thus the saving rate would not have a hump-shaped path. Note that this means that the capital-labor elasticity of substitution must satisfy $\pi > \frac{\eta}{\delta + n + \eta}$.

Antràs, 2004; Klump et al., 2007; León-Ledesma et al., 2010). The parameter ρ is set to 0.02, implying an annual discount rate of preference of 0.98, which is a widely used value in the literature. The depreciation rate δ is set to 0.1.

Table I: Parameter values

Parameter	Notation	Value
Rate of technical progress	η	0.02
Growth rate of population	n	0.01
Capital share	α	0.3479
Elasticity of substitution between capital and labor	π	0.84
Depreciation rate of capital stock	δ	0.1
Discount factor	ρ	0.02
Inverse of intertemporal elasticity of substitution	σ	6
Initial stock of capital in efficient units of labor	z_0	$0.01z^\star$

Finally, we set the value of σ as follows. Under the parameterization explained above, the two thresholds that delimit the values of σ necessary to obtain a hump-shaped path of saving rate are $\underline{\sigma} = 1.3453$ and $\overline{\sigma} = 7.3158$. However, as was explained in the previous section, the hump-shaped saving rate can only occur for a subset of values in the interval $(\underline{\sigma}, \overline{\sigma})$. For this reason, we follow Antras (2001) and we set the value of σ to 6, which guarantees that the hump-shaped path of the saving rate is similar to the one observed in the data. Although this value is higher than that generally considered in the macroeconomic literature, it is in the range of the values estimated, for instance, by Hall (1988), Ogaki and Reinhart (1998) or Yogo (2004).

We choose the initial level of the capital stock in efficient units of labor $z_0 = z^\star/100$. We simulate the equilibrium path of the calibrated model and we compute the path of the saving rate. Figure 2 shows that the calibrated economy has a hump-shaped dynamics in the saving rate. More importantly, since the latter dynamics require the elasticity of substitution between capital and labor π to be less than one, we also obtain a downward sloping path of the simulated capital income share.

We now examine whether the proposed theory is confirmed by the data. To do so, we first examine the extent to which the variation in the capital income share required by the model is observed in the data. Figure 3 plots the labor income share observed in the data. The downward sloping path of the simulated capital income share is not confirmed by the observed data. We observe that the unweighted average of the labor income share declined in the two samples for which the empirical literature has found a hump-shaped dynamics: OECD countries and the European countries that participated in the World War II (Austria, Belgium, France, Germany, Italy, Luxembourg and Netherlands). Assuming a competitive equilibrium framework, this empirical fact implies that the capital income share is increasing as this share is calculated as the fraction of aggregate GDP not allocated to labor compensation.

Finally, we also empirically evaluate the model by examining how it fits the relationship between the saving rate and the rates of technological progress and population growth. To this end, we note that Krusell and Smith (2015) and Alvarez-Cuadrado (2017) provide empirical evidence that the saving rate increases with the growth

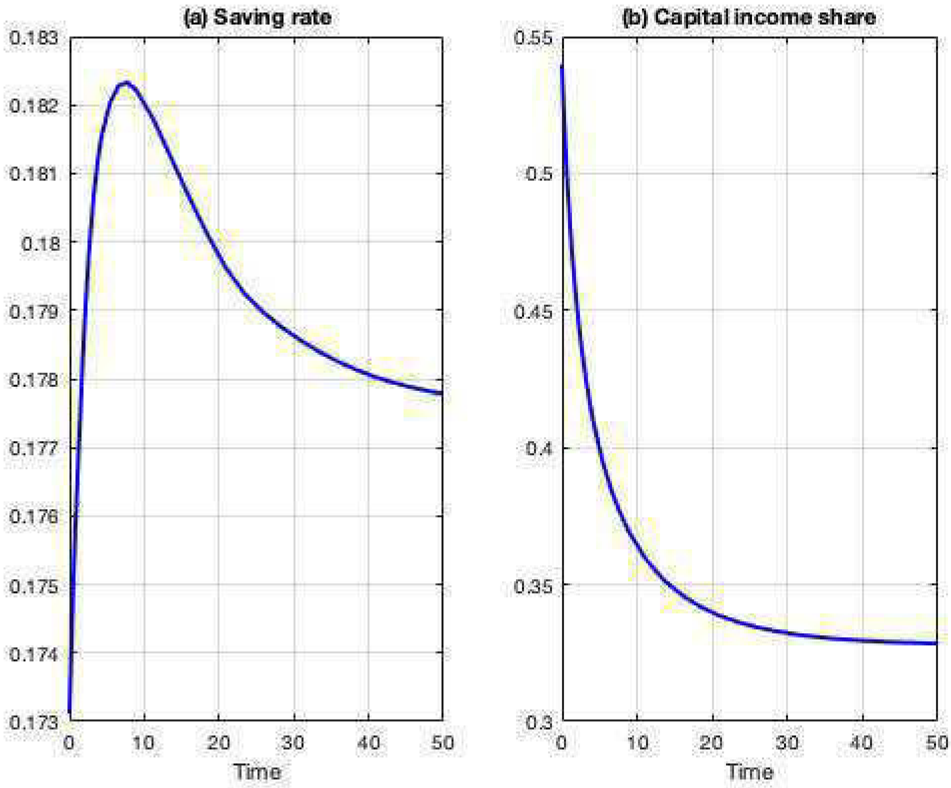
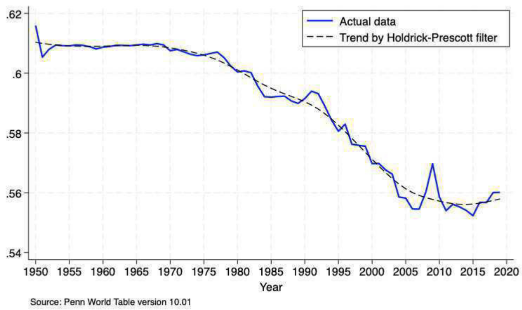
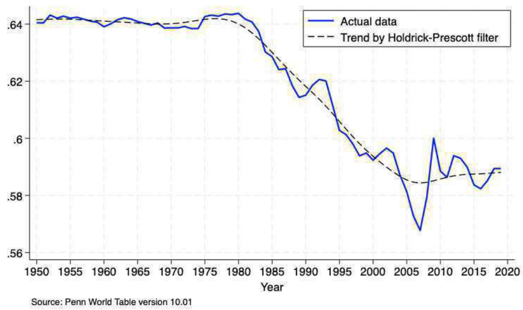


Figure 2: Time paths of the benchmark model



(a) OECD



(b) European countries

Figure 3: Evolution of unweighted-average labor income share

rate of output and the rate of population growth. In the debate over the implications of Piketty (2014)'s "second fundamental law of capitalism", Krusell and Smith (2015) evaluates this relationship in the steady state to assess the empirical plausibility of models of optimal saving. We use this strategy as a further test to validate the technological explanation of the hump shape followed by the saving rate. We obtain that the gross saving rate increases with the rate η of technological change at the BGP if and only if $\sigma < \underline{\sigma}$. Note that this threshold value corresponds to the lower bound $\underline{\sigma}$ of the interval of values that σ must take to have a hump-shaped saving rate. Thus, the capital-labor complementarity explanation for the behavior of the saving rate observed in the data is inconsistent with the estimated positive relationship between saving and growth rates.

6 Conclusion

We have characterized the dynamics of a neoclassical growth model with a CES production function. We have shown that this model is able to reproduce the hump-shaped path of the aggregate saving rate in the OECD countries over the period 1950-2020. We have shown that the key point to these dynamics is that the CES technology decouples the marginal and the average productivity of capital. Therefore, the income and substitution effects of capital accumulation decline at different rates. When capital and labor complements each other, the marginal productivity declines faster than the average productivity, so that a reversal in the ranking between the two effects can occur along the process of convergence to the BGP, even when the IES is constant.

We have also illustrated that this result comes at the cost of obtaining a increasing labor income share in stark contrast with the empirical evidence. This open an interesting research agenda consisting on looking for other mechanisms that may allow us to obtain the observed hump-shaped saving rate dynamics with decreasing labor income shares.

References

Alonso-Carrera, J., Bouché, S. and De Miguel, C. (2021), 'Revisiting the process of aggregate growth recovery after a capital destruction', *Journal of Macroeconomics* **68**.

Alonso-Carrera, J., Freire-Serén, M. and Raurich, X. (2024), 'Decomposing structural change', *acepted in The BE Journal of Macroeconomics* .

Alvarez-Cuadrado, F. (2008), 'Growth outside the stable path: Lessons from the european reconstruction', *European Economic Review* **52**(3), 568–588.

Alvarez-Cuadrado, F. (2017), 'Saving and growth in neoclassical growth models: A comment on "is piketty's "second law of capitalism" fundamental?"', *Economics Letters* **174**(128-131).

Antras, P. (2001), 'Transitional dynamics of the savings rate in the neoclassical growth model', *Working Paper, MIT* .

Antràs, P. (2004), 'Is the u.s. aggregate production function cobb-douglas? new estimates of elasticity of substitution', *B.E. Journal of Macroeconomics* **4**.

Christiano, L. J. (1989), 'Understanding japan's saving rate: The reconstruction hypothesis', *Federal Reserve Bank of Minneapolis. Quarterly Review-Federal Reserve Bank of Minneapolis* **13**(2), 10.

Gómez, M. A. (2008), 'Dynamics of the saving rate in the neoclassical growth model with ces production', *Macroeconomic Dynamics* **12**, 195–210.

Hall, R. (1988), 'Intertemporal substitution in consumption', *Journal of Political Economy* **96**, 339–357.

Hayashi, F. (1986), Why is japan's saving rate so apparently high?, *in* S. Fischer, ed., 'NBER Macroeconomics Annual 1986', Vol. 1, Cambridge, Mass.: MIT Press/National Bureau of Economic Research., pp. 147–210.

Hayashi, F. (1989), 'Is japan's saving rate high?', *Federal Reserve Bank of Minneapolis. Quarterly Review-Federal Reserve Bank of Minneapolis* **13**(2), 3.

Herrendorf, B., Rogerson, R. and Valentinyi, Á. (2015), 'Sectoral technology and structural transformation', *American Economic Journal: Macroeconomics* **7**, 104–133.

Klump, R., McAdam, P. and Willman, A. (2007), 'Factor substitution and factor-augmenting technical progress in the united states: A normalized supply-side approach', *Review of Economics and Statistics* **89**, 183–192.

Krusell, P. and Smith, A. (2015), 'Is piketty's second law of capitalism fundamental?', *Journal of Political Economy* **123**, 725–748.

León-Ledesma, M. A., McAdam, P. and Willman, A. (2010), 'Identifying the elasticity of substitution with biased technical change', *American Economic Review* **100**, 1330–1357.

Litina, A. and Palivos, T. (2010), 'The behavior of the saving rate in the neoclassical optimal growth model', *Macroeconomic Dynamics* **14**, 482–500.

Loayza, N., Schmidt-Hebbel, K. and Servén, L. (2000), 'Saving in developing countries: An overview', *World Bank Economic Review* **14**, 393–414.

Ogaki, M. and Reinhart, C. (1998), 'Measuring intertemporal substitution: The role of durable goods.', *Journal of Political Economy* **106**, 1078–1098.

Piketty, T. (2014), *Capital in the twenty-first century*, Translated by Arthur Golhammer. Belknap, Cambridge, MA.

Smetters, K. (2003), 'The (interesting) dynamic properties of the neoclassical growth model with CES production', *Review of Economic Dynamics* **6**, 697–707.

Valentinyi, Á. and Herrendorf, B. (2008), 'Measuring factor income shares at the sectoral level', *Review of Economic Dynamics* **11**, 820–835.

Yogo, M. (2004), 'Estimating the elasticity of intertemporal substitution when instruments are weak', *Review of Economics and Statistics* **86**, 797–810.

Appendix

A Uniqueness and existence of BGP

The stationary value of z and x are given by the intersection between (8) and (9). To determine the number of intersections, we first note that $H(0) = 1$, $H'(z) < 0$ and $G(0) = 1 - 1/\sigma < 1$. Furthermore, by using (9), we conclude that

$$G(\bar{z}^1) = \left(\frac{1}{\sigma}\right)\left[\frac{\delta+\rho+\sigma\eta}{f'(\bar{z}^1)} - 1\right],$$

since $H(\bar{z}^1) = 0$ by definition, By noting that $f'(z) < f(z)/z$ and $\rho + \sigma\eta > n + \eta$, we directly obtain that $G(\bar{z}^1) > 0$. We then distinguish the following cases:

1. Consider that $\pi = 1$ and $\alpha\sigma < \phi$. In this case, $G'(z) > 0$. Therefore, the exists a unique BGP since:

   (a) If $\sigma \geq 1$, then $G(0) \in [0, 1)$.

   (b) If $\sigma < 1$, then $G(0) < 1$, the uniqueness follows since $G(\bar{z}^1) > 0$.

2. Consider that $\pi = 1$ and $\alpha\sigma > \phi$. In this case, $G'(z) < 0$. Therefore, the existence of BGP depends on σ:

   (a) If $\sigma \leq 1$, then $G(0) \leq 0$, so that there does not exist a BGP.

   (b) If $\sigma > 1$, then then $G(0) \in [0, 1)$, so that there exists a unique BGP because $G(\bar{z}^1) > H(\bar{z}^1) = 0$.

3. Consider that $\pi < min\{1, \phi/\sigma\}$. In this case, $G'(z) > 0$. Hence, there exists a unique BGP because $G(0) < H(0)$.

4. Consider that $\pi \in (\phi/\sigma, 1)$. In this case, $G'(z) < 0$ when $z < \hat{z}$, whereas $G'(z) > 0$ when $z > \hat{z}$. Therefore, there exists a unique BGP.

5. Consider that $\pi > 1$. In this case, $\lim_{z \to 0} G(z) = 0$ and $G'(z) > 0$ when $z < \hat{z}$, whereas $G'(z) < 0$ when $z > \hat{z}$. Therefore, there exists a unique BGP because $\bar{z}^1 < \bar{z}^2$, where $\bar{z}^2$ is the root of equation $G(\bar{z}^2) = 0$.

B Stability property of BGP

Let J be the Jacobian matrix evaluated at the steady state of the system of difference equations formed by (4) and (5).[3] We obtain after some algebra that

$$J = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

where

$$a_{11} \equiv \frac{\partial \dot{z}}{\partial z} = (\delta + n + \eta)\left[m(z) - 1\right] < 0,$$

$$a_{12} \equiv \frac{\partial \dot{z}}{\partial x} = -(\delta + \rho + \sigma\eta) m(z) z < 0,$$

$$a_{21} \equiv \frac{\partial \dot{x}}{\partial z} = \left(\frac{1}{\sigma}\right)\left\{\frac{\phi x\left[1 - m(z)\right]}{z}\right\}\left[1 - \frac{\phi m(z)}{\sigma\pi}\right],$$

and

$$a_{22} \equiv \frac{\partial \dot{z}}{\partial x} = (\delta + \rho + \sigma\eta) x > 0.$$

[3]In this proof all the variables are evaluated at the BGP equilibrium. For simplicity, we then omit the asterisk notation.

After some algegra, we get that the determinant of J is given by

$$det(J) = \left(\frac{\phi x}{\sigma}\right)[1 - m(z)]\left\{m(z) - 1 - \frac{\phi m(z)^2}{\sigma^2 \pi}\right\},$$

which is negative. Therefore, the BGP is locally saddle-path stable.