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Linear risk tolerance and mean−variance preferences

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Abstract

We translate the property of linear risk tolerance (hyperbolical Arrow−Pratt index of risk aversion) from the expected−utility framework into a condition on the marginal rate of substitution between return and risk in the mean−variance approach.

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1. Introduction

Apart from being a convenient tool in the analysis of decision making under uncertainty, mean-variance- or (μ, σ)-specifications of preferences over lotteries are a perfect substitute for the standard expected utility (EU) approach if all attainable distributions differ from one another only by location and scale parameters. As Meyer (1987), Sinn (1983) and others convincingly argue, this location-scale assumption covers a wide range of economic decision problems.

When (μ, σ)- and EU-approach are perfect substitutes, a number of formal correspondences between them can be identified, relating, e.g., to absolute and relative risk aversion, absolute and relative prudence, risk vulnerability, standardness, or properness (see, e.g., Meyer, 1987; Lajeri and Nielsen, 2000; Lajeri-Chaherli, 2002; Wagener, 2002; Eichner and Wagener, 2003a,b, 2004). This paper adds to these results: It establishes a formal correspondence between linear risk tolerance (also called HARA-preferences) in the EU-setting and in the mean-variance framework.

The assumption of linear risk tolerance (i.e., the inverse of the Arrow-Pratt index of absolute risk aversion is linear in wealth) is powerful, e.g., in capital market models: If investors with homogeneous preferences face equal investment opportunities and the capital market is complete, the economy "aggregates" whenever risk tolerance is linear; average wealth in an economy entails enough information to predict aggregate variables, and equilibrium asset prices do not depend on the distribution of wealth. HARA-preferences imply the optimality of linear risk-sharing rules (Cass and Stiglitz, 1970) and of myopic investment strategies in dynamic problems of portfolio selection (Merton, 1969). HARA-preferences have mathematically convenient representations, making them the most commonly used functional forms in EU-analysis.

As HARA-preferences encompass a wide range of risk attitudes, our correspondence result for EU- and mean-variance approach generalizes several findings of the previous literature. We demonstrate that the mean-variance analogue entails the same behavioural implications as the EU-framework (in particular, linear investment rules). Moreover, we can utilize our result — which formally comes as a partial differential equation — to derive specific properties of mean-variance utility functions and the marginal rate of substitution between risk and return (which plays in important role in comparative static exercises).

2. Notation and Preliminaries

Consider a set $\mathcal{Y}$ of random variables that have support in a possibly unbounded real interval $\mathbf{Y}$ and that only differ from one another by location and scale parameters. I.e., if X is the random variable obtained by normalization of an arbitrary element of $\mathcal{Y}$, then any $Y \in \mathcal{Y}$ is equal in distribution to $\mu_y + \sigma_y X$, where μ_y and σ_y are the mean and the standard deviation of Y. We assume that $\mathcal{Y}$ contains all degenerate random variables on $\mathbf{Y}$. By $\mathbf{M} := \{(\mu,\sigma) \in \mathbb{R} \times \mathbb{R}_+ | \exists Y \in \mathcal{Y} : (\mu_y, \sigma_y) = (\mu, \sigma)\}$ we denote the set of all possible (μ,σ)-pairs that can be obtained for $Y \in \mathcal{Y}$. We assume that $\mathbf{M}$ is a convex set. In particular, $\mathbf{M}$ contains all pairs $(\mu_y, 0)$ where $y \in \mathbf{Y}$.

Let $u : \mathbb{R} \to \mathbb{R}$ be a von-Neumann-Morgenstern (vNM) utility index; for simplicity we assume that u is a smooth function. The expected utility from lottery Y can be written in terms of the mean and the standard deviation of Y as:

$$\mathbf{E}u(y) = \int_{\mathbf{Y}} u(\mu_y + \sigma_y x)\mathrm{d}F(x) =: U(\mu_y, \sigma_y) \tag{1}$$

where F is the distribution function of X and the mean and standard deviation of X are, respectively, zero and one by construction. We will refer to $U(\mu,\sigma)$ as two-parameter preferences. For $\sigma = 0$, eq. (1) implies that

$$U(\mu, 0) = u(\mu) \tag{2}$$

for all μ. From (1), $u(y)$ is increasing for all $y \in \mathbf{Y}$ if and only if $U(\mu,\sigma)$ is increasing in μ for all $(\mu,\sigma) \in \mathbf{M}$. As shown by Meyer (1987, Properties 1 and 2), risk aversion in $u(y)$ is tantamount to $U(\mu,\sigma)$ being decreasing in σ. Hence,

$$U_\mu(\mu,\sigma) > 0 > U_\sigma(\mu,\sigma) \quad \forall (\mu,\sigma) \in \mathbf{M} \qquad \Longleftrightarrow \qquad u'(y) > 0 > u''(y) \quad \forall y \in \mathbf{Y}, \tag{3}$$

where subscripts denote partial derivatives. The marginal rate of substitution between risk and return for two-parameter preferences U,

$$\alpha(\mu,\sigma) = -\frac{U_\sigma(\mu,\sigma)}{U_\mu(\mu,\sigma)}, \tag{4}$$

is the counterpart of the Arrow-Pratt risk-aversion measure

$$A(y) := -\frac{u''(y)}{u'(y)}$$

in the EU-framework. In particular, the monotonicity properties of the indexes for absolute and relative risk aversion and of α are related as follows (Meyer, 1987, Properties 5 and 6):

$$A'(y) \gtreqless 0 \quad \forall y \in \mathbf{Y} \quad \Longleftrightarrow \quad \frac{\partial \alpha(\mu,\sigma)}{\partial \mu} \gtreqless 0 \quad \forall (\mu,\sigma) \in \mathbf{M}; \tag{5}$$

$$[y \cdot A(y)]' \gtreqless 0 \quad \forall y \in \mathbf{Y} \quad \Longleftrightarrow \quad \frac{\partial \alpha(\lambda \cdot \mu, \lambda \cdot \sigma)}{\partial \lambda} \gtreqless 0 \quad \forall (\mu,\sigma) \in \mathbf{M}, \forall \lambda > 0. \tag{6}$$

The equality cases of constant absolute (CARA) or constant relative (CRRA) risk aversion will be particularly interesting in what follows.

3. HARA Preferences

A vNM-utility function exhibits hyberbolical absolute risk aversion (HARA) if there exist constants $a \geq 0$ and $b \in \mathbb{R}$ such that

$$A(y) = \frac{1}{a + b \cdot y} \tag{7}$$

for all y such that $(a + b \cdot y) \in \mathbf{Y}$. Defining risk tolerance as the inverse of the Arrow-Pratt index of risk aversion, HARA-preferences exhibit linear risk tolerance. It is well-known that (7) implies the following functional forms for u:

$$u(y) = u_H(y; a, b) := \begin{cases} \dfrac{(a + b \cdot y)^{1-\frac{1}{b}}}{b-1} & \text{if} \quad b \neq 0,\, b \neq 1 \\ -a \cdot \mathrm{e}^{(-y/a)} & \text{if} \quad b = 0 \\ \ln(a + y) & \text{if} \quad b = 1. \end{cases} \tag{8}$$

For $a = 0$ this yields constant relative risk aversion (CRRA), for $b = 0$ constant absolute risk aversion (CARA) emerges.

To derive the (μ,σ)-counterpart of HARA-preferences we define, for all $(\mu,\sigma) \in \mathbf{M}$, $a \geq 0$, and $b \in \mathbb{R}$,

$$G(\mu,\sigma; a, b) := (a + b \cdot \mu) \cdot \alpha_\mu(\mu,\sigma) + b \cdot \sigma \cdot \alpha_\sigma(\mu,\sigma). \tag{9}$$

Proposition 1. *Let $a \geq 0$ and $b \in \mathbb{R}$. The following are equivalent:*

- $G(\mu,\sigma; a, b) = 0$ *for all* $(\mu,\sigma) \in \mathbf{M}$;
- $A(y) = \dfrac{1}{a + b \cdot y}$ *for all* $y \in \mathbf{Y}$.

Proof: Calculate:

$$
\begin{aligned}
&G(\mu,\sigma;a,b) = \\
&= -\frac{1}{(U_\mu)^2}\cdot\left[(a+b\cdot\mu)\cdot(U_\mu U_{\mu\sigma}-U_{\mu\mu}U_\sigma)+b\cdot\sigma\cdot(U_\mu U_{\sigma\sigma}-U_{\mu\sigma}U_\sigma)\right] \\
&= -\frac{1}{(\int u'\mathrm{d}F(x))^2}\cdot\Bigg[(a+b\cdot\mu)\cdot\left(\int u'\mathrm{d}F(x)\int xu''\mathrm{d}F(x)-\int u''\mathrm{d}F(x)\int xu'\mathrm{d}F(x)\right) \\
&\qquad +b\cdot\sigma\cdot\left(\int u'\mathrm{d}F(x)\int x^2u''\mathrm{d}F(x)-\int xu''\mathrm{d}F(x)\int xu'\mathrm{d}F(x)\right)\Bigg] \\
&= -\Bigg[(a+b\cdot\mu)\cdot\left(\int x\frac{u''u'}{u'\int u'\mathrm{d}F}\mathrm{d}F-\int\frac{u''u'}{u'\int u'\mathrm{d}F}\mathrm{d}F\cdot\int x\frac{u'}{\int u'\mathrm{d}F}\mathrm{d}F\right) \\
&\qquad +b\cdot\sigma\cdot\left(\int x^2\frac{u''u'}{u'\int u'\mathrm{d}F}\mathrm{d}F-\int x\frac{u''u'}{u'\int u'\mathrm{d}F}\mathrm{d}F\cdot\int x\frac{u'}{\int u'\mathrm{d}F}\mathrm{d}F\right)\Bigg] \\
&= (a+b\cdot\mu)\cdot\Big[\mathbf{E}_\Phi(xA(y))-\mathbf{E}_\Phi x\mathbf{E}_\Phi A(y)\Big]+b\sigma\cdot\Big[\mathbf{E}_\Phi(x^2A(y))-\mathbf{E}_\Phi x\mathbf{E}_\Phi(xA(y))\Big] \\
&= (a+b\cdot\mu)\cdot\mathrm{Cov}_\Phi(x,A(y))+b\cdot\sigma\cdot\mathrm{Cov}_\Phi(x,xA(y)) \\
&= \mathrm{Cov}_\Phi(x,(a+b\cdot(\mu+\sigma x))\cdot A(y))=\mathrm{Cov}_\Phi(x,(a+b\cdot y)\cdot A(y)).
\end{aligned}
$$

Here, the argument of u is always $(\mu+\sigma x)=y$. The first equation comes from differentiating α with respect to μ and σ. In the second we used (1) and in the third we took $(\int u'\mathrm{d}F)^2$ into the integrals. In the fourth line, $\mathbf{E}_\Phi$ denotes the expectation operator with respect to distribution Φ defined by $\mathrm{d}\Phi=(u'/\int u'\mathrm{d}F)\mathrm{d}F$. The fifth line uses the definition of the covariance and the sixth follows from its additivity properties.
If the final expression is identically zero, $(a+by)\cdot A(y)$ must be constant in $y=\mu+\sigma x$. However, then $u(y)$ must satisfy (7). ■

HARA-preferences in the (μ,σ)-framework are, thus, characterized by the differential equation $G(\mu,\sigma;a,b)=0$ with G defined in (9). wo prominent special cases emerge:

- Constant absolute risk aversion: For $b=0$ we obtain the equality-case in (5): $G(\mu,\sigma;a,0)=a\cdot\alpha_\mu(\mu,\sigma)$ where the value of a is irrelevant.
- Constant relative risk aversion: For $a=0$, the equality-case in (6) emerges. To see this, verify that $\frac{\partial}{\partial\lambda}\left(-\frac{U_\sigma(\lambda\mu,\lambda\sigma)}{U_\mu(\lambda\mu,\lambda\sigma)}\right)=0$ for all $\lambda>0$ is tantamount to

$$-\frac{1}{(U_\mu)^2}\cdot\left[U_\mu\cdot(\mu U_{\mu\sigma}+\sigma U_{\sigma\sigma})-U_\sigma\left(\mu U_{\mu\mu}+\sigma U_{\mu\sigma}\right)\right]=0$$

  for all (μ,σ). This, in turn, is equivalent to $G(\mu,\sigma;0,b)=0$.

4. Application: Linear Investment Strategies

In terms of $\alpha(\mu, \sigma)$, the condition $G(\mu, \sigma; a, b) = 0$ constitutes a first-order partial differential equation whose general solution is provided in

Corollary 1. *If (μ, σ)-preferences exhibit linear risk-tolerance with parameters a and b, then there exists a differentiable function Γ such that*

$$\alpha(\mu, \sigma) = \Gamma\left(\frac{\sigma}{a + b\mu}\right). \tag{10}$$

Hence, the marginal rate of substitution between risk and return for HARA-preferences only depends on the ratio $\sigma/(a + b\mu)$. Naturally, the fact that α is defined through (4) imposes some additional structure on Γ. However, (10) already allows us to recover, in a straightforward way, some behavioural implications of linear risk tolerance for (μ, σ)-preferences. Consider, e.g., the following comparative static exercise for a standard portfolio problem with one safe and one risky asset:

An investor plans to allocate a certain initial wealth $w > 0$ to a riskfree asset with zero return and a risky asset with random return s. Denoting by q the amount invested in the risky asset, his final wealth is given by $y = w + q \cdot s$. Suppose that s has distribution F and $\mu_s := \mathbf{E}_F s$ and $\sigma_s := \sqrt{\mathbf{E}_F(s - \mu_s)^2}$ are the mean and the standard deviation of s. In the two-parameter framework, the investor chooses q as to maximize $U(\mu_y, \sigma_y)$ where $\mu_y = \mathbf{E}_F y = w + q \cdot \mu_s$ and $\sigma_y = q \cdot \sigma_s$. For HARA-preferences (10), the first-order condition for an optimal choice q^*, $\alpha(w + q^*\mu_s, q^*\sigma_s) = \mu_s/\sigma_s$, reads as:

$$\Gamma\left(\frac{\sigma_s q^*}{a + b(w + \mu_s q^*)}\right) = \frac{\mu_s}{\sigma_s}.$$

Given that w does not appear on the RHS, the argument of Γ on the LHS must not depend on w either. Hence, the optimal investment strategy can be written as

$$q^* = \beta(\mu_s, \sigma_s, a, b) \cdot (a + b \cdot w)$$

with β independent of w. This reproduces the linear investment strategy that emerges for HARA preferences in the EU-framework (see, e.g., Stiglitz, 1972). In particular, investment is wealth-independent if $b = 0$ (CARA) while the wealth elasticity of q is constant at one for $a = 0$ (CRRA).

Hence, also the behavioural implications of linear risk-tolerance can be transferred from the EU- to the (μ, σ)-framework.

5. Properties of Two-Parameter HARA-Functions

In the expected-utility framework, HARA-functions come in the forms listed in (7). We can now use Proposition 1 to elicit more information on HARA-type (μ, σ)-utility functions. In terms of $U(\mu, \sigma)$, the condition $G(\mu, \sigma; a, b) = 0$ (which defines HARA) constitutes a second-order partial differential equation:

$$(a + b\mu) \cdot (U_\mu U_{\sigma\mu} - U_\sigma U_{\mu\mu}) + b\sigma \cdot (U_\mu U_{\sigma\sigma} - U_\sigma U_{\mu\sigma}) = 0. \tag{11}$$

Clearly, if $U(\mu, \sigma)$ solves (11), then so does $f(U(\mu, \sigma))$ for any twice differentiable function $f(\cdot)$.

For non-constant absolute risk aversion (i.e., $b \neq 0$), a general class of solutions to (11) is given through:

$$U(\mu, \sigma) = V\left((a + b\mu) \cdot W\left(\frac{\sigma}{a + b\mu}\right)\right) \tag{12}$$

where V and W are twice differentiable functions. Without loss of generality, one can take $W(0) = 1$. Then V is determined by the initial condition (2) which for HARA-preferences reads: $V(a + b\mu) = u_H(\mu; a, b)$, as defined in (8).[1]

The CARA-case ($b = 0$) — which is not covered by (12) — can be solved more explicitly. According to (5) or Proposition 1, CARA is characterized by $\alpha_\mu(\mu, \sigma) = 0$. Using the separation-of-variables approach $U(\mu, \sigma) = g(\mu) \cdot h(\sigma)$, we arrange the resulting second-order partial differential equation for U as follows:

$$h(\sigma) \cdot h'(\sigma) \cdot \left[g'(\mu)^2 - g''(\mu)g(\mu)\right] = 0.$$

Except in the uninteresting case where $h(\sigma)$ is a constant, a solution to (13) requires $g(\mu) = c_1 \cdot \exp\{c_2 \cdot \mu\}$. Obeying initial condition (2) in conjunction with (8), we obtain that CARA-preferences in the two-parameter framework take the form:

$$U(\mu, \sigma) = -a \cdot h(\sigma) \cdot \mathrm{e}^{-\mu/a} \tag{13}$$

with $h(0) = 1$. The monotonicity conditions (3) require $h'(\sigma) > 0$ for all $\sigma > 0$. As expected, $\alpha(\mu, \sigma) = a \cdot h'(\sigma)/h(\sigma)$ does not depend on μ.

[1] One easily verifies that (12) indeed satisfies (10) with $\Gamma(x) = -\frac{1}{b} \cdot \frac{W'(x)}{W'(x) - xW(x)}$.

6. Conclusions

The property of linear risk tolerance in the expected-utility framework translates into a condition on the marginal rate of substitution between return and risk in the mean-variance approach. This condition does not only replicate behavioural implications (linear investment strategies) from the EU-approach but also implies certain properties for (μ, σ)-utility functions which might prove helpful in applications of mean-variance analysis.

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