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Heterogeneity in a Class of Two-Player Games

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Abstract

In two-player games with negative (positive) spillovers it is well-known that symmetric agents both overact (underact) at the Nash equilibria. We show that for heterogeneous agents this rule of thumb has to be amended if the game features strategic substitutability.

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1. Introduction

Many economic models are treated as two-player games. One can find examples in such fields as microeconomics (Bulow, Geanakoplos and Klemperer 1985), industrial organisation (Tirole 1992), international economics (Canzonery and Gray 1985, Casella 1992), macroeconomics (Cooper and John 1988) and environmental economics (Hanley and Folmer 1998). These are situations of interest whose transcription in the form of games allows us to predict and analyze the consequences of non-cooperative behaviors.

To underline the importance of externalities among agents the authors often compare the non-cooperative outcomes with the centralized ones. In such comparisons the level of actions matters to characterize inefficiency in term of over or under use/ investment of some observable quantities, and it turns out that both possibilities occur depending on the situation at hand. A similar logic applies when we look for efficiency improvements, such as when the challenge is to overcome pollution through emissions markets (for a game-theoretic analysis of this question, see Helm 2003).

Most of the time the emphasis is put on symmetric agents and in such a case when the game displays negative (positive) spillovers, players overact (underact) at a symmetric Nash equilibrium in comparison with the symmetric cooperative equilibrium. As a result one can expect a central authority to reduce (to increase) the actions of the agents. This prescription avoids for instance the over-exploitation of a common ressource (Hardin 1968) or the under-provision of a collective good (Olson 1965)[1].

The purpose of this note is to show that this conclusion no longer holds when the assumption of symmetric players is relaxed. Collective rationality needs to balance overall welfare improvements with the ability of each individual to contribute to the social goal. This may result in concentrating the effort of improvement on one individual while relaxing the effort on the other because its impact on the global welfare is less important. In the case of environemental problem, heterogeneity between regions may leads to a situation where the national level may consider that one region have to lessen its pressure on natural ressources and other have to increases it. We show that this result has its origins in the strategic nature of the interdependancies between players.

This work is organized as follows. The next section introduces a stylized two-player game. The third section presents our results. The last section is devoted to conclusions and comments.

[1] More recent expositions of these problems can be found in Ostrom (1991) and Moulin (1995).

2. The model

Consider a simple two-player game where the agent 1 takes the action s_1 and the agent 2 takes the action s_2. Their payoff functions U and V depend on both s_1 and s_2:

$$\begin{aligned} U &= U(s_1, s_2)\,, \\ V &= V(s_1, s_2)\,. \end{aligned} \tag{1}$$

The two variables s_1 and s_2 lie in compact and convex feasible sets $S_1 = [\underline{s}_1, \overline{s}_1]$ and $S_2 = [\underline{s}_2, \overline{s}_2]$. The functions U and V are assumed to be increasing and strictly concave in their own argument, twice continuously differentiable and globally concave. We denote U_i, V_i, $i = 1, 2$ the first order partial derivatives of the payoff functions with respect to the variable s_i and U_{ij}, V_{ij}, $i = 1, 2$ the second order partial derivatives of the payoff functions with respect to the variables s_i and s_j. To improve the exposition of the results, following Cooper and John (1988) we introduce four definitions:

[D1]: If $U_2 > 0$ and $V_1 > 0$, the game exhibits positive spillovers.

[D2]: If $U_2 < 0$ and $V_1 < 0$, the game exhibits negative spillovers.

[D3]: If $U_{12} > 0$ and $V_{21} > 0$, the game exhibits strategic complementarity.

[D4]: If $U_{12} < 0$ and $V_{21} < 0$, the game exhibits strategic substitutability.

In the sequel, we will maintain an assumption about payoffs which ensures that the absolute value of the slopes of the reaction functions is less than unity. It is a familiar sufficient condition for the uniqueness of a Nash equilibrium (Friedman 1988), whatever the nature of strategic relationships, complementary or substitutable.

[D5]: $0 < |U_{12}| < |U_{11}|$ and $0 < |V_{21}| < |V_{22}|$.

From our assumptions on $U(.,.)$ and $V(.,.)$ the Nash equilibrium exists (Nash 1951) and under [D5] it is unique.

We will assume that the Nash equilibrium is interior, so that it is given by:

$$N = \{(s_1^N, s_2^N) \in S_1 \times S_2 \;/\; U_1(s_1^N, s_2^N) = 0 \text{ and } V_2(s_1^N, s_2^N) = 0\}\,.$$

Since each agent ignores the effect of his action on the other agent, generally this equilibrium is inefficient. Pareto-optimal outcomes can be obtained with a central decision process that maximizes jointly the weighted sum of the payoff. To underline the role of heterogeneity between the agents we assume that this central authority gives to the agents the same weight in the common decision process. The objective of the central authority is:

$$W = W(s_1, s_2) = U(s_1, s_2) + V(s_1, s_2)\,.$$

Given the assumptions on the payoff functions, W is twice continuously differentiable and globally concave. We will also assume the centralized solution is interior, hence given by:

$$\begin{aligned} C = \{(s_1^C, s_2^C) \in S_1 \times S_2 \;/\; & U_1(s_1^C, s_2^C) + V_1(s_1^C, s_2^C) = 0 \\ & \text{and } U_2(s_1^C, s_2^C) + V_2(s_1^C, s_2^C) = 0\}\,. \end{aligned} \tag{2}$$

We are interested in comparing the levels of action of the Nash equilibrium with those obtained from the centralized problem. The benchmark case is given when the agents have symmetric payoff. In this framework, Cooper and John (1988) have shown that the agents underact at a symmetric Nash equilibrium if there are positive spillovers. Similarly it is easy to see that both agents overact when there are negative spillovers. This result does not depend on specific assumptions about strategic complementarity or substitutability.

3. Results in an asymmetric context

When the agents are heterogeneous we have to compare non symmetric Nash equilibrium with non-symmetric centralized solution. In this context the distinction between games that exhibit negative or positive spillovers is no longer sufficient to characterize the nature of inefficiency. Since we consider that the agents are asymmetric we can expect that the gaps between centralized and decentralized levels of action will be of different magnitudes. We go one step further. We claim that there exists situations where one player overacts whereas the other underacts. To prove our claim, we have to distinguish between games that exhibit strategic complementarity and games that exhibit strategic substitutability. This distinction motivates two propositions[2].

[2]In the following, we will limit our attention to the negative spillovers case. The results for the positive spillovers case (and underaction) can be deduced with the use of similar arguments.

Proposition 1. *Assume [D5]. When the game exhibits* **negative spillovers** *and* **strategic complementarity** *then* **agents overact** *at the unique Nash equilibrium.*

Proof. We use three lemma to facilitate the proof. Under [D2] and [D5] we establish:

- Lemma 1: the agents do not underact simultaneously.
- Lemma 2: the case where agent 1 underacts and agent 2 overacts cannot occur.
- Lemma 3: the case where agent 1 overacts and agent 2 underacts cannot occur.

The conjunction of these claims gives the result.

Lemma 1. *We cannot have:*

$$[s_1^N < s_1^C \textit{ and } s_2^N < s_2^C] \,.$$

Proof. Assume on the contrary that $s_1^N < s_1^C$ and $s_2^N < s_2^C$ and, without loss of generality, define $\epsilon = s_2^C - s_2^N \leq s_1^C - s_1^N$, then under [D5]:

$$0 = U_1(s_1^N, s_2^N) > U_1(s_1^N + \epsilon, s_2^N + \epsilon) = U_1(s_1^N + \epsilon, s_2^C) \;.$$

Indeed, when moving from left to right-hand member of the above inequality, both arguments of the payoff function have changed. Under [D5], the impact of the first argument outweighs that of the second argument.

And because $U(.,.)$ is strictly concave with respect to its first argument:

$$U_1(s_1^N + \epsilon, s_2^N + \epsilon) \geq U_1(s_1^C, s_2^C) = -V_1(s_1^C, s_2^C) > 0 \;.$$

Finally, bringing together all those inequalities:

$$0 = U_1(s_1^N, s_2^N) > U_1(s_1^N + \epsilon, s_2^N + \epsilon) \geq U_1(s_1^C, s_2^C) = -V_1(s_1^C, s_2^C) > 0 \;,$$

leads to a contradiction. □

Lemma 2. *We cannot have:*

$$[s_1^N < s_1^C \text{ and } s_2^N > s_2^C] \,.$$

Proof. Assume on the contrary that $s_1^N < s_1^C$ and $s_2^N > s_2^C$, then:

$$0 = U_1(s_1^N, s_2^N) > U_1(s_1^C, s_2^N) \,,$$

because function $U(.,.)$ is strictly concave with respect to its first argument. Also:

$$U_1(s_1^C, s_2^N) \geq U_1(s_1^C, s_2^C) = -V_1(s_1^C, s_2^C) > 0 \,,$$

under the assumption of strategic complementarity [D3]. Finally, merging all those inequalities

$$0 = U_1(s_1^N, s_2^N) > U_1(s_1^C, s_2^N) \geq U_1(s_1^C, s_2^C) = -V_1(s_1^C, s_2^C) > 0 \,,$$

leads to a contradiction. □

Lemma 3. *We cannot have:*

$$[s_1^N > s_1^C \text{ and } s_2^N < s_2^C] \,.$$

Proof. Assume on the contrary that $s_1^N > s_1^C$ and $s_2^N < s_2^C$, then the assumption of strategic complementarity [D3] implies:

$$0 = V_2(s_1^N, s_2^N) > V_2(s_1^C, s_2^N) \,.$$

And because function $V(.,.)$ is strictly concave with respect to its second argument:

$$V_2(s_1^C, s_2^N) \geq V_2(s_1^C, s_2^C) = -U_1(s_1^C, s_2^C) > 0 \,.$$

Finally, using all the above inequalities:

$$0 = V_2(s_1^N, s_2^N) > V_2(s_1^C, s_2^N) \geq V_2(s_1^C, s_2^C) = -U_1(s_1^C, s_2^C) > 0 \,,$$

leads to a contradiction. □

Lemma 1 to 3 imply the proposition. □

Proposition 2. *Assume [D5]. When the game exhibits* **negative spillovers** *and* **strategic substitutability** *then:*

- *(i) at least* **one agent overacts** *at the Nash equilibrium.*
- *(ii) in some cases* **the other agent underacts**.

Proof. (i) Let us begin with the first part of the proposition. Under [D2] and [D4] we have to prove that agents do not underact simultaneously, i.e. we cannot have:

$$[s_1^N < s_1^C \text{ and } s_2^N < s_2^C]\,.$$

Assume on the contrary that $s_1^N < s_1^C$ and $s_2^N < s_2^C$, then:

$$0 = U_1(s_1^N, s_2^N) > U_1(s_1^C, s_2^C) = -V_1(s_1^C, s_2^C) > 0\,.$$

Contradiction.

(ii) In order to establish the second part of the proposition we will show that we can set up an example where $[s_1^N > s_1^C$ and $s_2^N < s_2^C]$. Take a simple Cournot duopoly model with linear inverse demand function $p = a - (q_1 + q_2)$ and differentiated marginal costs. The payoffs are:

$$\begin{aligned} U(q_1, q_2) &= (a - q_1 - q_2)q_1 - \frac{1}{2}c_1\, q_1^2\,, \\ V(q_1, q_2) &= (a - q_1 - q_2)q_2 - \frac{1}{2}c_2\, q_2^2\,. \end{aligned} \tag{3}$$

It is easy to check that those payoffs satisfy Assumption [D5]. This is basically a situation with negative spillovers and strategic substitutability. The quantities at the Nash equilibrium are:

$$q_1^N = \frac{(1 + c_2)a}{(2 + c_1)(2 + c_2) - 1}\,, \qquad q_2^N = \frac{(1 + c_1)a}{(2 + c_1)(2 + c_2) - 1}\,.$$

The centralized solution (i.e. the joint monopoly point) gives:

$$q_1^C = \frac{c_2 a}{(2 + c_1)(2 + c_2) - 4}\,, \qquad q_2^C = \frac{c_1 a}{(2 + c_1)(2 + c_2) - 4}\,.$$

Our claim is that for possible values of the parameters one duopolist underacts whereas his rival overacts. Consider the following hypothesis on the marginal costs parameters:

$$c_2 < \frac{c_1}{(c_1 + 2)} < c_1\,.$$

From the expressions of q_2^C and q_2^N we have that:

$$\text{Sign } [q_2^C - q_2^N] = \text{Sign } [\frac{c_1}{(c_1+2)} - c_2] \, .$$

so that $q_2^C > q_2^N$. Besides, from the part (i) we get necessarily that $q_1^C < q_2^N$. The comparisons of the expressions q_1^C and q_1^N with the assumptions on the marginal cost parameters would have led us to the same result. □

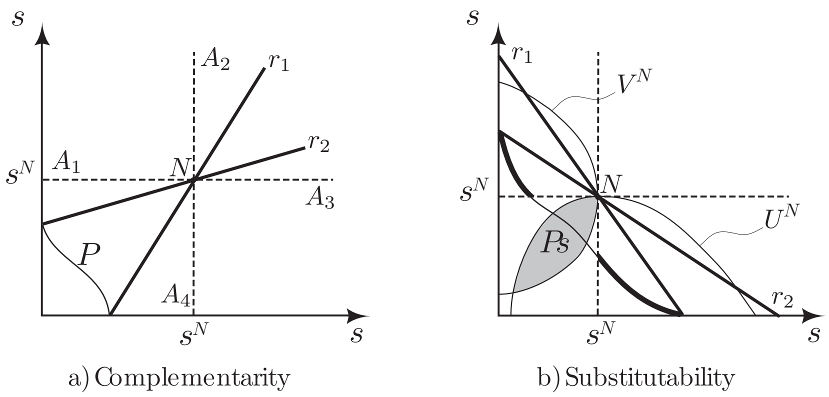

Figure 1: Asymmetric case with negative spillovers

Propositions 1 and 2 are illustrated in figure 1. The Nash equilibrium N is at the intersection of the best reply functions r_1 and r_2. For commodity of exposition we draw linear best reply functions. The curves U^N and V^N are the loci of the plane that give the same payoffs as in the Nash equilibrium. The results can be described by noting that the contract curve P is the set that contains all the possible Pareto-optimal outcomes of the game. The form of the contract curve depend on the specific functional form we adopt for $U(.,.)$ and $V(.,.)$. We have $C \subset P$, but its position on P depends on the asymmetries between the payoff functions. If the payoffs are symmetric then C is located at the intersection between P and the 45 degrees line. The lines $s_2 = s_2^N$ and $s_1 = s_1^N$ divide the (s_1, s_2) space into quarters A_i, $i = 1,..4$. In the quarter A_4 both players have lower level of action than in the Nash equilibrium: if C is in A_4 then both players overact. Proposition 1 says that $C \subset A_4$ if there are negative spillovers and limited strategic complementarity. This result is not surprising because we have $P \subset A_4$. Proposition 2 deals with a situation where P lies for a part in the quarter A_4 but also have non empty intersections with quarters A_1 and A_3. These intersections are depicted by the bold segments of P in figure (1b). If $C \subset (P \cap A_1)$ or $C \subset (P \cap A_3)$, then the central decision process decreases the level of action of one agent while increasing

the level of action of the other. This situations seems to occur when the degree of heterogeneity between agents is sufficiently high, as we can see in the cost-differentiated Cournot duopoly example. Differences are such that the central authority increases the level of action of the more productive agent and decreases the level of the other because it induces a higher aggregate payment.
We can see in Figure 1b that case (ii) in Proposition 2 implies that the centralized solution does not Pareto dominates the Nash equilibrium. This result can be explained by noting that the definition of a Nash equilibrium and the concavity assumptions on the payoffs imply that the set of outcomes that Pareto dominate the Nash equilibrium noted Ps are such that $Ps \subset A_4$.

We can remark that imposing individual rationality constraints leads to centralized solutions where both players overact. Extending the result of Cooper and John (1988) for two players asymmetric games leads to modifications in the common decision process.

4. Conclusions

In a class of two-player games we have stressed on the role of heterogeneity between agents when comparing Nash equilibria with centralized outcomes. It turns out that under the assumption of limited strategic complementarity, the heterogeneity does not change the qualitative results already known for the symmetric agents case. On the contrary, when the game features strategic substitutability the noncooperative actions of the players could lie in opposite sides of their respective centralized actions. Thus, only one agent over-exploits the common ressource or under-subscribes to a public good. This phenomenon seems more likely to occur as differences between agents increase, and implies that the centralized solution violates individual rationality (see Wilson 1991, and Hwang and Choe 1995, for examples in the fiscal competition literature with asymmetric jurisdictions). If the central authority takes care about individual rationality constraints it has to give more weight to some players in the common decision process. This phenomenon is of particular interest for decision process in institutions that gather different countries like EU (Casella 1992).

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