Economics Bulletin

Volume 31, Issue 1

A solution concept for housing market problems with externalities

Alfredo Salgado-Torres
Universidad Carlos III de Madrid

Abstract

Since the core of a housing market with externalities may be empty, we propose as a solution concept the set of allocations that satisfy two basic properties: Pareto efficiency and the No-regret condition. Our main result shows that for any instance of the housing market problem, there always exists at least one allocation that satisfies both properties.

The author is grateful to an anonymous referee and the editor of the Economics Bulletin for helpful comments. The comments from Cecilia Vives Coscojuela and the FPI grant of the Spanish Ministry of Science and Innovation are gratefully acknowledged.
Citation: Alfredo Salgado-Torres, (2011) "A solution concept for housing market problems with externalities", *Economics Bulletin*, Vol. 31 no.1 pp. 623-630.
Submitted: Dec 24 2010. **Published:** February 17, 2011.

1 Introduction

An allocation problem consists in assigning a set of objects to a set of agents, that is the case of two-sided matching, housing market problems, etc. In standard allocation problems, agents have complete, transitive and strict preferences over the set of objects and it is assumed that agents' valuations are independent, i.e. there are no externalities in the problem.

It is not difficult to find applications of the allocation problem where the absence of externalities seems unreasonable. For instance, in the case of professional sports leagues, where clubs compete to win a tournament, it is very reasonable to think that the chance of winning a league may depend on the complete matching between sportsmen and clubs. Another example regards housing markets, a house may be more valuable if very important people live in the same neighborhood (the case of Bervelly hills), this implies that the value of a house may depend on who live next door.

Recently, important contributions have emerged in the study of cooperative games with externalities. Clippel and Serrano (2008) and Macho-Stadler, et. al. (2007) characterize the *Shapley Value* in the presence of external effects; Hafalir (2009) and Sasaki and Toda (1996) analyze the stability of the matching in the *marriage problem* with externalities; Salgado-Torres (2010) analyzes the existence of stable matchings in many to one matching problems with externalities; Hafalir (2007) analyzes the problem of efficiency in *coalition form games* with externalities; Mumcu and Saglam (2007) analyze the existence of the **core** in housing market problems with externalities.

One of the main problems in cooperative setups with externalities is the existence of equilibrium allocations. Following that line, this note deals with the existence of a reasonable solution concept for housing market problems with externalities.

It is well known that the **core** of any standard housing market problem is a singleton and always exists. In addition, it is always possible to find the **core** of any housing market by means of the Top Trading Cycle Algorithm (TTC) (Roth and Postlewaite, 1977; Roth, 1982). In contrast, when we consider the presence of externalities, the **core** may contain more than one allocation and it may be empty for some instances of the housing market (Mumcu and Saglam, 2007).

Given this problem, there are two possible approaches. Either, it is possible to seek reasonable constrains over the preferences which guarantee the existence of the **core** or we could find an alternative solution concept that satisfies a set of minimum reasonable properties. In the first case, it may be impossible to show how large the constrained domain of preferences is, hence it is possible that the **core** only exists in a very small domain of preferences. In the second approach, the solution concept is weakened trying to keep as many desirable properties as possible. However, in this case we would be able to ensure that the solution concept is general enough, since it would be applicable to any instance of the problem. Since the **core** is a strong solution concept and we want to maintain the general conditions of the problem, in this note we follow the second approach.

First, we analyze the **bargaining set** already defined in the one to one matching problem (Klijn and Massó, 2003). Even when this solution concept is weaker than the **core**, we show that the **bargaining set** may be empty for some instances of the housing market with externalities.

After that, we consider an alternative solution concept that takes into account two basic properties of the TTC. First of all, any allocation attained by the TTC is **Pareto**

efficient, this is a minimum desirable property of a solution concept. Secondly, one of the main properties of the TTC is that exchange has to be profitable, i.e. no agent **regrets** having exchanged his initial allocation at the end of the algorithm.

Pareto efficiency and the **No-regret condition** are properties that a reasonable solution concept has to satisfy. The first property implies that the economy is in the frontier of consumption possibilities. The second one is a minimum requirement of non-myopic behavior of agents. We show that any allocation in the **core** always satisfies both properties.

Our main result shows that for any housing market problem with externalities, there always exists at least one allocation that satisfies both properties. Hence, the solution that associates to any problem the set of allocations that satisfy **Pareto efficiency** and the **No-regret condition** is always nonempty.

The rest of the paper is organized as follows. In section 2, we introduce the model; in section 3, we present our results; in section 4, we conclude.

2 The model

Let $A = \{a_1, ..., a_n\}$ be a finite set of agents and let $H = \{h_1, ..., h_m\}$ be a finite set of houses and assume that $|A| = |H| \geq 2$. An allocation ν is a bijection from A to H. Let $\mathcal{M}$ be the set of all possible allocations given the sets of agents and houses. We denote an allocation by a vector of n entries $\nu = (h, h', ..., h'')$ such that, $\nu(a_1) = h$, $\nu(a_2) = h'$,..., $\nu(a_n) = h''$.

Each agent $a \in A$ has a transitive, strict and complete preference relation P_a over the set of allocations $\mathcal{M}$. For each $a \in A$, let R_a denote the weak preference relation associated to P_a, so for all $v, v' \in \mathcal{M}$, $vR_a v'$ means either $vP_a v'$ or $v = v'$. Note that no agent is indifferent between two different allocations, even when these allocations assign the same house to that agent. This implies that the value of a house depends on the complete allocation. A housing market with externalities or simply a market is a four-tuple (A, H, P, μ), where $P = \{P_a\}_{a \in A}$ is the profile of preferences and $\mu \in \mathcal{M}$ is an exogenous initial endowment. In what follows, we fix the sets of agents and houses A and H, hence a market is completely described by a preference profile and an initial endowment, i.e. a tuple (P, μ).

With some abuse of notation, P_a denotes a preference list from the best to the worst possible allocation for each $a \in A$. For instance, $P_a = \nu, \nu', ..., \nu''$ means that ν and ν'' are, respectively, the most and the least preferred allocations for a, $\nu P_a \nu'$ and so on. Let $\mu(A') = \{h \in H : \mu(a) = h \text{ some } a \in A'\}$ denote the set of houses initially assigned to agents $A' \subset A$.

In the standard housing market problem, agents have rational and strict preferences over the set of houses and it is assumed that preferences are independent each other. This means that each agent is able to order houses from the best one to the worst one, and this ranking is independent of the complete allocation between agents and houses.

In the presence of externalities, preferences are interdependent, i.e. there is no a unique way to order the set of houses, since agents' values depend on the complete allocation. For instance, for some agents, a house may be the most preferred if their parents live next door, but the worst one if their neighbors are unfriendly and unknown people.

Example 1 *Consider a 3x3 housing market with* $A = \{a_1, a_2, a_3\}$, $H = \{h_1, h_2, h_3\}$, *the set of all feasible allocations is,*

$\nu_1 = (h_1, h_2, h_3)$;
$\nu_2 = (h_1, h_3, h_2)$;
$\nu_3 = (h_2, h_1, h_3)$;
$\nu_4 = (h_2, h_3, h_1)$;
$\nu_5 = (h_3, h_1, h_2)$;
$\nu_6 = (h_3, h_2, h_1)$.

Consider the next preferences:

$P_{a_1} = \nu_2, \nu_6, \nu_3, \nu_4, \nu_5, \nu_1$.
$P_{a_2} = \nu_2, \nu_3, \nu_4, \nu_1, \nu_6, \nu_5$.
$P_{a_3} = \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6$.

The initial endowment is $\mu = \nu_5$.

Clearly, under ν_2 *the agent* a_1 *considers that* h_1 *is his top choice. However, under* ν_1 *where* a_2 *and* a_3 *exchange their houses,* a_1 *considers that* h_1 *is his last choice. Hence, the* a_1*'s value of* h_1 *depends on the complete allocation.*

The usual solution concept in standard housing market problems is the **core,** which is a single value correspondence that always exists (Roth and Postlewaite, 1977; Roth, 1982). It is possible to extend this notion of equilibrium to environments with externalities. Formally,

Definition 1 *Given any housing market* (P, μ), *a coalition* $A' \subset A$ *blocks the allocation* ν *via another allocation* $\nu' \neq \nu$ *if:*

1. $\nu'(a) \neq \nu(a)$ *for all* $a \in A'$;
2. $\nu'(a) \in \mu(A')$ *for all* $a \in A'$; *and*
3. $\nu' R_a \nu$ *for all* $a \in A'$ *and* $\nu' P_a \nu$ *for some* $a \in A'$.

An allocation ν is in the **core** of a market (P, μ), if it is not blocked by any coalition via another allocation. Let $\mathcal{C}(P, \mu)$ denote the **core** of a market (P, μ). Unlike the standard problem, in the presence of externalities the **core** of a housing market may be empty and it may not be a singleton (Mumcu and Saglam, 2007).

3 Results

In the presence of external effects, the **core** of housing markets is a too demanding solution concept, since it requires the existence of an allocation not blocked by any coalition. We relax this solution and we consider the notion of the **bargaining set**, already applied in one to one matching problems (Klijn and Massó, 2003). The notion of the **bargaining set** only considers credible deviations. In order to define this solution concept in our setup, we require some additional definitions.

Definition 2 *Given a housing market* (P, μ), *a coalition* $A' \subset A$ *is able to enforce an allocation* ν *if, for all* $a \in A'$, *if* $\nu(a) \neq \mu(a)$ *implies that* $\nu(a) \in \mu(A' \setminus \{a\})$.

Note that the previous definition always takes into account the initial endowment of the market and it is independent of the preference profile. Further any agent is always able to enforce the initial endowment for any market (P, μ).

Definition 3 *An objection against an allocation ν is a pair (A', ν') such that $\nu' \in \mathcal{M}$ can be enforced by $A' \subset A$, and $\nu' P_a \nu$ for all $a \in A'$.*

Definition 4 *A counter-objection against an objection (A', ν'), is a pair (T, ν'') where $\nu'' \in \mathcal{M}$ can be enforced by $T \subset A$ and it is satisfied:*

1. *$T \diagdown A' \neq \phi$, $A' \diagdown T \neq \phi$ and $T \cap A' \neq \phi$; and*
2. *$\nu'' P_a \nu$ for all $a \in T \diagdown A'$ and $\nu'' P_a \nu'$ for all $a \in T \cap A'$.*

If there is a counter-objection against an objection (A', ν'), implies that the objection of agents $T \cap A'$ is not credible since $\nu'' P_a \nu'$ for all $a \in T \cap A'$. Further, no agent in $T \diagdown A'$ suffers when T enforces ν'', since $\nu'' P_a \nu$ for all $a \in T \diagdown A'$.

An objection (A', ν') against an allocation ν is justified if there is no counter-objection against (A', ν'), i.e. only credible objections are justified. The **bargaining set** is the set of allocations that have no justified objections. Let $B(P, \mu)$ denote the **bargaining set** of a market (P, μ). Note that an objection made by either one agent or the grand coalition is always justified.

It is possible to show that the **bargaining set** may be empty for some instances of problem, as we show in the next result.

Proposition 1 *There exists a housing market with externalities (P, μ), such that $B(P, \mu)$ is empty.*

Proof. Consider a 3x3 housing market with $A = \{a_1, a_2, a_3\}$ and $H = \{h_1, h_2, h_3\}$. Agents' preferences are,

$P_{a_1} = \nu_6, \nu_3, \nu_2, \nu_1, \nu_4, \nu_5$.
$P_{a_2} = \nu_3, \nu_2, \nu_6, \nu_1, \nu_4, \nu_5$.
$P_{a_3} = \nu_2, \nu_1, \nu_6, \nu_3, \nu_4, \nu_5$.

The initial endowment is $\mu = \nu_1$. We show that there is a justified objection for each allocation:

- ν_1 has the justified objection $(\{a_1, a_2, a_3\}, \nu_2)$;
- ν_2 has the justified objection $(\{a_1, a_2\}, \nu_3)$, since only $(\{a_1, a_3\}, \nu_6)$ may form a counter-objection but $\nu_2 P_{a_3} \nu_6$ with $\{a_3\} = \{a_1, a_3\} \diagdown \{a_1, a_2\}$;
- ν_3 has the justified objection $(\{a_1, a_3\}, \nu_6)$, since only $(\{a_2, a_3\}, \nu_2)$ may form a counter-objection but $\nu_3 P_{a_2} \nu_2$ with $\{a_2\} = \{a_2, a_3\} \diagdown \{a_1, a_1\}$;
- ν_4 has the justified objection $(\{a_1, a_2, a_3\}, \nu_6)$;
- ν_5 has the justified objection $(\{a_1, a_2, a_3\}, \nu_2)$; and
- ν_6 has the justified objection $(\{a_2, a_3\}, \nu_2)$, since only $(\{a_1, a_3\}, \nu_3)$ may form a counter-objection but $\nu_6 P_{a_1} \nu_3$ with $\{a_1\} = \{a_1, a_3\} \diagdown \{a_2, a_3\}$.

Then the set of allocations $B(P, \mu)$ is empty. ■

Since neither the **core** nor the **bargaining set** always exist, we relax these solution concepts by taking into account two basic properties of the Top Trading Cycle Algorithm (TTC). First of all, any allocation attained by the TTC is **Pareto efficient**, which is a very desirable property since it implies that it is not possible to improve one agent without harming another one. Formally,

Definition 5 *Given any market* (P, μ) *an allocation* ν *dominates another one* ν', *if* $\nu R_a \nu'$ *for all* $a \in A$ *and* $\nu P_a \nu'$ *for some* a.

An allocation ν is **Pareto efficient** if it is not dominated by any other allocation. Let $\mathcal{PE}(P, \mu)$ denote the set of **Pareto efficient** allocations.

Secondly, at each step of the TTC agents point to the owner of his most favorite house and agents (in a cycle) exchange their initial endowments, i.e. exchange is profitable. At the final step of the TTC no agent regrets having exchanged his initial endowment. Hence, a second natural property of any allocation in the solution of the problem is the **No-regret condition**. Formally,

Definition 6 *Given any market* (P, μ), *an allocation* ν *satisfies the* ***No-regret condition*** *if* $\nu' R_a \mu$ *for all* $a \in A$ *such that* $\nu'(a) \neq \mu(a)$.

Let $\mathcal{NR}(P, \mu)$ denote the set of allocations that satisfy the **No-regret condition** given a market (P, μ). Define $S(P, \mu) = \mathcal{PE}(P, \mu) \cap \mathcal{NR}(P, \mu)$.

In the next result, we show that any allocation in the **core** of a housing market with externalities satisfies both properties.

Claim 1 *Let* (P, μ) *be any instance of the housing market with externalities, then* $\mathcal{C}(P, \mu) \subset S(P, \mu)$.

Proof. If $\mathcal{C}(P, \mu) = \phi$ the proposition is trivially satisfied, hence assume that the core of the market is not empty. Assume that there is some allocation $\nu \in \mathcal{C}(P, \mu)$ but $\nu \notin \mathcal{PE}(P, \mu) \cap \mathcal{NR}(P, \mu)$, by definition ν is a Pareto efficient allocation. Hence, by assumption $\nu \notin \mathcal{NR}(P, \mu)$. This implies that there exists an agent $a \in A$ such that $\nu(a) \neq \mu(a)$ and $\mu P_a \nu$. Set $S = \{a\}$ and $\nu' = \mu$, then ν is blocked by the coalition $\{a\}$ via ν', a contradiction. ■

Example 2 *Consider a 3x3 housing market with* $A = \{a_1, a_2, a_3\}$, $H = \{h_1, h_2, h_3\}$ *and the next preferences:*

$P_{a_1} = \nu_2, \nu_1, \nu_3, \nu_4, \nu_5, \nu_6.$

$P_{a_2} = \nu_2, \nu_3, \nu_4, \nu_1, \nu_6, \nu_5.$

$P_{a_3} = \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6.$

The initial endowment is $\mu = \nu_4$.

It is possible to show that, $\mathcal{PE}(P, \mu) = \{\nu_1, \nu_2\}$ *and* $\mathcal{NR}(P, \mu) = \{\nu_2, \nu_3, \nu_4\}$, *whereas* $S(P, \mu) = \mathcal{C}(P, \mu) = \{\nu_2\}$.

In Example 2, we show that neither $\mathcal{PE}(P, \mu) \nsubseteq \mathcal{NR}(P, \mu)$ nor $\mathcal{NR}(P, \mu) \nsubseteq \mathcal{PE}(P, \mu)$. In the next result, we show that for any instance of the problem it is always possible to find at least one allocation that satisfies both **Pareto efficiency** and the **No-regret condition**.

Proposition 2 *For any instance of the housing market with externalities* (P, μ)*, the set of allocation* $S(P, \mu)$ *is always nonempty.*

Proof. Let (P, μ) be any housing market, if μ is Pareto efficient the proposition is trivially satisfied, since $\mu \in \mathcal{NR}(P, \mu)$ by definition.

Assume that μ is not Pareto efficient, hence there exists an allocation $\nu \neq \mu$ that dominates μ. If ν is Pareto efficient the proposition is proven, since $\nu \in \mathcal{NR}(P, \mu)$ given μ, otherwise there exist another $\nu' \neq \nu$ that dominates ν, and so on. Define the sequence of allocations $\{\mu, \nu, \nu', ...\}$, such that ν dominates μ, ν' dominates to ν, and so on.

Let $\succeq$ denote a partial order over the set $\mathcal{M}$, such that $\nu \succeq \nu'$ if and only if $\nu R_a \nu'$ for all $a \in A$. Under the order $\succeq$, the sequence $\{\mu, \nu, \nu' ...\}$ is increasing in $\mathcal{M}$. Hence, $\{\mu, \nu, \nu' ...\}$ has to converge to a Pareto efficient allocation ν^*, since $\mathcal{M}$ is a finite set. By construction $\nu^* \neq \mu$ and $\nu^* R_a \mu$ for all $a \in A$, hence $\nu^* \in \mathcal{NR}(P, \mu)$. This completes the proof. ■

The previous result has two main implications. First of all, the converse of Claim 1 does not hold. Secondly, unlike the **core** the set of allocations that satisfy **Pareto efficiency** and the **No-regret condition** is always nonempty. Hence, we are able to ensure that the solution that associates the set of allocations $S(P, \mu)$ to any housing market problem with externalities (P, μ) is always nonempty.

4 Conclusions

Mumcu and Saglam (2007) show that the **core** of a housing market with externalities may not exist. We show that a weaker solution concept, i.e. the **bargaining set** has the same problem, it is possible to find at least one instance of the problem where the **bargaining set** is empty.

We propose as a solution concept the set of allocations that satisfy two desirable properties: **Pareto efficiency** and the **No-regret condition**. The first property does not need to be justified, this is a minimum requirement for any economic problem. The second one implies that agents are not myopic, in the sense that no agent regrets having exchanged his initial endowment. We argue that both properties are desirable for a reasonable solution concept for housing markets in the presence of externalities. Our main result shows that for any instance of the problem, there always exists at least one allocation that satisfies **Pareto efficiency** and the **No-regret condition**.

References

[1] Abdülkadiroglu, A. and Sömez, T. (1998) "Random serial dictatorship and the core from random endowments in house allocation problems" Econometrica **66**, 689-701.

[2] De Clippel, G. and Serrano, R. (2008) "Marginal contributions and externalities in the value" Econometrica **76**, 1413-1436.

[3] Gale, D. and Shapley, L. (1962) "College admission and the stability of marriage" American Mathematical Monthly **69,** 9-15.

[4] Macho-Stadler, I., Pérez-Castrillo, D. and Wettstein, D. (2007) "Sharing the surplus: An extension of the Shapley value for environments with externalities" Journal of Economic Theory **135**, 339-356.

[5] Hafalir, I. (2008) "Stability of marriage with externalities" International Journal of Game Theory **37**, 353-369.

—— (2007) "Efficiency in coalitional games with externalities" Games and Economic Behavior **61**, 242-258.

[6] Klijn, F. and Massó, J. (2003) "Weak stability and the bargaining set for the marriage model" Games and Economic Behavior **42**, 91-100.

[7] Mumcu, Ayse and Saglam, Ismail (2006) "One to one matching with interdependent preferences" MPRA Paper No. **1908**, 1-28.

—— (2007) "The core of housing market with externalities" Economics Bulletin vol. 3 No. **55**, 1-5.

[8] Roth, A. (1982) "Incentive compatibility in a market with indivisible goods" Economic Letters **9**, 127-132.

[9] Roth, A. and Postlewaite, A. (1977) "Weak versus strong domination in a market with indivisible goods" Journal of Mathematical Economics **4**, 131-137.

[10] Salgado-Torres, Alfredo (2011) "Many to one matching: Externalities and stability" Mimeo Universidad Carlos III de Madrid, 1-30.

[11] Sasaki, H. and Toda, M. (1996) "Two-sided matching problems with externalities" Journal of Economic Theory **70**, 93-108.

[12] Shapley, L. and Shubik, M. (1969) "On the core of an economic system with externalities" The American Economic Review **59**, 678-684.