Economics Bulletin

Volume 32, Issue 2

Revenue Equivalence in Sequential Auctions

Srobonti Chattopadhyay
Jawaharlal Nehru University

Rittwik Chatterjee
National Institute of Science, Technology and Development Studies

Abstract

The revenue equivalence theorem is an widely known result in Auction Theory. This note generalize that theorem for the case of Sequential Auctions. Our results show that under a class of Sequential Auction, if an symmetric and increasing equilibrium bidding strategy exists, then the revenue equivalence still holds for that class of Sequential Auctions.

We are thankful to Krishnendu Ghosh Dastidar, for some useful suggestions.
Citation: Srobonti Chattopadhyay and Rittwik Chatterjee, (2012) "Revenue Equivalence in Sequential Auctions", *Economics Bulletin*, Vol. 32 No. 2 pp. 1272-1281.
Contact: Srobonti Chattopadhyay - srobonti@gmail.com, Rittwik Chatterjee - rittwik@gmail.com.
Submitted: July 14, 2011. **Published:** April 24, 2012.

1 Introduction

This note compares the expected revenue of a seller for a class of sequential auctions. The results show that the existence of a symmetric, increasing equilibrium bidding strategy will ensure revenue equivalence for that class of sequential auctions. We will extend the famous revenue equivalence theorem by Riley and Samuelson (1981)[6] for single-unit (single-stage) standard auction to a multi-unit sequential auction (where one unit is sold at each stage and there is single-unit demand by each bidder) and state that with all other assumptions remaining the same, as long as the expected payment of a bidder with value zero is zero, any symmetric and increasing equilibrium of any standard sequential auction yields the same expected revenue to the seller.

Milgrom and Weber (1982)[4] consider a single object auction and show that for a given particular auction mechanism, under the assumption of independent private values, the bidders adopt strategies which constitute a noncooperative equilibrium. If at that equilibrium, the bidder who values the object most highly is certain to receive it and any bidder whose valuation for the object is the lowest, has a zero expected payment, then the expected revenue generated for the seller by the mechanism is precisely the expected value of the object to the second highest evaluator. Therefore, they establish, that at the symmetric equilibria of the English, Dutch, first price and second price auctions, the expected selling price is the same[1].

Weber (1983)[7] considers sequential auctions involving multiple objects and multiple risk-neutral bidders, where the number of objects are less than the number of bidders. He considers single-unit demand for every bidder and establishes that under the symmetric, independent, private valuations assumptions, the sequential first and second price sealed bid auctions, for which only one object is sold at each stage and the winner from every stage exits before the next stage, yield the same expected revenue to the seller.

Engelbrecht-Wiggans (1988)[1] generalizes the results of Milgrom and Weber (1982)[4] for single object auctions to the context of multiple object auctions. He assumes multiple unit demand for the bidders in a symmetric, independent, private valuations framework, where each bidder is risk-neutral. For a single stage multiple unit auction, he establishes that, for sufficiently regular allocation functions and distribution function (of valuation vectors) of the concatenation of other bidders, if the expected amount paid by a bidder i at equilibrium is to be continuous in his/her own value vectors, then the equilibrium expected payment of that bidder as a function of his/her true valuation depends only on the allocation function, the distribution of others' values and the value of expected payment as a function of some fixed x_0 (where x denotes the valuation vector for the bidders).

Klemperer (2004)[2] also establishes revenue equivalence for a general class of single stage multiple unit auctions. He considers a situation where each of n risk-neutral potential buyers has a privately known value independently drawn from a common distribution $F(v)$ that is strictly increasing and atomless on $[\underline{v}, \overline{v}]$ and no buyer wants more than one of the k available identical indivisible objects. For such a framework, he suggests that, any auction mechanism in which (i) the objects always go to the k buyers with the highest values, and (ii) any bidder with value $\underline{v}$ expects zero surplus,

[1] In their paper, Milgrom and Weber (1982)[4], also analyze auctions involving common values and affiliated values. Since our paper strictly deals with symmetric independent private values framework, the discussion on the analysis of common and affiliated values auctions, presented by Milgrom and Weber (1982)[4] is forgone here.

yields the same expected revenue, and results in a buyer with value v making the same expected payment.

The results in this note, as stated above, deal with sequential auctions with single-unit demand. In contrast to Weber (1983)[7] we consider here a more general framework, which includes first-price and second-price sealed bid sequential auctions as special cases. It is more of a generalization of the result given by Weber (1983), since it establishes revenue equivalence for a broader class of auctions. Our study also differs from Milgrom and Weber (1982)[4], Engelbrecht-Wiggans (1988)[1] and Klemperer (2004)[2] because all of them study single stage multi-object auctions and our analysis is based on sequential multi-object auctions.

2 Structure of the Model

- A seller possesses $K > 1$ units of a homogeneous commodity.
- There are N buyers and $N > K$.
- Each buyer has demand for one unit of the commodity.
- The objective of the seller is to ensure efficient allocation of this commodity in the sense that the units of the homogeneous commodity should end up going to those who value it the most. This process therefore amounts to a multi-unit auction with single-unit demand. Here the buyers are the bidders.
- We assume the symmetric, independent, private values (or SIPV) framework.
- The private value vector for bidder i in this case can be written as $X^i = (X_1^i, 0, ..., 0)$ (since bidder i has a non-negative valuation for the first unit and from the second unit onwards to every unit, bidder i definitely attaches a value zero). For simplicity's sake we denote X_1^i as X_i.
- Each X_i is independently and identically distributed on some interval $[0, \omega]$. According to the increasing distribution function F. We have assumed that F has a continuous density and $f \equiv F^/$ and has full support.
- Bidder i knows the realization x_i of X_i only and also knows that the valuations of the other bidders are independently distributed according to F.
- Bidders are risk neutral - they seek to maximize their expected profits.
- All components of the model other than the realized values are assumed to be commonly known to all bidders. In particular, the distribution F is common knowledge, as is the number of bidders. And after each period the highest bid of that period also becomes common knowledge to the rest of the bidders.
- It is also assumed that bidders are not subject to any liquidity or budget constraints.

- In this game the strategy for each bidder is his/her bid. This means to say that every bidder can choose to report a bid in order to maximize his/her expected payoff. The bidding strategy for a bidder is a function $\beta_k : [0, \omega] \to R_+$ which determines his or her bid for any value[2]. $\beta_k(.)$ is assumed to be an increasing function of valuation and differentiable in its domain, for all k.
- It is assumed that the seller has a valuation 0 for each of the objects.
- The seller designs multi-stage sequential auctions to sell her commodities, where at each stage the highest bidder of the previous stage gets eliminated. In each period only one unit of a particular homogeneous, indivisible commodity is offered for sale through standard auction.
- Finally we assume that β constitutes a symmetric and increasing equilibrium of the auction.

3 Revenue Equivalence in Sequential Standard Auctions

First we consider a case where only two units are sold ($K = 2$), so that a symmetric equilibrium consists of two functions (β_1, β_2), denoting the equilibrium bidding strategies in the first and second periods, respectively. The first-period bidding strategy is a function $\beta_1 : [0, \omega] \to R_+$ that depends only on the bidder's value. The bid in the second period depends on both the bidder's value and the valuation of the winning bidder of the first auction. We denote by Y_1^{N-1} the highest of $(N-1)$ values (the highest order statistic), by $Y_2^{N-1}$the second highest, and so on and similarly by $Y_1^{N-2}$the highest of the $(N-2)$ values, by $Y_2^{N-2}$the second highest and so on. Let $G_j^{[1]}$ and $G_j^{[2]}$ be the distributions of the j^{th} highest order statistics in the first and the second periods respectively.

Since the first period bidding strategy $\beta_1(.)$ is assumed to be invertible, the value of the winning bidders in the first period is commonly known in the second period; it is just $y_1 = \beta_1^{-1}(p_1)$, where p_1 is the price paid by the winning bidder in the first stage. Thus the second period strategy can be thought of as a function $\beta_2 : [0, \omega] \times [0, \omega] \to R_+$, so that a bidder with value x bids an amount $\beta_2(x, y_1)$ if $Y_1^{N-1} = y_1$. We are interested in equilibria that are sequentially rational - that is, equilibria with the property that following any outcome of the first-period auction, the strategies in the second period form an equilibrium.

We begin with the second period, considering the decision problem facing a particular bidder, say i whose value is x. Let us suppose that all other bidders follow the equilibrium strategy $\beta_2(., y_1)$in the second stage of the auction. Since the bidders competing against bidder i in the second auction have values $Y_2^{N-1}, Y_3^{N-1}, ..., Y_{N-1}^{N-1}$ and in equilibrium, $Y_1^{N-2} < y_1$, it makes no sense for bidder i to bid an amount greater than $\beta_2(y_1, y_1)$. His expected payoff in the second auction if he bids $\beta_2(z, y_1)$for some $z \leq y_1$can be written as:

$$\begin{aligned}
\Pi_2(z, x, y_1) &= \left(x - M_1^{[2]}(\beta_2(z))\right) \tfrac{G_1^{[2]}(z)}{F(y_1)^{N-2}} - \sum_{j=2}^{N-1} M_j^{[2]}(\beta_2(z)) \tfrac{G_j^{[2]}(z)}{F(y_1)^{N-2}} \\
&= x \tfrac{G_1^{[2]}(z)}{F(y_1)^{N-2}} - \sum_{j=1}^{N-1} M_j^{[2]}(\beta_2(z)) \tfrac{G_j^{[2]}(z)}{F(y_1)^{N-2}}
\end{aligned}$$

[2] k denotes the stage.

where, $G_j^{[2]}(z) = {}^{N-2}C_{j-1} F(z)^{(N-2)-(j-1)} (1-F(z))^{j-1}$ denotes the probability distribution of the j^{th} highest order statistic for the $(N-2)$ bidders other than bidder i in the second period, with the corresponding density being

$$\begin{aligned} g_j^{[2]}(z) &= {}^{N-2}C_{j-1} \Big[\{(N-2)-(j-1)\} F(z)^{(N-2)-(j-1)-1} (1-F(z))^{j-1} f(z) \\ &\qquad - (j-1) F(z)^{(N-2)-(j-1)} (1-F(z))^{j-2} f(z) \Big] \end{aligned}$$

and $\frac{G_j^{[2]}(z)}{F(y_1)^{N-1}}$ denotes the conditional probability that bidder i is the j^{th} highest bidder in the second period, given that the winning valuation in the first period is y_1(which becomes a common knowledge in the second stage). Here $M_j^{[2]}(\beta_2(z))$denotes the payment when bidder i is the j^{th} highest bidder in the second period. From the first order condition of maximization of the expected payoff, we obtain,

$$\begin{aligned} &\frac{\partial \Pi_2(z,x,y_1)}{\partial z} = 0 \\ &\Rightarrow x\frac{g_1^{[2]}(z)}{F(y_1)^{N-2}} - \sum_{j=1}^{N-1} \left[M_j^{[2]}(\beta_2(z)) \frac{g_j^{[2]}(z)}{F(y_1)^{N-2}} + M_j^{[2]/}(\beta_2(z)) \beta_2^{/}(z) \frac{G_j^{[2]}(z)}{F(y_1)^{N-2}} \right] = 0 \\ &\Rightarrow x g_1^{[2]}(z) = \sum_{j=1}^{N-1} M_j^{[2]}(\beta_2(z)) g_j^{[2]}(z) + \sum_{j=1}^{N-1} M_j^{[2]/}(\beta_2(z)) \beta_2^{/}(z) G_j^{[2]}(z) \end{aligned}$$

At a symmetric equilibrium $z = x$ and therefore we have

$$\begin{aligned} &x g_1^{[2]}(x) = \sum_{j=1}^{N-1} M_j^{[2]}(\beta_2(x)) g_j^{[2]}(x) + \sum_{j=1}^{N-1} M_j^{[2]/}(\beta_2(x)) \beta_2^{/}(x) G_j^{[2]}(x) \\ &\Rightarrow \int_0^x y g_1^{[2]}(y)\, dy = \sum_{j=1}^{N-1} M_j^{[2]}(\beta_2(x)) G_j^{[2]}(x) \\ &\Rightarrow x G_1^{[2]}(x) - \int_0^x y G_1^{[2]}(y)\, dy = \sum_{j=1}^{N-1} M_j^{[2]}(\beta_2(x)) G_j^{[2]}(x) \end{aligned}$$

For our note, we have already assumed that $\beta_2^{/}(x) > 0$. We have,

$$\begin{aligned} &\Pi_2(z,x,y_1) \\ &= x\frac{G_1^{[2]}(z)}{F(y_1)^{N-2}} - \sum_{j=1}^{N-1} M_j^{[2]}(\beta_2(z)) \frac{G_j^{[2]}(z)}{F(y_1)^{N-2}} \\ &= x\frac{G_1^{[2]}(z)}{F(y_1)^{N-2}} - \frac{1}{F(y_1)^{N-2}} \int_0^z y g_1^{[2]}(y)\, dy \\ &= (x-z)\frac{G_1^{[2]}(z)}{F(y_1)^{N-2}} + \frac{1}{F(y_1)^{N-2}} \int_0^z G_1^{[2]}(y)\, dy \end{aligned}$$

so that,

$$\begin{aligned}
&\Pi_2(x,x,y_1)-\Pi_2(z,x,y_1)\\
&=\tfrac{1}{F(y_1)^{N-2}}\int\limits_0^x G_1^{[2]}(y)\,dy-(x-z)\tfrac{G_1^{[2]}(z)}{F(y_1)^{N-2}}-\tfrac{1}{F(y_1)^{N-2}}\int\limits_0^z G_1^{[2]}(y)\,dy\\
&=\tfrac{1}{F(y_1)^{N-2}}\int\limits_z^x yG_1^{[2]}(y)\,dy-(x-z)\tfrac{G_1^{[2]}(z)}{F(y_1)^{N-2}}
\end{aligned}$$

which is positive irrespective of $x \geq z$ or $x \leq z$. Thus if the bidding function is increasing in valuations, then the existence of a symmetric equilibrium is always ensured in the second stage.

The expected payoff in the first stage of auction can be written as:

$$\begin{aligned}
\Pi_1(z,x)&=\left(x-M_1^{[1]}(\beta_1(z))\right)G_1^{[1]}(z)-\sum_{j=2}^{N}M_j^{[1]}(\beta_1(z))\,G_j^{[1]}(z)\\
&\quad+\left(1-G_1^{[1]}(z)\right)E\left[\Pi_2(x,x)\,|Y_1^{N-1}>z\right]\\
&=xG_1^{[1]}(z)-\sum_{j=1}^{N}M_j^{[1]}(\beta_1(z))\,G_j^{[1]}(z)+\left(1-G_1^{[1]}(z)\right)E\left[\Pi_2(x,x)\,|Y_1^{N-1}>z\right]
\end{aligned}$$

where $G_j^{[1]}(z)={}^{N-1}C_{j-1}F(z)^{N-1}(1-F(z))^{j-1}$denotes the probability distribution of the j^{th}highest order statistic for the $(N-1)$ bidders other than bidder i in the first period, with the corresponding density being

$$\begin{aligned}
g_1^{[1]}(z)={}^{N-1}C_{j-1}[&((N-1)-(j-1))\,F(z)^{(N-1)-(j-1)-1}(1-F(z))^{j-1}f(z)\\
&-(j-1)\,F(z)^{(N-1)-(j-1)}(1-F(z))^{j-2}f(z)
\end{aligned}$$

Here, $E\left[\Pi_2(x,x)\,|Y_1^{N-1}>z\right]$ denotes the expected payoff in the second period, conditional on the fact that $Y_1^{N-1}>z$ i.e. the highest order statistic among the $(N-1)$bidders other than bidder i exceeds z.

Now,

$$\begin{aligned}
&E\left[\Pi_2(x,x)\,|Y_1^{N-1}>z\right]\\
&=E\left[\tfrac{1}{F\left(Y_1^{N-1}\right)^{N-2}}\int\limits_0^x G_1^{[2]}(y)\,dy|Y_1^{N-1}>z\right]\\
&=\left(\int\limits_0^x G_1^{[2]}(y)\,dy\right)E\left[\tfrac{1}{F\left(Y_1^{N-1}\right)^{N-2}}|Y_1^{N-1}>z\right]\\
&=\left(\int\limits_0^x G_1^{[2]}(y)\,dy\right)\tfrac{1}{1-G_1^{[1]}(z)}\int\limits_0^\omega \tfrac{(N-1)F\left(Y_1^{N-1}\right)^{N-2}f\left(Y_1^{N-1}\right)}{F\left(Y_1^{N-1}\right)^{N-2}}dY_1^{N-1}\\
&=\left(\int\limits_0^x G_1^{[2]}(y)\,dy\right)\tfrac{(N-1)}{1-G_1^{[1]}(z)}(1-F(z))
\end{aligned}$$

Therefore the expected payoff function for the first stage of auction can be written as

$$\Pi_1(z,x)=xG_1^{[1]}(z)-\sum_{j=1}^{N}M_j^{[1]}(\beta_1(z))\,G_j^{[1]}(z)+(N-1)(1-F(z))\int\limits_0^x G_1^{[2]}(y)\,dy$$

From the first order condition for maximization of $\Pi_1(z,x)$ we obtain:

$$\frac{\partial \Pi_1(z,x)}{\partial z} = 0$$
$$\Rightarrow x g_1^{[1]}(z) - \sum_{j=1}^{N} \left[M_j^{[1]}(\beta_1(z)) \right] g_1^{[1]}(z) + M_j^{[1]/}(\beta_1(z))\, \beta^{/}(z)\, G_j^{[1]}(z) - (N-1) f(z) \int_0^x G_1^{[2]}(y)\, dy = 0$$
$$\Rightarrow x g_1^{[1]}(z) = \sum_{j=1}^{N} \left[M_j^{[1]}(\beta_1(z))\, g_1^{[1]}(z) + M_j^{[1]/}(\beta_1(z))\, \beta^{/}(z)\, G_j^{[1]}(z) \right] - (N-1) f(z) \int_0^x G_1^{[2]}(y)\, dy$$

At a symmetric equilibrium $z = x$, so that

$$x g_1^{[1]}(x) = \sum_{j=1}^{N} \left[M_j^{[1]}(\beta_1(x))\, g_1^{[1]}(x) + M_j^{[1]/}(\beta_1(x))\, \beta_1^{/}(x)\, G_j^{[1]}(x) \right] - (N-1) f(x) \int_0^x G_1^{[2]}(y)\, dy$$
$$\Rightarrow \sum_{j=1}^{N} M_j^{[1]}(\beta_1(x))\, G_j^{[1]}(x) = \int_0^x y g_1^{[1]}(y)\, dy - (N-1) \int_0^x \left(\int_0^v G_1^{[2]}(y)\, dy \right) f(v)\, dv$$
$$\Rightarrow \sum_{j=1}^{N} M_j^{[1]}(\beta_1(x))\, G_j^{[1]}(x) = x G_1^{[1]}(x) - \int_0^x G_1^{[1]}(y)\, dy - \int_0^x \left(\int_0^v G_1^{[2]}(y)\, dy \right) f(v)\, dv$$

We have already assumed that $\beta_1^{/}(x) > 0$. We have,

$$\Pi_1(z,x)$$
$$= x G_1^{[1]}(z) - \sum_{j=1}^{N} M_j^{[1]}(\beta_1(z))\, G_j^{[1]}(z) + (N-1)(1 - F(z)) \int_0^x G_1^{[2]}(y)\, dy$$
$$= x G_1^{[1]}(z) - z G_1^{[1]}(z) + \int_0^z G_1^{[1]}(y)\, dy + \int_0^z \left(\int_0^v G_1^{[2]}(y)\, dy \right) f(v)\, dv + (N-1)(1 - F(z)) \int_0^x G_1^{[2]}(y)\, dy$$
$$= (x - z)\, G_1^{[1]}(z) + \int_0^z G_1^{[1]}(y)\, dy + \int_0^z \left(\int_0^v G_1^{[2]}(y)\, dy \right) f(v)\, dv + (N-1)(1 - F(z)) \int_0^x G_1^{[2]}(y)\, dy$$

So we have

$$\Pi_1(x,x) - \Pi_1(z,x) = \int_z^x G_1^{[1]}(y)\, dy + \int_z^x \left(\int_0^v G_1^{[2]}(y)\, dy \right) f(v)\, dv$$
$$+ (N-1)(F(z) - F(x)) \int_0^x G_1^{[2]}(y)\, dy - (x - z)\, G_1^{[1]}(z)$$

which is positive irrespective of $x \geq z$ or $x \leq z$. Thus if the bidding function is increasing in valuations, then the existence of a symmetric equilibrium is always ensured in the first stage. Note that the expected payments to the seller at equilibrium, in both the stages, $\sum_{j=1}^{N} M_j^{[k]}(\beta_k(z))\, G_j^{[k]}(z)$ $\forall k = 1, 2$ do not depend on the payment rules set by the seller. Therefore, we have found that the revenue equivalence theorem holds for each individual stage, and hence for the entire sequential auction.

We have checked that if a symmetric and increasing equilibrium exists, then for a two-stage sequential auction, the expected payment of a bidder with value zero is zero in both the stages of a two-stage sequential standard auction. To check whether it holds for any finite sequential standard auction, we resort to the method of induction. We will assume that in each of the stages of the sequential auction an increasing, symmetric equilibrium bidding strategy exists. It is routine to check that the revenue equivalence holds for the final stage of the sequential auctions[3]. We will run the induction backward. So, supposing that revenue equivalence holds for the S^{th}stage, where $S < K$, we are interested in checking whether it holds for the $(S-1)^{th}$stage as well.

Let us define

$$H_S(x,x) = \left[xG_1^{[S]}(x) - \sum_{j=1}^{N-S+1} M_j^{[S]}(\beta_S(x))\, G_j^{[S]}(x) + (N-S)\, H_{S+1}(x,x)\,(1-F(x)) \right] \quad \forall S \in [1, K-1]$$

and

$$H_K(x,x) = \left[xG_1^{[K]}(x) - \sum_{j=1}^{N-K+1} M_j^{[K]}(\beta_K(x))\, G_j^{[K]}(x) \right]$$

In the S^{th}stage , there are $(N-S+1)$ bidders, and in the $(S-1)^{th}$ stage there are $(N-S+2)$ bidders. The expected payoff in the $(S-1)^{th}$stage be $\Pi_S(x,x) = \frac{H_S(x,x)}{F\left(Y_1^{N-S+1}\right)^{N-S}} = \frac{H_S(x,x)}{F(y_1)^{N-S}}$ where $Y_1^{N-S+1} = y_1$ (say). Note that

$$\begin{aligned}\Pi_{S-1}(z,x) = \tfrac{1}{F\left(Y_1^{N-S+2}\right)^{N-S+1}} \Bigg[&\left(x - M_1^{[S-1]}(\beta_{S-1}(z))\right) G_1^{[S-1]}(z) - \sum_{j=2}^{N-S+2} M_j^{[S-1]}(\beta_{S-1}(z))\, G_j^{[S-1]}(z) \\ &+ \left(1 - F(z)^{N-S+1}\right) E\left[\Pi_S(x,x) \,\middle|\, Y_1^{N-S+1} > z\right] \Bigg]\end{aligned}$$

where $\frac{1}{F\left(Y_1^{N-S+2}\right)^{N-S+1}}$ is equal to 1 if $S = 2$[4]. Note that $\forall k \in [2, K] \quad Y_1^{k-1} > Y_1^k$, as we are assuming an increasing equilibrium, therefore expected profit function of a bidder in the k^{th} stage depends only on the value of Y_1^{k-1} and does not depend on $Y_1^1, \cdots, Y_1^{k-2}$.

Now it is clear that $\arg\max_z \Pi_{S-1}(z,x) = \arg\max_z \Pi^*_{S-1}(z,x)$ where

$$\begin{aligned}\Pi^*_{S-1}(z,x) = &\left(x - M_1^{[S-1]}(\beta_{S-1}(z))\right) G_1^{[S-1]}(z) - \sum_{j=2}^{N-S+2} M_j^{[S-1]}(\beta_{S-1}(z))\, G_j^{[S-1]}(z) \\ &+ \left(1 - F(z)^{N-S+1}\right) E\left[\Pi_S(x,x) \,\middle|\, Y_1^{N-S+1} > z\right]\end{aligned}$$

Now,

[3]The formal proof is very similar to the proof of revenue equivalence in the 2^{nd} stage as presented above

[4]i.e. stage 1.

$$\begin{aligned}
\Pi^*_{S-1}(z,x) &= \left(x - M_1^{[S-1]}\left(\beta_{S-1}(z)\right)\right) G_1^{[S-1]}(z) - \sum_{j=2}^{N-S+2} M_j^{[S-1]}\left(\beta_{S-1}(z)\right) G_j^{[S-1]}(z) \\
&\quad + \left(1 - F(z)^{N-S+1}\right) E\left[\Pi_S(x,x) \left| Y_1^{N-S+1} > z\right.\right] \\
&= xG_1^{[S-1]}(z) - \sum_{j=1}^{N-S+2} M_j^{[S-1]}\left(\beta_{S-1}(z)\right) G_j^{[S-1]}(z) + \left(1 - F(z)^{N-S+1}\right) E\left[\frac{H_S(x,x)}{F\left(Y_1^{N-S+1}\right)^{N-S}} \left| Y_1^{N-S+1} > z\right.\right] \\
&= xG_1^{[S-1]}(z) - \sum_{j=1}^{N-S+2} M_j^{[S-1]}\left(\beta_{S-1}(z)\right) G_j^{[S-1]}(z) \\
&\quad + \left(1 - F(z)^{N-S+1}\right) \frac{H_S(x,x)}{\left(1-F(z)^{N-S+1}\right)} \int_z^{\omega} \frac{(N-S+1)F\left(Y_1^{N-S+1}\right)^{N-S} f\left(Y_1^{N-S+1}\right)}{F\left(Y_1^{N-S+1}\right)^{N-S}} dY_1^{N-S+1} \\
&= xG_1^{[S-1]}(z) - \sum_{j=1}^{N-S+2} M_j^{[S-1]}\left(\beta_{S-1}(z)\right) G_j^{[S-1]}(z) + (N-S+1) H_S(x,x)\left(1 - F(z)\right)
\end{aligned}$$

Therefore, from the first order condition of a maximization we obtain

$$\begin{aligned}
&\frac{\partial \Pi^*_{S-1}(z,x)}{\partial z} = 0 \\
&\Rightarrow xg_1^{[S-1]}(z) - \sum_{j=1}^{N-S+2} \left[M_j^{[S-1]}\left(\beta_{S-1}(z)\right)\right] g_1^{[S-1]}(z) - \beta^{/}_{S-1}(z) \sum_{j=1}^{N-S+2} M_j^{[S-1]/}\left(\beta_{S-1}(z)\right) G_j^{[S-1]}(z) \\
&\quad - (N-S+1) H_s(x,x) f(z) = 0 \\
&\Rightarrow \sum_{j=1}^{N-S+2} \left[M_j^{[S-1]}\left(\beta_{S-1}(z)\right)\right] g_1^{[S-1]}(z) + \beta^{/}_{S-1}(z) \sum_{j=1}^{N-S+2} M_j^{[S-1]/}\left(\beta_{S-1}(z)\right) G_j^{[S-1]}(z) \\
&\quad = xg_1^{[S-1]}(z) - (N-S+1) H_s(x,x) f(z)
\end{aligned}$$

At a symmetric equilibrium $z = x$ and we thus obtain

$$\begin{aligned}
&\sum_{j=1}^{N-S+2} \left[M_j^{[S-1]}\left(\beta_{S-1}(x)\right)\right] g_1^{[S-1]}(x) + \beta^{/}_{S-1}(x) \sum_{j=1}^{N-S+2} M_j^{[S-1]/}\left(\beta_{S-1}(x)\right) G_j^{[S-1]}(x) \\
&\quad = xg_1^{[S-1]}(x) - (N-S+1) H_s(x,x) f(x) \\
&\Rightarrow \sum_{j=1}^{N-S+2} M_j^{[S-1]}\left(\beta_{S-1}(x)\right) G_j^{[S-1]}(x) = \int_0^x yg_1^{[S-1]}(y)\,dy - (N-S+1) \int_0^x H_s(y,y) f(y)\,dy
\end{aligned}$$

Note that as we have assumed that the revenue equivalence holds up to S^{th} stage, we know that $H_s(.,.)$ is independent of $M^{[i]}(.)\quad \forall i \in [S,K]$. The final expression of the above equation shows that revenue equivalence holds for the $(S-1)^{th}$stage as well. Therefore, we can infer that the revenue equivalence holds for each individual stage of a sequential standard auction with finite number of stages, if the bid function is increasing in valuations in each stage (and therefore the existence of a symmetric equilibrium is ensured) and the expected payment of bidder with value zero is zero.

4 Conclusion

This paper makes an attempt to generalize the existing result for revenue equivalence of Weber (1983) for first and second price sealed bid sequential auctions with single-unit demand by proving that revenue equivalence holds for any standard sequential auction, in a SIPV framework, with finite number of stages and single-unit demand whenever a symmetric and increasing equilibrium exists and the expected payment of a bidder with valuation zero becomes zero. Further scope of research in this direction lies in analyzing the multiple unit sequential auctions, where only one unit is sold at each stage, involving multiple unit demand, thus generalizing the context suggested by Engelbrecht-Wiggans (1988)[1] to the case of sequential auctions.

References

[1] Engelbrecht-Wiggans, R. (1988): "Revenue Equivalence in Multi-Object Auctions", *Economics Letters*, **26**, 15-19.

[2] Klemperer, P. (2004): *Auctions: Theory and Practice*, Princeton University Press, Princeton, USA.

[3] Krishna, V. (2010): *Auction Theory*, Academic Press, California, USA.

[4] Milgrom, P. and R.J. Weber (1982): "A Theory of Auctions and Competitive Bidding", *Econometrica,***50**, 1089-1122 .

[5] Myerson, R. (1981): "Optimal Auction Design", *Mathematics of Operations Research*, **6**, 58-73.

[6] Riley, J. and W. Samuelson (1981): "Optimal Auctions", *American Economic Review,* **71**,381-392.

[7] Weber, R.J. (1983): "Multiple Object Auctions" in R. Engelbrecht-Wiggans, M. Shubik and R. Stark (eds.), *Auctions, Bidding and Contracting: Uses and Theory*, New York, NY: New York University Press, 165-191.