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Imitation, local interaction, and efficiency: reappraisal

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Abstract

We revisit the model of Alos-Ferrer and Weidenholzer (2006) but under the assumption that risk-dominant equilibria are Pareto efficient. It is found that risk-dominant equilibria, non-risk-dominant equilibria, and some non-monomorphic states can emerge in the long run when players interact with their immediate neighbors only.

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1. Introduction

Which Nash equilibria in coordination games (hereafter CG) would emerge in the long run has been intensively studied in the literature of evolutionary games. Risk-dominant equilibria are predicted by many works (e.g., Blume (1993, 1995), Ellison (1993), Kandori et al. (1993), Young (1993), Sandholm (1998)). Under imitation dynamics and local interaction, Alós-Ferrer and Weidenholzer (2006) show that risk-dominant equilibria survive uniquely in the long run when players interact with their immediate neighbors only. But payoff-dominant equilibria will be selected when players' interactions are neither global nor limited to their immediate neighbors. In Alós-Ferrer and Weidenholzer (2006), it is assumed that risk-dominant equilibria are not Pareto efficient. Here we revisit Alós-Ferrer and Weidenholzer's (2006) model but under the assumption that risk-dominant equilibria are Pareto efficient. We find that risk-dominant equilibria, non-risk-dominant equilibria, and some non-monomorphic states all can emerge in the long run when players interact with their immediate neighbors only. The intuition is simple. When risk-dominant equilibria are not Pareto efficient, payoff-dominant-strategy takers can clump together and expand. Then, it is costly for risk-dominant equilibria to jump out of their basin of attraction such that they survive uniquely. In contrast, if risk-dominant equilibria are Pareto efficient, the expansion force of non-risk-dominant strategy takers is weakened. Then, it is less costly for risk-dominant equilibria to jump out of their basin of attraction so that some non-monomorphic states can survive as well.

2. The Model and Results

Let $N = \{1,\ 2, \ldots, n\},\ n \geq 5$, be the set of players. Players are assumed to sit sequentially and equally spaced around a circle. Each individual has exactly two neighbors. For $i \in N$, let $N_i = \{i-1,\ i+1\}$ be the set of player i's neighbors. At each time period $t \in \{1,\ 2,\ 3, \ldots\}$, players meet each of their two neighbors once to play 2×2 symmetric CG below.

	A	B
A	$a,\ a$	$b,\ c$
B	$c,\ b$	$d,\ d$

where $a > c$ and $d > b$ such that both $(A,\ A)$ and $(B,\ B)$ are strict Nash equilibria. Alós-Ferrer and Weidenholzer (2006) further assume that $d > a$ and $a + b > c + d$ so that $(A,\ A)$ is risk dominant and $(B,\ B)$ is Pareto efficient. Here we assume that $a \geq d$ and $a+b > c+d$ so that $(A,\ A)$ is both Pareto efficient and risk-dominant, and $(B,\ B)$ is non-risk-dominant. As in Alós-Ferrer and Weidenholzer (2006), we normalize the above game as

$$\begin{array}{c|c|c|} & A & B \\ \hline A & 1,1 & 0,\alpha \\ \hline B & \alpha,0 & \beta,\beta \\ \hline \end{array}$$

where $\alpha = \frac{c-b}{a-b}$ and $\beta = \frac{d-b}{a-b}$. Hence,

$$\alpha < 1,\ 0 < \beta \leq 1 \text{ and } \alpha + \beta < 1. \tag{1}$$

Our state space $S \equiv \{A,\ B\}^N$ is a set containing all players' strategy profiles. Denote $\vec{A} = (A,\ A, \ldots, A)$ and $\vec{B} = (B,\ B, \ldots, B)$ the states in which all players take strategies A and B, respectively. At the beginning of each period, players' actions and payoffs occured (after revision) in the last period are observable to their neighbors. And players are assumed to imitate the strategies earning the highest total payoff among their neighbors and themselves. Given state $\vec{s} = (s_1,\ s_2, \ldots, s_n) \in S$, let $z_i(\vec{s})$ be player i's total payoff after playing with his neighbors. Therefore,

$$z_i(\vec{s}) = \begin{cases} n_i^A(\vec{s}) & \text{if } s_i = A, \\ \alpha \cdot n_i^A(\vec{s}) + \beta \cdot (2 - n_i^A(\vec{s})) & \text{if } s_i = B, \end{cases}$$

where $n_i^A(\vec{s}) = |\{j \in N_i : s_j = A\}|$ is the number of player i's neighbors taking strategy A. Then, player i's next-period rational choice, $r_i(\vec{s})$, will satisfy

$$r_i(\vec{s}) \in \arg \max_{j \in N_i \cup \{i\}} z_j(\vec{s}). \tag{2}$$

Whenever there is a tie, strategy A or B will be taken with strictly positive probability. At the end of each period, all players are allowed to revise their rational choices with probability $\epsilon > 0$. For fixed ϵ, our dynamic system is a Markov chain on S. Let Q_0 and Q_ϵ be the transition probability matrices for the rational and revised processes respectively. Define $r(\vec{s}) = (r_1(\vec{s}), \ldots, r_n(\vec{s}))$. Being a perturbation of Q_0, we have $Q_\epsilon(\vec{s}, \vec{u}) \approx \text{constant} \cdot \epsilon^{U(\vec{s}, \vec{u})}$ for any $\vec{s}, \vec{u} \in S$, where $U(\vec{s}, \vec{u}) = \min_{r(\vec{s})} d(r(\vec{s}), \vec{u})$ and $d(r(\vec{s}), \vec{u}) = |\{i \in N : r_i(\vec{s}) \neq u_i\}|$ counts the total number of player i revising his rational choice $r_i(\vec{s})$ at state $\vec{s}$.

Because $Q_\epsilon(\vec{s}, \vec{u}) > 0$ for all $\vec{s}, \vec{u} \in S$, the revision makes our dynamic system $\{X_t\}$ ergodic. Let μ_ϵ be the associated unique invariant distribution under Q_ϵ. We are interested in the limit probability distribution $\mu_* \stackrel{\text{def}}{=} \lim_{\epsilon \to 0} \mu_\epsilon$ and its support $S_* \equiv \{\vec{s} \in S\ :\ \mu_*(\vec{s}) > 0\}$. Each element in S_* is called a long run equilibrium (hereafter LRE). A non-monomorphic state consists of A-strings alternating with equal number of B-strings since all players sit around a circle as follows.

$$\cdots \underbrace{A \cdots A}_{a_k} \underbrace{B \cdots B}_{b_k} \underbrace{A \cdots A}_{a_1} \underbrace{B \cdots B}_{b_1} \underbrace{A \cdots A}_{a_2} \underbrace{B \cdots B}_{b_2} \cdots, \tag{3}$$

where a_i, b_i are the lengths of its i-th A-string and B-string respectively. Let $M_{\geq m,\, p} \stackrel{\text{def}}{=} \{\vec{s} \in S :\ \text{all } a_i \geq m \text{ and } b_j = p \text{ in (3)}\}$ consisting of non-monomorphic states with all A-strings of length $\geq m$ and all B-strings of length p. The LREs of our dynamic system are given below.

Theorem 1: *Under the imitation rule* (2), $S_* = \{\vec{A}\}$ *except the following two cases:*
(i) *When* $\alpha > 1/2$, $S_* = \{\vec{B}\}$ *if* $5 \le n \le 6$, $S_* = \{\vec{A}, \vec{B}\} \cup M_{\ge 3,\,1}$ *if* $7 \le n \le 12$, *and* $S_* = \{\vec{A}\} \cup M_{\ge 3,\,1}$ *if* $n \ge 13$.
(ii) *When* $\alpha = 1/2$, $S_* = \{\vec{A}, \vec{B}\}$ *if* $5 \le n \le 6$, *and* $S_* = \{\vec{A}\}$ *if* $n \ge 7$.
Proof. See the Appendix A.

Theorem 1 shows that risk-dominant equilibria, non-risk-dominant equilibria, and some non-monomorphic states can be LREs. The payoff structure and population size determine which equilibria will emerge in the long run. For large population, risk-dominant equilibrium $\vec{A}$ is in favor. This is no wonder. But it is not the unique LRE as shown by Theorem 1(i). Due to the space limit, we provide the intuition of Theorem 1(i) below.

Since $S_* \subseteq S_0$ (i.e., the set of all ergodic states under Q_0), the first step is to determine S_0. Certainly, $\{\vec{A}, \vec{B}\} \subseteq S_0$. Moreover, all non-monomorphic states in $M_{\ge 3,\,1}$ are absorbing under Q_0 as well. It is easy to check that

$$\begin{array}{lccccccccccccc} \text{state } \ldots & B & A & B & B & A & A & A & B & A & A & B & B & B \ \ldots \\ \text{total payoff } \ldots & \ldots & 0 & \alpha+\beta & \alpha+\beta & 1 & 2 & 1 & 2\alpha & 1 & 1 & \alpha+\beta & 2\beta & \ldots \ \ldots \end{array}$$

When $\alpha > 1/2$, we have $\alpha > \beta$ and $2\beta < 1$ by (1). Thus, a single A-player will change to strategy B in the next period under Q_0, while a single B-player will retain his strategy in the next period iff he is isolated or confronted with an isolated A-player. Moreover, each string of A-player with length ≥ 3 can hold and expand until it is surrounded by singleton B-players. Thus, $M_{\ge 3,\,1} \subseteq S_0$. However, which of $\vec{A}$, $\vec{B}$ and $M_{\ge 3,\,1}$ are LREs are determined by which states can be reached from the others at the minimum cost. Let $\vec{s}_0 \xrightarrow{k} \vec{s}_1$ represent that state $\vec{s}_0$ can reach state $\vec{s}_1$ by k mutants, and $\vec{s}_0 \overset{k}{\leftrightarrow} \vec{s}_1$ indicates that state $\vec{s}_1$ can reach $\vec{s}_0$ by k mutants as well. Then, any two states in $\{\vec{A}\} \cup M_{\ge 3,\,1}$ can communicate with each other by a sequence of one-mutant transitions because

$$\cdots \underbrace{A \cdots A}_{\ge 3} \dot{B} \underbrace{A \cdots A}_{\ge 3} \cdots \overset{1}{\leftrightarrow} \cdots \underbrace{A \cdot \dot{A} \cdot A}_{\ge 5} \cdots \quad \text{and} \quad \vec{A} \overset{1}{\leftrightarrow} \underbrace{A \cdots A}_{n-1} B.$$

Next, since an A-string with length 2 can grow until a single B left or $\vec{A}$ reached, the minimum cost from $\vec{B}$ to any state in $\{\vec{A}\} \cup M_{\ge 3,\,1}$ is 2. In contrast, the minimum cost of states in $\{\vec{A}\} \cup M_{\ge 3,\,1}$ reaching $\vec{B}$ depends on population sizes. For $n = 5$, we have

$$AAAAB \xrightarrow{1} ABAAB \xrightarrow{0} \vec{B},$$

and

$$AAAAAB \xrightarrow{1} AABAAB \xrightarrow{0} \vec{B}$$

for $n = 6$. It means that $\vec{B}$ is the unique LRE for $n = 5,\ 6$. However, for $n \ge 7$, the minimum-cost path to $\vec{B}$ from states in $\{\vec{A}\} \cup M_{\ge 3,\,1}$ must first attain a non-monomorphic state alternating A-strings with length 5 and B-strings with length 1.

Then, by adding one mutant in the middle of each A-string of this state, the non-monomorphic state will reach $\vec{B}$ at zero cost, i.e.,

$$\cdots \underbrace{B}_{1}\underbrace{AA\overset{\bullet}{A}AA}_{5}\underbrace{B}_{1}\cdots \xrightarrow{1} \cdots \underbrace{B}_{1}\underbrace{AA\overset{\bullet}{B}AA}_{5}\underbrace{B}_{1}\cdots \xrightarrow{0} \cdots \underbrace{B\ldots B}_{7}\cdots.$$

Thus, the total of $\lfloor \frac{n}{6} \rfloor$ mutants are needed if n is a multiple of 6. Otherwise, an extra mutant is needed to eliminate the remaining block containing some A's. Accordingly, $\lceil \frac{n}{6} \rceil$ mutants are required to reach $\vec{B}$ from states in $\{\vec{A}\} \cup M_{\geq 3,\,1}$. Thus, the relative sizes of 2 and $\lceil \frac{n}{6} \rceil$ determine which stationary states are LREs. For $7 \leq n \leq 12$, $\vec{A}$, $\vec{B}$, and $M_{\geq 3,\,1}$ are all LREs, and $\{\vec{A}\} \cup M_{\geq 3,\,1}$ will be the LREs for $n \geq 13$.

3. Conclusion

In conclusion, under imitation dynamics, Alós-Ferrer and Weidenholzer (2006) show that selecting risk-dominant equilibria is sensitive to players' interacting ways. Our results further demonstrate that the selection is sensitive to games' payoff structures as well.

Appendix A

Proof of Theorem 1. Only the case of $\alpha > 0$ is considered, the rest can be treated similarly. Ellison's (2000) Radius and Coradius Theorem is adopted when $|S_*| = 1$, while the Freidlin-Wentzell Method (1984) is used when S_* is complicated as in case (i). Since $S_* \subseteq S_0$, the set of all ergodic states under Q_0, the first step is to determine S_0. Certainly, $\{\vec{A}, \vec{B}\} \subseteq S_0$. Let $M \overset{\text{def}}{=} S_0 \setminus \{\vec{A}, \vec{B}\}$ be the set of non-monomorphic ergodic states. Using $\alpha > 0$ and (1), we have $0 = \min\{1, 0, \alpha, \beta\} < 1 = \max\{1, 0, \alpha, \beta\}$. Since $r_i(\vec{s})$ depends only on the strategies $(s_{i-2}, s_{i-1}, s_i, s_{i+1}, s_{i+2})$ taken by five consecutive players from $i-2$ to $i+2$ and are independent of the time t and the label of player i, we define $r(s_{i-2}, s_{i-1}, s_i, s_{i+1}, s_{i+2}) \overset{\text{def}}{=} r_i(\vec{s})$ for brevity. Figure A in the Appendix B implies that with $* = A$ or B independently,

$$r(*, B, A, B, *) = B, \tag{4}$$

which means that an isolated A-player would change to strategy B in the next period under Q_0. The following classifications are used to determine other strategy-updating rules under Q_0.

Case (i) $\alpha > 1/2$. For $2\alpha \geq 1$, we can use $\alpha + \beta < 1$ in (1) to get $\alpha > \beta$ and $2\beta < 1$. Under Q_0, we get from Figures B and BB that

$$r(*, A, B, A, *) = B,\ r(B, A, B, B, *) = B \text{ and } r(A, A, B, B, *) = A. \tag{5}$$

Certainly, $r(*,B,B,A,B) = B$ and $r(*,B,B,A,A) = A$ by symmetry. Eq (5) means that a B-player will keep his strategy B in the next period iff he is isolated or confronted with an isolated A-player. As to a non-isolated A-player, Figure AA shows that

$$r(A,B,A,A,B) = B \text{ and } r(*,B,A,A,A) = r(B,B,A,A,B) = A. \tag{6}$$

Using (4)-(6) and the definition of S_0, we have observations as follows:
(O1) If $\vec{s} \in M$ and $\vec{t}$ is reachable from $\vec{s}$ under Q_0, then $\vec{t}$ is ergodic as well.
(O2) Any A-string with length ≥ 3 can hold and grow until it is surrounded by singleton B-strings, which can hold under Q_0. In particular, $M_{\geq 3,\,1} \subseteq M$.
(O3) A singleton A-string will be absorbed into a larger B-string under Q_0 and the singleton A will not be recovered afterwards. Hence, $a_i \geq 2$ for all $\vec{s} \in M$.
(O4) Since $a_i \geq 2$ for all $\vec{s} \in M$ by (O3), any B-string with length ≥ 2 in $\vec{s} \in M$ will shrink until it disappears or becomes a singleton. Note that the length of any its neighboring A-strings does not decrease in the process. When encountered by some A-string of length 2, a singleton B-string could expand under Q_0 to length 2 or 3 in the next period. By (O1) and (O3), it will disappear in the next period under Q_0 in the former case. In the latter case it will shrink back to be singleton under Q_0. Hence, $b_i = 1$ or 3 for all $\vec{s} \in M$.
(O5) Any A-string of length 2 will be eliminated in the next period when surrounded by singleton B-strings. By (O2) and (O4), we deduce that if $\vec{s} \in M$ has some A-string with length ≥ 3, then $\vec{s} \in M_{\geq 3,\,1}$.
(O6) Let $M^* = M \setminus M_{\geq 3,\,1}$. By (O5), all $a_i = 2$ for $\vec{s} \in M^*$. Using (O2) and (O4) again, two successive B-strings in $\vec{s}$ must have length 1 and 3 respectively. Say, $b_{i-1} = 3$ and $b_i = 1$. Because $a_{i-1} = a_{i+1} = 2$, the same argument shows $b_{i-2} = 1$ and $b_{i+1} = 3$. Repeating over and over, we conclude that if exists, any $\vec{s} \in M^*$ must have the following periodic structure:

$$\vec{s} = \underbrace{\dot{A}\,ABB\,\mathring{B}\,AAB}_{\text{repeat } \frac{n}{8} \text{ times}} \cdots \overset{0}{\leftrightarrow} \underbrace{\dot{B}\,AAB\,\mathring{A}\,ABB}_{\text{repeat } \frac{n}{8} \text{ times}} \cdots = \vec{s}'. \tag{7}$$

Hereafter, $\vec{u} \overset{c}{\rightarrow} \vec{v}$ means $U(\vec{u},\vec{v}) = c$ and $\vec{u} \overset{c}{\leftrightarrow} \vec{v}$ means $U(\vec{v},\vec{u}) = c$ as well. It follows from (7) that $M^* \neq \emptyset$ iff $8|n$.

Next, we need to find $v(\vec{s})$, the minimum cost among all spanning trees rooted at $\vec{s}$. Certainly, only $\vec{s} \in S_0$ needs to be considered. Write $M_{\geq 3,1} = \cup_{k\geq 1} M_k$, where k is the number of A-strings in representation (3) for $\vec{s}$.

Step 1. For convenience, define $M_0 = \{\vec{A}\}$. The following diagram shows that any $\vec{s} \in M_k$ with $k \geq 1$ can reach some state in M_{k-1} at the minimum cost 1 and vice versa :

$$\cdots \underbrace{B}_{1} \underbrace{A\cdots A}_{a_i \geq 2} \underbrace{\dot{B}}_{1} \underbrace{A\cdots A}_{a_{i+1}\geq 2} \underbrace{B}_{1} \cdots \overset{1}{\leftrightarrow} \cdots \underbrace{B}_{1} \underbrace{A\cdots A\,\dot{A}\,A\cdots A}_{a_i+1+a_{i+1}} \underbrace{B}_{1} \cdots. \tag{8}$$

Since $|M_0| = 1$, (8) implies that all states in $\{\vec{A}\} \cup M_{\geq 3,\,1}$ can reach any $\vec{s} \in \{\vec{A}\} \cup M_{\geq 3,\,1}$ at cost 1 for each state. So, the total cost is $|M_{\geq 3,\,1}|$.

Step 2. By (4)-(6), the most economical path for $\vec{B}$ to reach $\{\vec{A}\}\cup M_{\geq 3,1}$ is as follows. Depending on whether $2|n$ or not, we get as in (O4) that

$$\vec{B} \xrightarrow{2} \cdots BB\ \overset{\bullet}{A}\overset{\bullet}{A}\ BB \cdots \xrightarrow{0} \cdots BA\ \overset{\bullet}{A}\overset{\bullet}{A}\ AB \cdots \xrightarrow{0} \cdot\cdot \xrightarrow{0} \vec{A} \text{ or } AA\cdots AAAAB \in M_1.$$

Step 3. When $8|n$, states $\vec{s}, \vec{s'}$ in (7) form an irreducible class under Q_0 as $Q_0(\vec{s}, \vec{s'}) = Q_0(\vec{s'}, \vec{s}) = 1$. The following path shows an optimal way for the class to reach out:

$$\vec{s'} \overset{0}{\leftrightarrow} \vec{s} = \underbrace{AAB\ \overset{\bullet}{B}\ BAAB}\cdot\cdot \xrightarrow{1} \underbrace{BAA\ \overset{\bullet}{A}\ AABB}\cdot\cdot \xrightarrow{0} \underbrace{AAA\ \overset{\bullet}{A}\ AAAB}\cdot\cdot\cdot\cdot\cdot \xrightarrow{0} \vec{A}$$

as the newly formed A-string absorbs its neighboring B's until it reaches $\vec{A}$. Since $|M^*|$ =8, all states in M^* can reach $\vec{A}$ at a minimum total cost of $8/2 = 4$.

Step 4. When $8|n$ so $M^* \neq \emptyset$, (O2) indicates that the following path is optimal to reach M^* from $\{\vec{A}\} \cup M_{\geq 3,\,1}$:

$$M_{\frac{n}{4}} \ni \underbrace{AA\ \overset{\bullet}{A}\ B\ \overset{\bullet}{A}\ AAB}_{\text{repeat}\,\frac{n}{8}\,\text{times}}\cdots \xrightarrow{\frac{n}{4}} \underbrace{AA\ \overset{\bullet}{B}\ B\ \overset{\bullet}{B}\ AAB}_{\text{repeat}\,\frac{n}{8}\,\text{times}}\cdots = \vec{w} \in M^*. \tag{9}$$

Step 5. We now find an optimal path from $\{\vec{A}\} \cup M_{\geq 3,\,1}$ to $\{\vec{B}\}$. To avoid A-strings with length ≥ 3 which can hold under Q_0 as shown in (O2), it saves to start from some $\vec{s} \in M_{\geq 3,\,1}$ which has as many B's as possible. Moreover, it takes at least ℓ revisions under Q_ϵ to eliminate an A-string with length $\geq 3\ell+2$ in $\vec{s} \in M_{\geq 3,\,1}$. Since an A-string in $\vec{s} \in M_{\geq 3,\,1}$ needs at least one revision to be eliminated under Q_ϵ, some calculation shows that it is the most economical to have block $BAAAAA$ duplicated in $\vec{s} \in M_{\geq 3,\,1}$ up to the maximum allowed $\left\lfloor \frac{n}{6} \right\rfloor$ times and that one revision is enough to eliminate the five A's in such blocks. As to the remaining blocks with length $r = n - 6\left\lfloor \frac{n}{6} \right\rfloor$, an optimal choice for being both in $M_{\geq 3,\,1}$ and economical is $\emptyset, A, AA, AAA, BAAA$ and $BAAAA$ for $r = 0, 1, 2, 3, 4$ and 5 respectively. Of course, an extra mutation is needed if $r \geq 1$. Let $\vec{s} \in M_{\geq 3,\,1}$ be such a state, an optimal path from $\{\vec{A}\} \cup M_{\geq 3,\,1}$ to $\vec{B}$ is as follows:

$$\vec{s} \xrightarrow{\left\lceil \frac{n}{6} \right\rceil} \underbrace{BAA\ \overset{\bullet}{B}\ AA}_{\text{repeat}\left\lfloor \frac{n}{6} \right\rfloor\text{times}}\cdots(\emptyset, \overset{\bullet}{B}, \overset{\bullet}{B}\ A, \overset{\bullet}{B}\ AA, A\ \overset{\bullet}{B}\ AA, ABA\ \overset{\bullet}{B}\ A) \xrightarrow{0} \vec{B}.$$

Since $\frac{n}{4} \geq \left\lceil \frac{n}{6} \right\rceil$ for $8|n$, it is also an optimal path from $\{\vec{A}\} \cup M_{\geq 3,\,1}$ to $\{\vec{B}\}$.

All together, we have $v(\{\vec{B}\}) = |M_{\geq 3,\,1}| + \left\lceil \frac{n}{6} \right\rceil + 4 \cdot \chi_{\{8|n\}}$ and $v(\{\vec{s}\}) = |M_{\geq 3,\,1}| + 2 + 4 \cdot \chi_{\{8|n\}}$ for $\vec{s} \in \{\vec{A}\} \cup M_{\geq 3,\,1}$. If $8|n$, (9) shows that $v(\{\vec{w}\}) = |M_{\geq 3,\,1}| + 2 + \left\lceil \frac{n}{4} \right\rceil + 3$ for $\vec{w} \in M^*$. Since $S_* = \{\vec{s} \in S_0 : v(\vec{s}) = \min_{\vec{w} \in S_0} v(\vec{w})\}$ by Theorems 4.1 in Chen and Chow (2009), the conclusion follows by comparing $\left\lceil \frac{n}{6} \right\rceil$ with 2. For instance, if $n \geq 13$ then $\left\lceil \frac{n}{6} \right\rceil > 2$ and $S_* = \{\vec{A}\} \cup M_{\geq 3,\,1}$.

Case (ii) $\alpha = 1/2$. All the updating rules in Case (i) remain valid except the first rules in both (5) and (6) are revised as follows:

$$0 < \mathrm{Prob}(\mathrm{r}(\mathrm{A}, \mathrm{A}, \underline{\mathrm{B}}, \mathrm{A}, *) = \mathrm{B}) < 1 \text{ and } 0 < \mathrm{Prob}(\mathrm{r}(\mathrm{A}, \mathrm{B}, \underline{\mathrm{A}}, \mathrm{A}, \mathrm{B}) = \mathrm{B}) < 1.$$

Consequently, we have that, with positive probability under Q_0,
(O7) an A-string with length ≥ 2 can hold and grow until it reaches $\vec{A}$.

By (4) and (5), any non-monomorphic state without an A-string of length ≥ 2 belongs to the basin of attraction of $\vec{B}$. Hence, $S_0 = \{\vec{A}, \vec{B}\}$. By (4) and (O7), the following path shows $v(\{\vec{A}\}) = 2$:

$$\vec{B} \xrightarrow{2} \cdot\cdot BB \, \dot{A}\dot{A} \, BB \cdot\cdot \xrightarrow{0} \cdot\cdot BA \, \dot{A}\dot{A} \, AB \cdot\cdot \xrightarrow{0} \cdot\cdot \xrightarrow{0} \vec{A}. \tag{10}$$

Because we still have $\mathrm{Prob}(\mathrm{r}(\mathrm{A}, \mathrm{A}, \mathrm{A}, \mathrm{B}, *) = \mathrm{A}) = 1$, an A-string of length ≥ 3 should be avoided in order to reach $\vec{B}$ from $\vec{A}$. Hence, the following path is optimal:

$$\vec{A} \xrightarrow{\lceil \frac{n}{3} \rceil} \underbrace{AA\,\dot{B}}_{\text{repeat} \lfloor \frac{n}{3} \rfloor \text{times}} \cdot\cdot (\emptyset, \dot{B}, A\,\dot{B}) \xrightarrow{0} \vec{B}, \text{ thus } v(\{\vec{B}\}) = \left\lceil \frac{n}{3} \right\rceil.$$

The conclusion follows by comparing 2 with $\left\lceil \frac{n}{3} \right\rceil$. Note that $n \geq 5$ by assumption.

Case (iii) $\alpha < 1/2$ and $\beta < 1/2$. While (4) remains valid, Figure BB implies

$$r(B, A, B, B, *) = B, \; r(A, A, B, B, A) = A \text{ and } r(A, A, B, B, B) = A \tag{11}$$

under Q_0. Moreover, Figures AA and B imply that a non-isolated A-player and an isolated B-player would rationally update their strategies in the next period respectively according to

$$r(*, A, A, B, *) = A, \; r(B, A, B, A, B) = B \text{ and } r(A, A, B, A, *) = A. \tag{12}$$

The first rule above and (11) imply that (O7) holds with probability 1 under Q_0 . Therefore, $S_0 = \{\vec{A}, \vec{B}\}$ as the basin of attraction at $\vec{B}$ remains the same as that in Case (ii). The path of (10) shows $CR(\{\vec{A}\}) = 2$. Because of (O7), an A-string of length ≥ 2 should be avoided in order to escape from $\vec{A}$. Hence, $R(\{\vec{A}\}) \geq 3 > CR(\{\vec{A}\})$. The definitions of radius ($R(\vec{s})$) and coradius ($CR(\vec{s})$) at state $\vec{s}$ can be found in Ellison (2000). Then, by Ellison's (2000) Radius and Coradius Theorem, $S_* = \{\vec{A}\}$ as claimed in the theorem.

Case (iv) $\alpha < 1/2 \leq \beta$. Similar to Case (i), we have $\beta > \alpha$. All updating rules in Case (iii) remain valid, except that the last rule in (11) needs to be modified. Depending on $\beta = 1/2$ or $\beta > 1/2$, we have $0 < \mathrm{Prob}(\mathrm{r}(\mathrm{A}, \mathrm{A}, \mathrm{B}, \mathrm{B}, \mathrm{B}) = \mathrm{A}) < 1$ or $\mathrm{Prob}(\mathrm{r}(\mathrm{A}, \mathrm{A}, \mathrm{B}, \mathrm{B}, \mathrm{B}) = \mathrm{B}) = 1$. As in Case (iii), we have $S_* = \{\vec{A}\}$ for $\beta = 1/2$.

For $\beta > 1/2$, $\mathrm{Prob}(\mathrm{r}(\mathrm{A}, \mathrm{A}, \mathrm{B}, \mathrm{B}, \mathrm{B}) = \mathrm{B}) = 1$ means that a B-string of length ≥ 3 can hold when surrounded by A-strings of length ≥ 2. By (O3) and (12), it is not difficult to show that $M = M_{\geq 2,\geq 3}$. Write $M_{\geq 2,\geq 3} = \cup_{k\geq 1} M_k$ and define $M_0 = \{\vec{A}\}$ as in Case (i). By shrinking B-strings at the cost 1 of each move, any $\vec{s} \in M_k$ with $k \geq 1$ can move within M_k and reach $M_{\geq 2,3}$ as shown below:

$$\cdot\cdot\underbrace{A\cdot\cdot A}_{a_i}\underbrace{B\cdot\cdot B\,\dot{B}}_{b_i\geq 3}\underbrace{A\cdots A}_{a_{i+1}}\cdot\cdot \overset{1}{\leftrightarrow} \cdot\cdot\underbrace{A\cdot\cdot A}_{a_i}\underbrace{B\cdot\cdot BB}_{b_i-1}\underbrace{\dot{A}\cdot\cdot A}_{a_{i+1}+1}\cdot\cdot \overset{1}{\leftrightarrow} \cdot\cdot \overset{1}{\leftrightarrow} \cdot\cdot\underbrace{A\cdot\cdot A}_{a_i}\underbrace{BBB}_{3}\ \underbrace{A\cdot\cdot A}_{a_{i+1}+b_i-3}\ \cdot\cdot.$$

Then by the second rule in (11),

$$\cdot\cdot\underbrace{A\cdots A}_{a_i\geq 2}\underbrace{BB\,\dot{B}}_{3}\underbrace{A\cdots A}_{a_{i+1}\geq 2}\cdot\cdot \overset{1}{\rightarrow} \cdot\cdot\underbrace{A\cdots A}_{a_i}\underbrace{BB}_{2}\underbrace{\dot{A}\ A\cdots A}_{a_{i+1}+1}\cdot\cdot \overset{0}{\rightarrow} \cdot\cdot\underbrace{A\cdots A\ \dot{A}\ A\cdots A}_{a_i+3+a_{i+1}}\cdot\cdot \in M_{k-1}.$$

Together with (10), this suggests that the modified coradius of $\vec{A}$ is $CR^*(\{\vec{A}\}) = 2$. Note that the state after $\vec{B}$ in (10) is in M_1. Since $R(\{\vec{A}\}) \geq 3$ by the first rule in (12), $S_* = \{\vec{A}\}$ as in Case (iii).

Appendix B

In Figures A, B, AA, BB, the state columns depict strategies adopted by five consecutive players $i-2,\ i-1,\ i,\ i+1$ and $i+2$.

Figure A

State $\vec{s}$	Total payoffs for players $i-1, i$, and $i+1$
$\cdots ABABA \cdots$	$z_{i-1}(\vec{s}) = 2\alpha,\ z_i(\vec{s}) = 0,\ z_{i+1}(\vec{s}) = 2\alpha$
$\cdots ABABB \cdots$	$z_{i-1}(\vec{s}) = 2\alpha,\ z_i(\vec{s}) = 0,\ z_{i+1}(\vec{s}) = \alpha + \beta$
$\cdots BBABB \cdots$	$z_{i-1}(\vec{s}) = \alpha + \beta,\ z_i(\vec{s}) = 0,\ z_{i+1}(\vec{s}) = \alpha + \beta$

Figure B

State $\vec{s}$	Total payoffs for players $i-1, i$, and $i+1$
$\cdots AABAA \cdots$	$z_{i-1}(\vec{s}) = 1,\ z_i(\vec{s}) = 2\alpha,\ z_{i+1}(\vec{s}) = 1$
$\cdots AABAB \cdots$	$z_{i-1}(\vec{s}) = 1,\ z_i(\vec{s}) = 2\alpha,\ z_{i+1}(\vec{s}) = 0$
$\cdots BABAB \cdots$	$z_{i-1}(\vec{s}) = 0,\ z_i(\vec{s}) = 2\alpha,\ z_{i+1}(\vec{s}) = 0$

Figure AA

State $\vec{s}$	Total payoffs for players $i-1, i$, and $i+1$
$\cdots AAABA \cdots$	$z_{i-1}(\vec{s}) = 2,\ z_i(\vec{s}) = 1,\ z_{i+1}(\vec{s}) = 2\alpha$
$\cdots AAABB \cdots$	$z_{i-1}(\vec{s}) = 2,\ z_i(\vec{s}) = 1,\ z_{i+1}(\vec{s}) = \alpha + \beta$
$\cdots BAABA \cdots$	$z_{i-1}(\vec{s}) = 1,\ z_i(\vec{s}) = 1,\ z_{i+1}(\vec{s}) = 2\alpha$
$\cdots BAABB \cdots$	$z_{i-1}(\vec{s}) = 1,\ z_i(\vec{s}) = 1,\ z_{i+1}(\vec{s}) = \alpha + \beta$

Figure BB

State $\vec{s}$	Total payoffs for players $i-1, i$, and $i+1$
$\cdots AABBA \cdots$	$z_{i-1}(\vec{s}) = 1,\ z_i(\vec{s}) = \alpha + \beta,\ z_{i+1}(\vec{s}) = \alpha + \beta$
$\cdots AABBB \cdots$	$z_{i-1}(\vec{s}) = 1,\ z_i(\vec{s}) = \alpha + \beta,\ z_{i+1}(\vec{s}) = 2\beta$
$\cdots BABBA \cdots$	$z_{i-1}(\vec{s}) = 0,\ z_i(\vec{s}) = \alpha + \beta,\ z_{i+1}(\vec{s}) = \alpha + \beta$
$\cdots BABBB \cdots$	$z_{i-1}(\vec{s}) = 0,\ z_i(\vec{s}) = \alpha + \beta,\ z_{i+1}(\vec{s}) = 2\beta$

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