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A backwardly solvable search equilibrium model

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Abstract

So-called "search equilibrium models" typically have multiple equilibria. In almost all studies on these models, only steady states are considered mainly because it is difficult to find non-stationary equilibria. This difficulty does not disappear even if we consider finite-horizon versions of these models. In this note, we propose an approach that might be useful to study non-stationary equilibria in these models. In particular, we consider a discrete-time and finite-horizon version of Diamond's (1982, JPE) model and show how to solve it backwardly. As an illustration, we compute a non-stationary equilibrium of a specific example, which exhibits a three-period cycle.

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1 Introduction

So-called "search equilibrium models" typically have multiple equilibria. In almost all studies on these models, only steady states are considered mainly because it is difficult to find non-stationary equilibria.[1] This difficulty does not disappear even if we consider finite-horizon versions of these models. In this note, we propose an approach that might be useful to study non-stationary equilibria in these models. In particular, we consider a discrete-time and finite-horizon version of Diamond's (1982, JPE) model, and show how to solve it backwardly.[2] As an illustration, we compute a non-stationary equilibrium of a specific example, which exhibits a three-period cycle.

2 The model

Consider a tropic island consisting of $\bar{n}$ people and many palm trees.[3] There are T periods with $T < \infty$. At the beginning of each period, each person is in one of the following two states: not carrying a coconut and looking for palm trees, or carrying a coconut and looking for other individuals with coconuts. Let n_t denote the number of people who have coconuts at the beginning of period t.

If a person without a coconut finds a palm tree, he or she can climb the tree and pick a coconut. This has a cost (in utility units), which is a random variable: with probability a_1, it is c_1; with probability a_2, it is c_2; and with probability $1 - a_1 - a_2$, it is ∞. We assume that $0 < c_1 < c_2$.

There is a taboo against the consumption of coconuts picked by themselves. If a person with a coconut meets another person with a coconut, they trade and eat each other's coconuts: this yields y units of utility for each of them. The probability of finding such a trading partner during period t is given by $\frac{n_t}{\bar{n}}$.

We assume that there is no aggregate risk in each period. Among $(\bar{n} - n_t)$ people who are looking for palm trees, $(a_1 + a_2)(\bar{n} - n_t)$ of them actually find palm trees. For $a_1(\bar{n} - n_t)$ of them, the cost of climbing the tree is c_1, and for

[1] See Rogerson, Shimer, and Wright (2005) for an extensive survey on these models.

[2] The model itself is introduced in Hokari, Iimura, and Onuma (2006). Our contribution here is to show that it can be solved backwardly.

[3] The description of the model follows that of Romer (2006, Exercise 6.8, page 342). Note that Exercise 6.8 in Romer (2006) considers one type of palm trees whereas we consider two types.

others this cost is c_2. Among n_t people who are looking for other individuals who have coconuts, exactly $\frac{n_t}{\bar{n}} \cdot n_t$ of them actually find such trading partners.

Let β denote the common discount factor. Given a sequence $\{n_\tau\}_{\tau=t}^{T}$ of expected numbers of people who have coconuts at the beginning of each period, let V_t denote the maximum expected present discounted value of lifetime utility for an individual not carrying a coconut at the beginning of period t, and W_t the maximum expected present discounted value of lifetime utility for an individual carrying a coconut at the beginning of period t. Clearly,

$$\begin{aligned} V_T &= 0, \\ W_T &= \frac{n_T y}{\bar{n}}. \end{aligned}$$

For all $t < T$, the following Bellman-type equations hold:

$$\begin{aligned} V_t &= a_1 \max\{\beta V_{t+1}, -c_1 + \beta W_{t+1}\} + a_2 \max\{\beta V_{t+1}, -c_2 + \beta W_{t+1}\} \\ &\quad +(1 - a_1 - a_2)\beta V_{t+1}, \\ W_t &= \frac{n_t}{\bar{n}}\left[y + \beta V_{t+1}\right] + \left(1 - \frac{n_t}{\bar{n}}\right)\beta W_{t+1}. \end{aligned}$$

Note that the βV_{t+1} term in the first equation corresponds to a decision not to climb the tree to obtain a coconut.

Assuming that everyone has the same expectation for future values of n_t's, the relation between n_t and n_{t+1} can be described as follows:

(i) If $\beta V_{t+1} \leq -c_2 + \beta W_{t+1}$, then

$$n_{t+1} = (a_1 + a_2)(\bar{n} - n_t) + \left(1 - \frac{n_t}{\bar{n}}\right) n_t.$$

(ii) If $\beta V_{t+1} \leq -c_1 + \beta W_{t+1}$ and $\beta V_{t+1} > -c_2 + \beta W_{t+1}$, then

$$n_{t+1} = a_1(\bar{n} - n_t) + \left(1 - \frac{n_t}{\bar{n}}\right) n_t.$$

(iii) If $\beta V_{t+1} > -c_1 + \beta W_{t+1}$, then

$$n_{t+1} = \left(1 - \frac{n_t}{\bar{n}}\right) n_t.$$

Let $\texttt{IF}(\cdot)$ be a function such that for any statement A,

$$\texttt{IF}(A) \equiv \begin{cases} 1 & \text{if } A \text{ is true,} \\ 0 & \text{if } A \text{ is false.} \end{cases}$$

Let $b_t \equiv \frac{n_t}{\bar{n}}$, and

$$\alpha_t \equiv a_1 \mathtt{IF}(\beta V_{t+1} \leq -c_1 + \beta W_{t+1}) + a_2 \mathtt{IF}(\beta V_{t+1} \leq -c_2 + \beta W_{t+1}).$$

Then, the relation between b_t and b_{t+1} can be described by the following single equation:

$$b_{t+1} = \alpha_t(1 - b_t) + (1 - b_t)b_t.$$

To summarize, we are interested in solving the following system of equations: for all $t < T$,

$$\begin{aligned} V_t &= a_1 \max\{\beta V_{t+1}, -c_1 + \beta W_{t+1}\} + a_2 \max\{\beta V_{t+1}, -c_2 + \beta W_{t+1}\} \\ &\quad +(1 - a_1 - a_2)\beta V_{t+1}, & (1) \\ W_t &= b_t [y + \beta V_{t+1}] + (1 - b_t)\beta W_{t+1}, & (2) \\ \alpha_t &= a_1 \mathtt{IF}(\beta V_{t+1} \leq -c_1 + \beta W_{t+1}) + a_2 \mathtt{IF}(\beta V_{t+1} \leq -c_2 + \beta W_{t+1}), & (3) \\ b_{t+1} &= \alpha_t(1 - b_t) + (1 - b_t)b_t, & (4) \end{aligned}$$

and for $t = T$,

$$\begin{aligned} V_T &= 0, & (5) \\ W_T &= b_T y. & (6) \end{aligned}$$

Note that there are $(4T-1)$ variables, $\{V_t\}_{t=1}^T$, $\{W_t\}_{t=1}^T$, $\{\alpha_t\}_{t=1}^{T-1}$, and $\{b_t\}_{t=1}^T$, whereas there are $(4T-2)$ equations. So, we have to specify the value of one variable to solve the above system of equations. A natural choice for such a variable would be b_1. Then, the problem can be described as follows:

Problem 1. Given a value of $b_1 \in [0, 1]$, find $\{V_t\}_{t=1}^T$, $\{W_t\}_{t=1}^T$, $\{\alpha_t\}_{t=1}^{T-1}$, and $\{b_t\}_{t=2}^T$ that satisfy from (1) to (6).

Unlike usual dynamic programing problems with a single decision maker, this problem cannot be solved backwardly. It turns out if a value of b_T is given instead, then the corresponding problem can be solved backwardly.

Problem 2. Given a value of $b_T \in [0, 1]$, find $\{V_t\}_{t=1}^T$, $\{W_t\}_{t=1}^T$, $\{\alpha_t\}_{t=1}^{T-1}$, and $\{b_t\}_{t=1}^{T-1}$ that satisfy from (1) to (6).

Let us illustrate how to solve Problem 2. From (4), we get

$$b_t = \frac{1 - \alpha_t \pm \sqrt{(1 - \alpha_t)^2 - 4(b_{t+1} - \alpha_t)}}{2}.$$

We have to check whether $(1-\alpha_t)^2-4(b_{t+1}-\alpha_t) \geq 0$ and whether $0 \leq b_t \leq 1$. Suppose that $(1-\alpha_t)^2-4(b_{t+1}-\alpha_t) \geq 0$. Then

$$\begin{aligned} & \frac{1-\alpha_t+\sqrt{(1-\alpha_t)^2-4(b_{t+1}-\alpha_t)}}{2} \leq 1 \\ \Leftrightarrow \quad & 1-\alpha_t+\sqrt{(1-\alpha_t)^2-4(b_{t+1}-\alpha_t)} \leq 2 \\ \Leftrightarrow \quad & \sqrt{(1-\alpha_t)^2-4(b_{t+1}-\alpha_t)} \leq 1+\alpha_t \\ \Leftrightarrow \quad & (1-\alpha_t)^2-4(b_{t+1}-\alpha_t) \leq (1+\alpha_t)^2 \\ \Leftrightarrow \quad & b_{t+1} \geq 0. \end{aligned}$$

Thus,

$$(1-\alpha_t)^2-4(b_{t+1}-\alpha_t) \geq 0 \quad \Rightarrow \quad \frac{1-\alpha_t+\sqrt{(1-\alpha_t)^2-4(b_{t+1}-\alpha_t)}}{2} \leq 1.$$

Also, we have

$$\left.\begin{array}{c} (1-\alpha_t)^2-4(b_{t+1}-\alpha_t) \geq 0 \\ b_{t+1} \geq \alpha_t \end{array}\right\} \quad \Rightarrow \quad 0 \leq \frac{1-\alpha_t-\sqrt{(1-\alpha_t)^2-4(b_{t+1}-\alpha_t)}}{2}.$$

Thus,

$$\alpha_t \leq b_{t+1} \leq \frac{(1+\alpha_t)^2}{4} \quad \Rightarrow \quad 0 \leq \frac{1-\alpha_t \pm \sqrt{(1-\alpha_t)^2-4(b_{t+1}-\alpha_t)}}{2} \leq 1.$$

If $b_{t+1} < \alpha_t$, then $(1-\alpha_t)^2-4(b_{t+1}-\alpha_t) \geq 0$ and

$$\frac{1-\alpha_t-\sqrt{(1-\alpha_t)^2-4(b_{t+1}-\alpha_t)}}{2} < 0 \leq \frac{1-\alpha_t+\sqrt{(1-\alpha_t)^2-4(b_{t+1}-\alpha_t)}}{2} \leq 1.$$

Suppose that α_t and b_{t+1} are given. If $\frac{(1+\alpha_t)^2}{4} < b_{t+1}$, then there is no $b_t \in [0,1]$ that satisfies (4). If $\alpha_t \leq b_{t+1} \leq \frac{(1+\alpha_t)^2}{4}$, there are two values of $b_t \in [0,1]$ that satisfy (4): $b_t = \frac{1-\alpha_t \pm \sqrt{(1-\alpha_t)^2-4(b_{t+1}-\alpha_t)}}{2}$. If $b_{t+1} < \alpha_t$, then there is only one value of $b_t \in [0,1]$ that satisfies (4): $b_t = \frac{1-\alpha_t+\sqrt{(1-\alpha_t)^2-4(b_{t+1}-\alpha_t)}}{2}$.

Consider the following example.

Example 1.

$$\begin{aligned} a_1 &= 0.25, \\ a_2 &= 0.575, \\ c_1 &= 10, \\ c_2 &= 29, \\ \beta &= 0.9, \\ y &= 40, \\ b_T &= 0.375. \end{aligned}$$

In the Appendix, we compute a unique solution to Problem 2. It turns out that this solution exhibits a three-period cycle.

3 A concluding remark

We tend think it is reasonable to assume that the state variable of the initial period is known. As a result, we tend to think Problem 1 is more natural than Problem 2. However, it is also natural to face a problem in which we can set the initial value of a variable, and we are given a value of the variable that should be attained when the process ends. In such a situation, we solve the system backwardly to find what initial value is needed just like in Problem 2. Furthermore, as mentioned above, there is no systematic way to solve Problem 1 other than simple "guess-and-verify" approach. In this note, we have argued that we should shift our attention from Problem 1 to Problem 2, which can be solved backwardly. If our primary concern is to understand what kind of equilibria are possible, then Problem 2 does not look so unnatural. It should be also noted that our approach can be applied to many other models. Applying our approach to these models, and investigating what kind of equilibria (including non-stationary ones) are possible will be our next project.

References

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[3] Rogerson, R., R. Shimer, and R. Wright (2005), "Search-theoretic models of the labor market: a survey" *Journal of Economic Literature* **43**, 959–988.

[4] Romer, D. (2006) *Advanced Macroeconomics*, 3rd edition, McGraw-Hill/Irwin.

Appendix

Consider the following example:

Example 1.

$$\begin{aligned} a_1 &= 0.25, \\ a_2 &= 0.575, \\ c_1 &= 10, \\ c_2 &= 29, \\ \beta &= 0.9, \\ y &= 40, \\ b_T &= 0.375. \end{aligned}$$

We compute a unique solution to Problem 2.

From (5) and (6), $V_T = 0$ and $W_T = b_T y = 0.375 \times 40 = 15$. Note that

$$\begin{aligned} -c_1 + \beta W_T - \beta V_T &= -10 + 0.9 \times 15 - 0 = 3.5 > 0, \\ -c_2 + \beta W_T - \beta V_T &= -29 + 0.9 \times 15 - 0 = -15.5 < 0. \end{aligned}$$

Thus, from (1),

$$\begin{aligned} V_{T-1} &= a_1(-c_1 + \beta W_T) + (1 - a_1)\beta V_T = 0.25 \times 3.5 + 0 = 0.875, \\ \alpha_{T-1} &= a_1 = 0.25, \\ \frac{(1 + \alpha_{T-1})^2}{4} &= \frac{(1 + 0.25)^2}{4} = 0.390625, \end{aligned}$$

so that we have

$$\alpha_{T-1} < b_T < \frac{(1 + \alpha_{T-1})^2}{4}.$$

Thus, there are two values of $b_{T-1} \in [0, 1]$ that satisfy (4):

$$b_{T-1} = \frac{1 - \alpha_{T-1} \pm \sqrt{(1 - \alpha_{T-1})^2 - 4(b_T - \alpha_{T-1})}}{2}.$$

Since $\sqrt{(1 - \alpha_{T-1})^2 - 4(b_T - \alpha_{T-1})} = \sqrt{(1 - 0.25)^2 - 4(0.375 - 0.25)} = \sqrt{0.0625} = 0.25$, $b_{T-1} \in \{0.5, 0.25\}$.

If we choose $b_{T-1} = 0.5$, then

$$\begin{aligned} W_{T-1} &= 0.5 \times (40+0) + (1-0.5) \times 0.9 \times 15 = 26.75, \\ -c_1 + \beta W_{T-1} - \beta V_{T-1} &= -10 + 0.9 \times 26.75 - 0.9 \times 0.875 = 13.2875 > 0, \\ -c_2 + \beta W_{T-1} - \beta V_{T-1} &= -29 + 0.9 \times 26.75 - 0.9 \times 0.875 = -5.7125 < 0, \\ \alpha_{T-2} &= a_1 = 0.25, \\ \frac{(1+\alpha_{T-2})^2}{4} &= \frac{(1+0.25)^2}{4} = 0.390625, \end{aligned}$$

so that we have

$$\frac{(1+\alpha_{T-2})^2}{4} < b_{T-1}.$$

Thus, if we choose $b_{T-1} = 0.5$, then there is no b_{T-2} that precedes it.

If we choose $b_{T-1} = 0.25$, then

$$\begin{aligned} W_{T-1} &= 0.25 \times (40+0) + (1-0.25) \times 0.9 \times 15 = 20.125, \\ -c_1 + \beta W_{T-1} - \beta V_{T-1} &= -10 + 0.9 \times 20.125 - 0.9 \times 0.875 = 17.325 > 0, \\ -c_2 + \beta W_{T-1} - \beta V_{T-1} &= -29 + 0.9 \times 20.125 - 0.9 \times 0.875 = -1.675 < 0, \\ \alpha_{T-2} &= a_1 = 0.25, \\ \frac{(1+\alpha_{T-2})^2}{4} &= \frac{(1+0.25)^2}{4} = 0.390625, \end{aligned}$$

so that we have

$$\alpha_{T-2} = b_{T-1} < \frac{(1+\alpha_{T-2})^2}{4}.$$

Thus, if we choose $b_{T-1} = 0.25$, then there are two values of b_{T-2} that satisfy (4):

$$b_{T-2} = \frac{1-\alpha_{T-2} \pm \sqrt{(1-\alpha_{T-2})^2 - 4(b_{T-1} - \alpha_{T-2})}}{2}.$$

Since $\sqrt{(1-\alpha_{T-2})^2 - 4(b_{T-1} - \alpha_{T-2})} = \sqrt{(1-0.25)^2 - 4(0.25-0.25)} = 0.75$, $b_{T-2} \in \{0.75, 0\}$.

Since

$$\begin{aligned} V_{T-2} &= a_1(-c_1 + \beta W_{T-1}) + (1-a_1)\beta V_{T-1} \\ &= 0.25 \times (-10 + 0.9 \times 20.125) + 0.75 \times 0.9 \times 0.875 \\ &= 2.61875, \end{aligned}$$

if we choose $b_{T-2} = 0$, then

$$\begin{aligned} W_{T-2} &= 0 + 0.9 \times 20.125 = 18.1125, \\ -c_1 + \beta W_{T-2} - \beta V_{T-2} &= -10 + 0.9 \times 18.1125 - 0.9 \times 2.61875 = 23.944375 > 0, \\ -c_2 + \beta W_{T-2} - \beta V_{T-2} &= -29 + 0.9 \times 18.1125 - 0.9 \times 2.61875 = -15.055625 < 0, \\ \alpha_{T-3} &= a_1 = 0.25, \\ \frac{(1+\alpha_{T-3})^2}{4} &= \frac{(1+0.25)^2}{4} = 0.390625, \end{aligned}$$

so that we have $b_{T-2} < \alpha_{T-3}$. Thus, there is a unique $b_{T-3} \in [0,1]$ that precedes $b_{T-2} = 0$:

$$b_{T-3} = \frac{1 - \alpha_{T-3} + \sqrt{(1-\alpha_{T-3})^2 - 4(b_{T-2} - \alpha_{T-3})}}{2} = 1.$$

But then $b_{T-3} = 1 > 0.83265625 = \frac{(1+a_1+a_2)^2}{4} \geq \frac{(1+\alpha_{T-4})^2}{4}$, so that there is no $b_{T-4} \in [0,1]$ that precedes $b_{T-3} = 1$. Thus, if we choose $b_{T-2} = 0$, then $b_{T-3} = 1$ but there is no $b_{T-4} \in [0,1]$ that precedes it.

If we choose $b_{T-2} = 0.75$, then

$$\begin{aligned} W_{T-2} &= 0.75 \times (40 + 0.9 \times 0.875) + 0.25 \times 0.9 \times 20.125 = 35.11875, \\ -c_1 + \beta W_{T-2} - \beta V_{T-2} &= -10 + 0.9 \times 35.11875 - 0.9 \times 2.61875 = 19.25 > 0, \\ -c_2 + \beta W_{T-2} - \beta V_{T-2} &= -29 + 0.9 \times 35.11875 - 0.9 \times 2.61875 = 0.25 > 0, \\ \alpha_{T-3} &= a_1 + a_2 = 0.25 + 0.575 = 0.825, \\ \frac{(1+\alpha_{T-3})^2}{4} &= \frac{(1+0.825)^2}{4} = 0.83265625, \end{aligned}$$

so that we have $b_{T-2} < \alpha_{T-3}$. Thus, if $b_{T-2} = 0.75$,

$$b_{T-3} = \frac{1 - \alpha_{T-3} + \sqrt{(1-\alpha_{T-3})^2 - 4(b_{T-2} - \alpha_{T-3})}}{2} = 0.375$$

is the unique $b_{T-3} \in [0,1]$ that precedes it.

So far, we have shown that given $b_T = 0.375$, $(b_{T-1}, b_{T-2}, b_{T-3}) = (0.25, 0.75, 0.375)$ is the only path that survives.

Note that

$$\begin{aligned}
V_{T-3} &= -a_1c_1 - a_2c_2 + (a_1+a_2)\beta W_{T-2} + (1-a_1-a_2)\beta V_{T-2} \\
&= -0.25 \times 10 - 0.575 \times 29 + 0.825 \times 0.9 \times 35.11875 \\
&\quad +0.175 \times 0.9 \times 2.61875 \\
&= 7.313125, \\
W_{T-3} &= 0.375 \times (40 + 0.9 \times 2.61875) + 0.625 \times 0.9 \times 35.11875 \\
&= 35.638125, \\
-c_1 + \beta W_{T-3} - \beta V_{T-3} &= -10 + 0.9 \times (35.638125 - 7.313125) = 15.4925 > 0, \\
-c_2 + \beta W_{T-3} - \beta V_{T-3} &= -29 + 0.9 \times (35.638125 - 7.313125) = -3.5075 < 0, \\
\alpha_{T-4} &= a_1 = 0.25, \\
\frac{(1+\alpha_{T-4})^2}{4} &= \frac{(1+0.25)^2}{4} = 0.390625,
\end{aligned}$$

so that we have

$$\alpha_{T-4} < b_{T-3} < \frac{(1+\alpha_{T-4})^2}{4}.$$

Thus, there are two values of $b_{T-4} \in [0,1]$ that satisfy (4): $b_{T-4} \in \{0.5, 0.25\}$.

Since

$$\begin{aligned}
V_{T-4} &= a_1(-c_1 + \beta W_{T-3}) + (1-a_1)\beta V_{T-3} \\
&= 0.25 \times (-10 + 0.9 \times 35.638125) + 0.75 \times 0.9 \times 7.313125 \\
&= 10.4549375,
\end{aligned}$$

if we choose $b_{T-4} = 0.5$, then

$$\begin{aligned}
W_{T-4} &= 0.5 \times (40 + 0.9 \times 7.313125) + 0.5 \times 0.9 \times 35.638125 \\
&= 39.3280625, \\
-c_1 + \beta W_{T-4} - \beta V_{T-4} &= -10 + 0.9 \times (35.638125 - 10.4549375) = 12.66486875 > 0, \\
-c_2 + \beta W_{T-4} - \beta V_{T-4} &= -29 + 0.9 \times (35.638125 - 10.4549375) = -6.33513125 < 0, \\
\alpha_{T-5} &= a_1 = 0.25, \\
\frac{(1+\alpha_{T-5})^2}{4} &= \frac{(1+0.25)^2}{4} = 0.390625,
\end{aligned}$$

so that we have

$$\frac{(1+\alpha_{T-5})^2}{4} < b_{T-4}.$$

Thus, if we choose $b_{T-4} = 0.5$, then there is no b_{T-5} that precedes it.

If we choose $b_{T-4} = 0.25$, then

$$\begin{aligned} W_{T-4} &= 0.25 \times (40 + 0.9 \times 7.313125) + 0.75 \times 0.9 \times 35.638125 \\ &= 35.7011875, \\ -c_1 + \beta W_{T-4} - \beta V_{T-4} &= -10 + 0.9 \times (35.7011875 - 10.4549375) = 12.721625 > 0, \\ -c_2 + \beta W_{T-4} - \beta V_{T-4} &= -29 + 0.9 \times (35.7011875 - 10.4549375) = -6.278375 < 0, \\ \alpha_{T-5} &= a_1 = 0.25, \\ \frac{(1+\alpha_{T-5})^2}{4} &= \frac{(1+0.25)^2}{4} = 0.390625, \end{aligned}$$

so that we have

$$\alpha_{T-5} = b_{T-4} < \frac{(1+\alpha_{T-5})^2}{4}.$$

Thus, if we choose $b_{T-4} = 0.25$, then there are two values of b_{T-5} that satisfy (4): $b_{T-5} \in \{0.75, 0\}$.

Since

$$\begin{aligned} V_{T-5} &= a_1(-c_1 + \beta W_{T-4}) + (1-a_1)\beta V_{T-4} \\ &= 0.25 \times (-10 + 0.9 \times 35.7011875) + 0.75 \times 0.9 \times 10.4549375 \\ &= 12.58975, \end{aligned}$$

if we choose $b_{T-5} = 0$, then

$$\begin{aligned} W_{T-5} &= 0 + 0.9 \times 35.7011875 = 32.13106875, \\ -c_1 + \beta W_{T-5} - \beta V_{T-5} &= -10 + 0.9 \times (32.13106875 - 12.58975) = 7.587186875 > 0, \\ -c_2 + \beta W_{T-2} - \beta V_{T-2} &= -29 + 0.9 \times (32.13106875 - 12.58975) = -11.41281313 < 0, \\ \alpha_{T-6} &= a_1 = 0.25, \\ \frac{(1+\alpha_{T-6})^2}{4} &= \frac{(1+0.25)^2}{4} = 0.390625, \end{aligned}$$

so that we have $b_{T-5} < \alpha_{T-6}$. Thus, there is a unique $b_{T-6} \in [0,1]$ that precedes $b_{T-5} = 0$: $b_{T-6} = 1$. But then $b_{T-6} = 1 > 0.83265625 = \frac{(1+a_1+a_2)^2}{4} \geq \frac{(1+\alpha_{T-7})^2}{4}$, so that there is no $b_{T-7} \in [0,1]$ that precedes $b_{T-6} = 1$. Thus, if we choose $b_{T-5} = 0$, then $b_{T-6} = 1$ but there is no $b_{T-7} \in [0,1]$ that precedes it.

If we choose $b_{T-5} = 0.75$, then

$$\begin{aligned}
W_{T-5} &= 0.75 \times (40 + 0.9 \times 10.4549375) + 0.25 \times 0.9 \times 35.7011875 \\
&= 45.08985, \\
-c_1 + \beta W_{T-5} - \beta V_{T-5} &= -10 + 0.9 \times (45.08985 - 12.58975) = 19.25009 > 0, \\
-c_2 + \beta W_{T-5} - \beta V_{T-5} &= -29 + 0.9 \times (45.08985 - 12.58975) = 0.25009 > 0, \\
\alpha_{T-6} &= a_1 + a_2 = 0.25 + 0.575 = 0.825, \\
\frac{(1+\alpha_{T-6})^2}{4} &= \frac{(1+0.825)^2}{4} = 0.83265625,
\end{aligned}$$

so that we have $b_{T-5} < \alpha_{T-6}$. Thus, if we choose $b_{T-2} = 0.75$,

$$b_{T-6} = \frac{1 - \alpha_{T-6} + \sqrt{(1-\alpha_{T-6})^2 - 4(b_{T-5} - \alpha_{T-6})}}{2} = 0.375$$

is the unique $b_{T-6} \in [0,1]$ that precedes it.

Thus, we have shown that given $b_T = 0.375$,

$$(b_{T-1}, b_{T-2}, b_{T-3}, b_{T-4}, b_{T-5}, b_{T-6}) = (0.25, 0.75, 0.375, 0.25, 0.75, 0.375)$$

is the only path that survives. Note that the corresponding path of α_t's is

$$(\alpha_{T-1}, \alpha_{T-2}, \alpha_{T-3}, \alpha_{T-4}, \alpha_{T-5}, \alpha_{T-6}) = (0.25, 0.25, 0.825, 0.25, 0.25, 0.825).$$

Thus, both the path of b_t's and that of α_t's exhibit a three-period cycle.

It turns out that these paths can be extended for longer periods.[4]

[4] We have used Excel to check that this claim is valid at least for 50 periods. The Excel file is available from the authors on request.