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Preferences with Open Graphs: A New Result

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Abstract

We provide a new characterization of binary relations that have an open graph. This result is applicable to any binary relation on any topological space. The relationship of this new theorem to several known results for binary relations used for preference representation, namely strict partial orders (SPOs), weak orders (WOs), and preorders is described.

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1. Introduction

In the representation of preferences, various types of binary relations (on some set X), including, in particular, transitive relations such as strict partial orders (SPOs) and weak orders (WOs), play an important role; see e.g. Bridges & Mehta (1995). When X is a topological space, such orders – and binary relations in general – may have certain nice properties, e.g. an open-graph, or open-sections (defined below).

The open-graph property is stronger than the open-section property: if a binary relation B on a space X has an open graph, then B also has the open-sections property, but not conversely, in general. Precise circumstances or additional conditions under which a binary relation with the open-sections property will also then have the open-graph property have been established. Thus we know that in a WO the two properties are equivalent; and if R is an SPO that is either order-dense, or has a transitive complement R^{C}, then the two properties are also equivalent for R.

Recently Gerasimou (2013) has shown that for a preorder J on a space X, J^{C} has an open graph if and only if J^{C} has open sections and if J^{C} has a special property called "expansion of local transitivity". See Theorem 4 of Gerasimou (2013).

In this paper we prove a new characterization theorem applicable to any binary relation on any topological space X. The relationship of our theorem to each of the previous results cited above is described.

2. Preliminaries

DEFINITIONS.

A **binary relation** B on a set (resp. space) X is a subset (resp. subspace) of the set (resp. product space) $\mathrm{X} \times \mathrm{X}$, i.e. $B \subseteq (\mathrm{X} \times \mathrm{X})$. In the sequel, the relation $(x, y) \in B$ is often denoted xBy.

A binary relation B is **transitive** if for all x, y, z in X, xBy & yBz implies xBz.

A **strict partial order** ("SPO") is an asymmetric, transitive binary relation on a set X.

A **weak order** ("WO") W on a set X is a SPO in which W^{C}, the complement of W in $\mathrm{X} \times \mathrm{X}$, is transitive.

A **preorder** on a set X is a reflexive and transitive binary relation J on X.

Let B be a binary relation on a topological space X. We say that:

- B has **open sections** (or the open-sections property) if for each x in X, both the *upper section* $U_B(x) = \{y \in \mathrm{X}\text{: } yBx\}$ as well as the lower section $L_B(x) = \{y \in \mathrm{X}\text{: } xBy\}$ are X - open.

- B has an **open graph** (or the open-graph property) if B is an open subset of the product space X×X.
- B is **order dense** if for all $(x, y) \in B$ there exists some $z \in X$ such that xBz and zBy.

We denote by $N(x)$ a neighborhood of x, and by Int(B) the interior of B.

For elaboration of definitions and of the underlying ideas, see Bridges & Mehta (1995) and any standard text on general topology, e.g. Munkres, (2000).

3. The Results

We turn to a characterization of the open-graph property for arbitrary binary relations.

REMARK 1. Let B be a binary relation on a space X. If B has an open graph then B has open sections. See e.g. p.265 of Bergstrom *et al.* (1976).

REMARK 2. For an example of a binary relation R with open sections that fails to have an open graph, see p.266 of Bergstrom *et al.* (1976).

REMARK 3. Let P be a transitive binary relation with open sections. If xPz and zPy, then $(x, y) \in \text{Int}(P)$. See p.265, Bergstrom *et al.* (1976)

We now introduce

CONDITION (‡). A binary relation R on a space X satisfies Condition (‡) if:

For each $(x,y) \in R$ and sections $U_R(y)$ and $L_R(x)$, there exist X – open sets A and B in X with the property that

$$(x,y) \in ([U_R(y) \cap \mathrm{A}] \times [L_R(x) \cap \mathrm{B}]) \subseteq R,$$

LEMMA I. Let R be a binary relation on a space X.

(a) If R has an open graph, then R satisfies Condition (‡).

(b) If R is transitive, has open sections, and if R^{C} is transitive, then R satisfies Condition (‡).

(c) If R is a weak order with open sections, then R satisfies Condition (‡).

(d) If R is transitive, order-dense and has open sections, then R satisfies Condition (‡)

Proof. (a) Fix $(x,y) \in R$. By remark 1, each section is X - open, implying that $[U_R(y) \times L_R(x)] \cap R$ is open in $\mathrm{X} \times \mathrm{X}$ with respect to the product topology. Since $(x,y) \in [U_R(y) \times L_R(x)] \cap R$, it follows that there exist X - open sets $N(x)$ and $N(y)$ such that $(x,y) \in N(x) \times N(y) \subseteq [U_R(y) \times L_R(x)] \cap R$. Therefore $U_R(y) \times N(x)$ and $L_R(x) \times N(y)$ are X – open, and

$$(x,y) \in [U_R(y) \cap N(x)] \times [L_R(x) \cap N(y)] \subseteq R$$

implying that Condition (‡) is satisfied.

(b) Fix an arbitrary $(x,y) \in R$. Since $(x,y) \in U_R(y) \times L_R(x)$, and $U_R(y)$ and $L_R(x)$ are X – open, it follows that Condition (‡) will obviously be satisfied if $[U_R(y) \times L_R(x)] \setminus R = \varnothing$. So suppose that $[U_R(y) \times L_R(x)] \setminus R \neq \varnothing$ and choose $(x^*, y^*) \in [U_R(y) \times L_R(x)] \setminus R$. Then $(x, x^*) \in R$. (Otherwise $(x, x^*) \in R^{\mathrm{C}}$ so that, in conjunction with $(x^*, y^*) \in R^{\mathrm{C}}$, it follows that $(x, y^*) \in R^{\mathrm{C}}$ by transitivity of R^{C}, contradicting the assumption that $y^* \in L_R(x)$). Since $x^* \in U_R(y)$, it follows from remark 3 that $(x,y) \in \mathrm{Int}(R)$. Choose X – open sets $N(x)$ and $N(y)$ such that $(x,y) \in N(x) \times N(y) \subseteq R$. Then $U_R(y) \cap N(x)$ and $L_R(x) \cap N(y)$ are X – open and

$$(x,y) \in [U_R(y) \cap N(x)] \times [L_R(x) \cap N(y)] \subseteq R$$

implying that Condition (‡) is satisfied.

(c) This is just a special case of (b).

(d) Follows from remark 3 and definition of order-denseness. □

Here is our main result.

THEOREM. The following are equivalent for any binary relation R on space X:

(i) R has an open graph.

(ii) R has open sections, and satisfies condition (‡).

Proof. (i)$\Rightarrow$(ii): Follows from Lemma I(a) and remark 1 above.

(ii)$\Rightarrow$(i): Choose $(x,y) \in R$. Applying Condition (‡), there exist X – open sets A, B in X with the property that

$$(x,y) \in [U_R(y) \cap \mathrm{A}] \times [L_R(x) \cap \mathrm{B}] \subseteq R$$

Since $U_R(y)$ and $L_R(x)$ are X – open, it follows that $[U_R(y) \cap \mathrm{A}]$ and $[L_R(x) \cap \mathrm{B}]$ are X – open, and we conclude that R is X - open. □

COROLLARY. If R is a transitive binary relation, e.g. a SPO, then R has an open graph if and only if it has open sections and satisfies Condition (‡).

REMARK 4. If W is a weak order then (in particular) W and W^{C} are transitive, so we recover the following result of Bergstrom *et al.* (1976) from our Lemma I(b) and Theorem above: If W is a weak order, then W has an open graph if and only if W has open sections.

REMARK 5. Lemma I(d) together with our Theorem yields the following result of Bergstrom *et al.* (1976): If R is a transitive and order-dense binary relation on a space X, then: R has open graph if and only if R has open sections.

REMARK 6. The property of "expansion of local transitivity" for a preorder J, as defined in Gerasimou's Theorem 4 [p.162 of Gerasimou (2013)] implies that if J has this property, then J^{C} is transitive. Thus it follows from our

Lemma I(b), and the fact that $(J^C)^C = J$, that if J^C has open sections, then J^C satisfies our Condition (‡). Gerasimou's result then follows from our Theorem.

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