Economics Bulletin

Volume 37, Issue 3

A fractal analysis of world stock markets

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Abstract

This paper provides a fractal analysis of world stock prices. We find that the majority of stock prices are fractal.

Citation: Taro Ikeda, (2017) "A fractal analysis of world stock markets", *Economics Bulletin*, Volume 37, Issue 3, pages 1514-1532
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Submitted: December 22, 2016. **Published:** July 02, 2017.

1 Introduction

The recent financial crisis suggests the failure of the rational expectations of economic agents. Rational expectations are a key concept for the efficient market hypothesis (i.e., the random walk model), where rational agents process all available information to forming stock price forecasts, and therefore prices only change in response to new shocks, and are not correlated with past shocks.

This paper contributes to the macroeconomics literature using the fractal geometry established by Benoit, B., Mandelbrot.[1] The important difference between fractal geometry and the random walk is the assumption surrounding the distribution. A bell-shaped distribution for the random walk cannot account for the risk associated with very large deviations of stock prices from their mean values, and therefore the efficient market hypothesis fails to consider the big-break-down of the stock markets. However, the power-law distribution used in fractal geometry gives large deviations higher probabilities and can capture black swan events more easily than can the random walk model.

This paper provides a fractal analysis of world stock markets to test the efficient market hypothesis empirically. The results suggest that 82% of world stock prices are consistent with the fractal market hypothesis, rather than the efficient market hypothesis.

This paper proceeds as follows. Section 2 provides a brief introduction to fractal geometry and the estimation method of determining the fractal dimension. Section 3 presents the empirical results. Section 4 concludes.

2 Methodology

A fractional Brownian motion, formally introduced by Mandelbrot, generalizes the concept of the ordinary Brownian motion (corresponding to a random walk). In this section, we provide an intuitive explanation of fractional Brownian motion, and a brief summary of Hurst rescaled range analysis to estimate the fractal dimension.

Fractional Brownian motion and random walks

An increment of the fractional Brownian motion $\omega(t)$ for any two times t and t_0 is described as $|\omega(t) - \omega(t_0)|$. Then, the variance of the increment can be written as:

$$\langle |\omega(t) - \omega(t_0)|^2 \rangle \propto |t - t_0|^{2H}. \tag{1}$$

Here, $\langle\rangle$ implies an ensemble average, and H is the Hurst exponent. It is important to note that H takes a value between zero and one, and the random walk corresponds to the case of $H = 0.50$.

Accordingly, if $H = 0.50$, equation (1) reduces to the variance equation of the random walk as follows:

$$\langle |\omega(t) - \omega(t_0)|^2 \rangle \propto |t - t_0|. \tag{2}$$

It is apparent that the random walk is a special case of the fractional Brownian motion.

[1] The purpose of this paper is not to provide the best predictor of stock prices, but to determine whether stock prices are fractal.

By equations (1) and (2), the variance of the fractional Brownian motion diverges with time, and the rate is faster than the random walk for the case of $0.50 < H < 1.00$.

The statistical Hurst exponent and R/S analysis

As explained above, the Hurst exponent is a key factor in distinguishing between fractal series and random walk series. To obtain the statistical Hurst exponent, we use R/S analysis.[2][3]

Let the sequence of the first difference of a time series in logarithms be $x(t) \in \big(x(1), ..., x(T)\big)$. Then we can describe the accumulated sum of dispersion of $x(t)$ from its average by window size τ as follows:

$$x(t,\tau) = \sum_{t=1}^{T} \big(x(t) - \langle x(t)|\tau \rangle\big), \tag{3}$$

where $\langle x(t)|\tau \rangle$ is the average of $x(t)$, conditioned by the window scale τ.

Furthermore, let $R(\tau)$ be the range for the accumulated average $x(t,\tau)$ over the window:

$$R(\tau) = \max_{\tau(l-1)+1 \le t \le \tau l} x(t,\tau) - \min_{\tau(l-1)+1 \le t \le \tau l} x(t,\tau), \tag{4}$$

where $l = 1 : T/\tau$. Clearly, the range R is an increasing function of τ.

Then, the R/S statistics can be obtained by dividing equation (4) with the standard deviation S of $x(t)$ by window size τ, the distribution of which can be described as:

$$E_\tau(R(\tau)/S(\tau)) \propto \tau^H. \tag{5}$$

Here, the power H has a direct relationship with the self-affine fractal dimension $D = 2 - H$.[4]

Using Hurst's H, we test the fractal market hypothesis against the efficient market hypothesis (corresponding to $H = 0.50$). The fractal market hypothesis is valid if H takes a value of more than 0.50. The case $0.50 < H < 1.00$, labeled as *long memory dependence*, suggests a positive autocorrelation of the time series implying that shocks in the distant past never disappear, and affect the series forever. In this situation, an increasing trend can suddenly become decreasing, and the effect is massive because fractal geometry assumes a power distribution instead of the well-utilized bell-shaped distribution. In this sense, fractal geometry considers the possibility of a large decrease in stock prices following a black swan event.[5]

[2]Hurst (1955) examines the storage capacity of reservoirs for a large number of rivers using R/S analysis.

[3]See Booth et al. (1982) for a formal R/S analysis of exchange rates for European countries.

[4]I would like to thank Tamàs Vicsek, Shinichi Sato, and Jun-ichi Wakita for making this point.

[5]The case $0 < H < 0.50$ corresponds to *antipersistence*. In this state, an increase in the value of a series today implies a decrease tomorrow.

3 Empirical results

This section presents empirical Hurst exponents for almost all available stock indexes around the world. The data frequency is weekly to ensure a large sample size. All data are taken from the Thomson Reuters Datastream.[6]

Furthermore, we estimate nonparametric bootstrapped confidence intervals using 5555 resamples in the regression of H. This is because the variance for fBms asymptotically diverges when H is greater than 0.50. Here, we restrict ourselves to stating that the stock index is fractal (fBm) only if the 95% lower interval exceeds the value 0.50.

3.1 Benchmark results

Table 1 presents the estimates of the Hurst exponents. We find that 82% of the stock indexes are fractal.[7] The majority of the stock prices are well described by the fractal market hypothesis rather than the efficient market hypothesis. Actual stock prices are more persistent than random walks, but the corruptions of the markets are more drastic than suggested by the efficient market hypothesis. The ordinary risk measure increases quickly in the case of the power-law distribution (see Section 2).

3.2 τ-Dependence for the Hurst exponents

Following Feder (1988, Chap. 11), this subsection considers the possibility of time dependence in the Hurst exponent. This exercise is clearly important because the exponent specifies the shape of the power-law distribution. To show τ-dependence for the Hurst exponents, we run sequential ordinary least squares on τ.

The results are presented in Figure 1. One general finding here is that the Hurst exponents appear to be time varying for all stock indexes, implying that the shape of the power-law distribution of stock prices changes gradually over time.

4 Conclusion

This paper examined the fractal market hypothesis using world stock prices. We find that the majority of stock prices are fractal.

References

Booth, Geoffrey G., Kaen, Fred R., Koveos, Peter E., (1982) "R/S analysis of foreign exchange rates under two international monetary regimes." *Journal of Monetary Economics*, 10, 407–415.

Feder, Jens, (1988) *Fractals*, Plenum Press, New York.

Hurst, Edwin H., (1955) "Methods of using long-term storage in reservoirs." *Proceedings of the Institution of Civil Engineers*, part 1, 519–577.

[6] See the appendix for a summary of the data.

[7] The total number of indexes is 137.

Figures and Tables

Table 1: Hurst exponents by stock index

No.	Stock Indexes	Hurst exponent	Lower	Upper
1	AEX ALL SHARE	0.64541	0.56983	0.77422
2	AEX INDEX (AEX)	0.57232	0.51376	0.64122
3	AMMAN SE FINANCIAL MARKET	0.66556	0.56642	0.74331
4	ARGENTINA MERVAL	0.62060	0.50216	0.71938
5	ATHEX COMPOSITE	0.60772	0.52794	0.69870
6	ATX - AUSTRIAN TRADED INDEX	0.52467	0.42402	0.58503
7	BANGKOK S.E.T.	0.54209	0.43788	0.56297
8	BEL 20	0.56239	0.52357	0.97901
9	BIST NATIONAL 100	0.63146	0.51727	0.72228
10	BOTSWANA SE DMS COS. IDX.	0.75837	0.69564	0.83007
11	BUDAPEST (BUX)	0.54051	0.45160	0.56588
12	BULGARIA SE SOFIX	0.76258	0.66515	0.84824
13	CHILE SANTIAGO SE GENERAL (IGPA)	0.66872	0.63448	0.79869
14	COLOMBIA IGBC INDEX	0.73974	0.64365	0.84146
15	COLOMBO SE ALL SHARE	0.63117	0.52676	0.65627
16	CROATIA CROBEX	0.60144	0.55711	0.67995
17	CYPRUS GENERAL	0.74089	0.64006	0.83102
18	DAX 30 PERFORMANCE	0.52707	0.51396	0.62678
19	DJGL WORLD $	0.59757	0.54556	0.69497
20	DOW JONES COMPOSITE 65 STOCK AVE	0.56675	0.51787	0.59305
21	DOW JONES INDUSTRIALS	0.57509	0.53975	0.64258
22	DOW JONES UTIILITIES	0.58453	0.50436	0.67088
23	EGYPT HERMES FINANCIAL	0.67948	0.62106	0.77390
24	EURO STOXX	0.57708	0.52630	0.62804
25	EURO STOXX 50	0.61485	0.50225	0.73378
26	EURONEXT 100	0.58639	0.51640	0.79614
27	FRANCE CAC 40	0.58275	0.52486	0.69122
28	FTSE 100	0.55657	0.46819	0.61965
29	FTSE 250	0.47654	0.20570	0.60497
30	FTSE ALL SHARE	0.63475	0.55636	0.66618
31	FTSE ALL WORLD $	0.60501	0.56431	0.67413
32	FTSE BURSA MALAYSIA KLCI	0.47815	0.43336	0.57125
33	FTSE CHINA B 35	0.57947	0.50896	0.66874
34	FTSE EUROTOP 100 E	0.57231	0.51819	0.63764
35	FTSE GLOBAL 100 ($)	0.70264	0.62738	0.80222
36	FTSE LATIBEX ALL SHARE	0.67719	0.58372	0.80686
37	FTSE MIB INDEX	0.54894	0.51696	0.82517
38	FTSE MULTINATIONALS ($)	0.65933	0.63521	0.77639
39	FTSE TECHMARK FOCUS (£)	0.74248	0.62329	0.85032

40	FTSE/ATHEX LARGE CAP	0.62635	0.51075	0.72692
41	FTSE/JSE ALL SHARE	0.49640	0.46407	0.56932
42	FTSEUROFIRST 100 E	0.63286	0.52013	0.71142
43	FTSEUROFIRST 80 E	0.61251	0.56247	0.72648
44	HANG SENG	0.51986	0.49958	0.60337
45	HANG SENG CHINA AFFILIATED CORP	0.55469	0.49094	0.64298
46	HANG SENG CHINA ENTERPRISES	0.57626	0.46516	0.61981
47	IBEX 35	0.52901	0.51707	0.63051
48	IDX COMPOSITE	0.53536	0.44226	0.61714
49	IRELAND SE OVERALL (ISEQ)	0.57378	0.53879	0.62345
50	ISRAEL TA 100	0.56888	0.49412	0.63082
51	JAMAICA SE MAIN INDEX	0.67318	0.62662	0.78633
52	KARACHI SE 100	0.56049	0.49273	0.60242
53	KENYA NAIROBI SE (NSE20)	0.71988	0.59285	0.82057
54	KOREA SE COMPOSITE (KOSPI)	0.55174	0.52420	0.92293
55	KOREA SE KOSPI 200	0.51427	0.51061	0.79534
56	KUWAIT KIC GENERAL	0.30118	0.13788	0.49045
57	LEBANON BLOM	0.62967	0.61263	0.71615
58	LUXEMBOURG SE GENERAL	0.60302	0.53960	0.68352
59	MADRID SE GENERAL (IGBM)	0.64111	0.62686	0.76923
60	MALTA SE MSE	0.65344	0.57685	0.78248
61	MDAX FRANKFURT	0.58912	0.52646	0.87398
62	MEXICO IPC (BOLSA)	0.60782	0.55363	0.68730
63	MOROCCO ALL SHARE (MASI)	0.73346	0.59243	0.87033
64	MSCI AC WORLD	0.57798	0.55688	0.69312
65	MSCI EAFE	0.54373	0.44911	0.62590
66	MSCI EM	0.57591	0.48789	0.67036
67	MSCI EUROPE	0.53541	0.51961	0.58847
68	MSCI PACIFIC	0.59821	0.50431	0.69826
69	MSCI WORLD	0.56198	0.53005	0.64465
70	NASDAQ 100	0.65491	0.58426	0.69775
71	NASDAQ COMPOSITE	0.59915	0.50247	0.61201
72	NEXT 150	0.59920	0.52260	0.79240
73	NIFTY 500	0.56844	0.48326	0.61938
74	NIKKEI 225 STOCK AVERAGE	0.61743	0.53129	0.67277
75	NYSE COMPOSITE	0.55188	0.51262	0.60876
76	OMAN MUSCAT SECURITIES MKT.	0.65505	0.55296	0.78096
77	OMX COPENHAGEN (OMXC)	0.59010	0.50705	1.00262
78	OMX COPENHAGEN (OMXC20)	0.52952	0.50570	1.00550
79	OMX HELSINKI (OMXH)	0.67754	0.62085	0.72138
80	OMX ICELAND ALL SHARE	0.65200	0.53050	0.77509
81	OMX RIGA (OMXR)	0.62369	0.52668	0.74479
82	OMX STOCKHOLM (OMXS)	0.55607	0.52401	0.81710
83	OMX STOCKHOLM 30 (OMXS30)	0.58471	0.53290	0.64174
84	OMX TALLINN (OMXT)	0.57534	0.48226	0.68440

85	OMX VILNIUS (OMXV)	0.69925	0.57470	0.82348
86	OSLO EXCHANGE ALL SHARE	0.49521	0.43253	0.57104
87	PHILIPPINE SE I(PSEi)	0.60757	0.54001	0.64312
88	PORTUGAL PSI ALL-SHARE	0.64364	0.56414	0.75832
89	PORTUGAL PSI-20	0.60621	0.52726	0.70235
90	PRAGUE SE PX	0.64454	0.54547	0.70058
91	PRIME ALL SHARE (XETRA)	0.64896	0.55972	0.75311
92	ROMANIA BET (L)	0.69349	0.66948	0.80720
93	RUSSELL 2000	0.49274	0.41357	0.50475
94	RUSSIA RTS INDEX	0.57297	0.52255	0.63391
95	RUSSIAN MICEX INDEX	0.63818	0.57753	0.70810
96	S&P 500 130/30 STRATEGY INDEX	0.65415	0.54283	0.74827
97	S&P 500 BUYBACK $	0.58692	0.56447	0.90514
98	S&P 500 COMPOSITE	0.57387	0.52247	0.60314
99	S&P 500 COMPOSITE(16.00)	0.67383	0.57734	0.80208
100	S&P 500 DAILY RISK CONTROL 10%	0.55967	0.55887	0.94657
101	S&P 500 DIVIDENDS ARISTOCRATS	0.60985	0.52462	0.73136
102	S&P 500 DYNAMIC VEQTOR	0.59021	0.51777	0.70194
103	S&P 500 EQUAL WEIGHTED	0.54898	0.50656	0.60459
104	S&P 500 GROWTH	0.60230	0.52056	0.66921
105	S&P 500 HIGH BETA	0.65445	0.56558	0.75755
106	S&P 500 HIGH QUALITY	0.61059	0.53255	0.73094
107	S&P 500 INVERSE	0.54721	0.53391	0.77661
108	S&P 500 LOW VOLATILITY	0.57660	0.50296	0.62514
109	S&P 500 SECURITIES LENDING INDEX	0.62664	0.54462	0.75172
110	S&P 500 SHARIAH $	0.61591	0.51745	0.67656
111	S&P 500 TR (1970)	0.57560	0.54130	0.68151
112	S&P 500 VALUE	0.52702	0.51949	0.77971
113	S&P 500/CITIGROUP PURE GROWTH	0.66754	0.59464	0.75452
114	S&P 500/CITIGROUP PURE VALUE	0.62445	0.52965	0.68963
115	S&P EURO	0.59804	0.52885	0.68445
116	S&P/ASX 200	0.54018	0.52734	0.76951
117	S&P/ASX 300	0.54207	0.53082	0.77845
118	S&P/BVL GENERAL(IGBVL)	0.71793	0.60042	0.82305
119	S&P/NZX 50	0.64525	0.60385	0.77107
120	S&P/TSX 60 INDEX	0.49239	0.41578	0.54280
121	S&P/TSX COMPOSITE INDEX	0.44476	0.42661	0.52754
122	SBF 120	0.59208	0.52497	0.66174
123	SHANGHAI SE A SHARE	0.40227	0.17839	0.64382
124	SHENZHEN SE B SHARE	0.49868	0.40479	0.55163
125	SLOVAKIA SAX 16	0.65707	0.57783	0.76119
126	SLOVENIAN BLUE CHIP (SBI TOP)	0.78508	0.69695	0.90852
127	STOXX EUROPE 50	0.62898	0.58046	0.72670
128	STOXX EUROPE 600 E	0.57966	0.53838	0.64602
129	STRAITS TIMES INDEX L	0.59385	0.54298	0.87380

130	SWISS MARKET (SMI)	0.59917	0.51042	0.61623
131	TAIWAN SE WEIGHED TAIEX	0.53240	0.51494	0.62369
132	TOPIX	0.59869	0.54371	0.70997
133	TR GLOBAL $	0.60127	0.51978	0.70275
134	TSE SECOND SECTION	0.60241	0.55497	0.70363
135	TUNISIA TUNINDEX	0.68722	0.64150	0.78374
136	VENEZUELA SE GENERAL	0.59056	0.27334	0.72253
137	WARSAW GENERAL INDEX	0.67020	0.61012	0.72738

Note: 'Lower' and 'Upper' denote bootstrapped 95% confidence intervals.

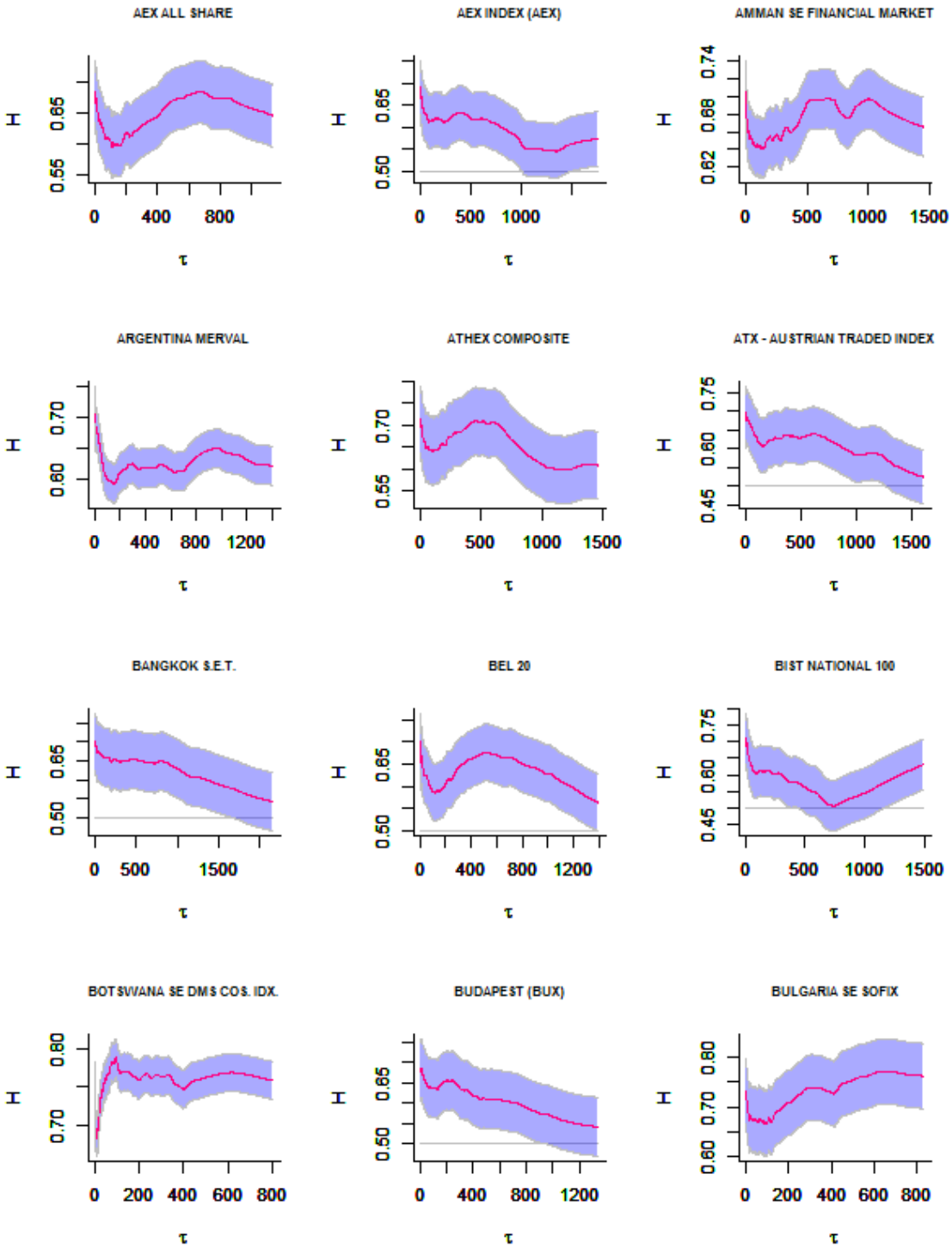
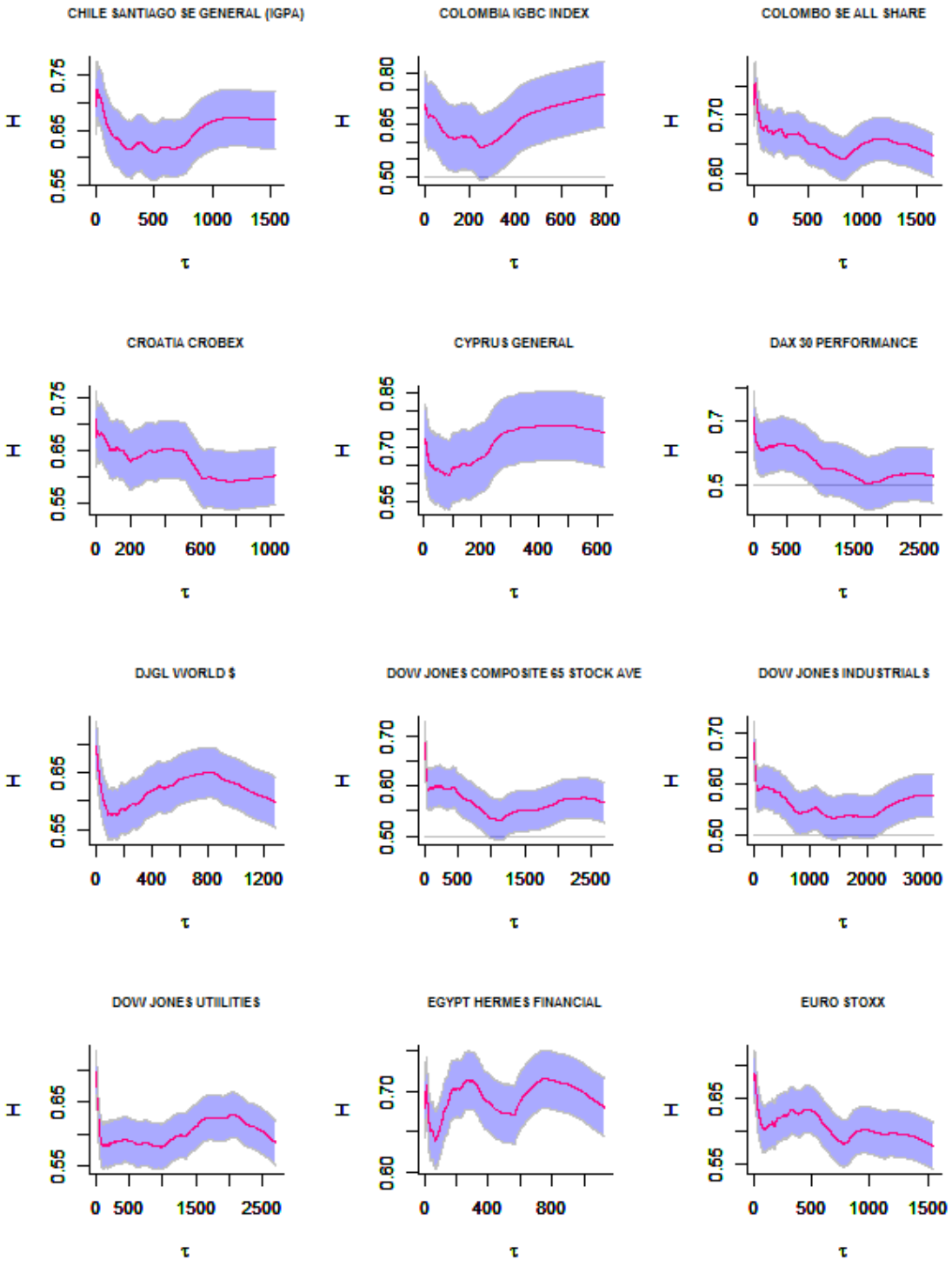
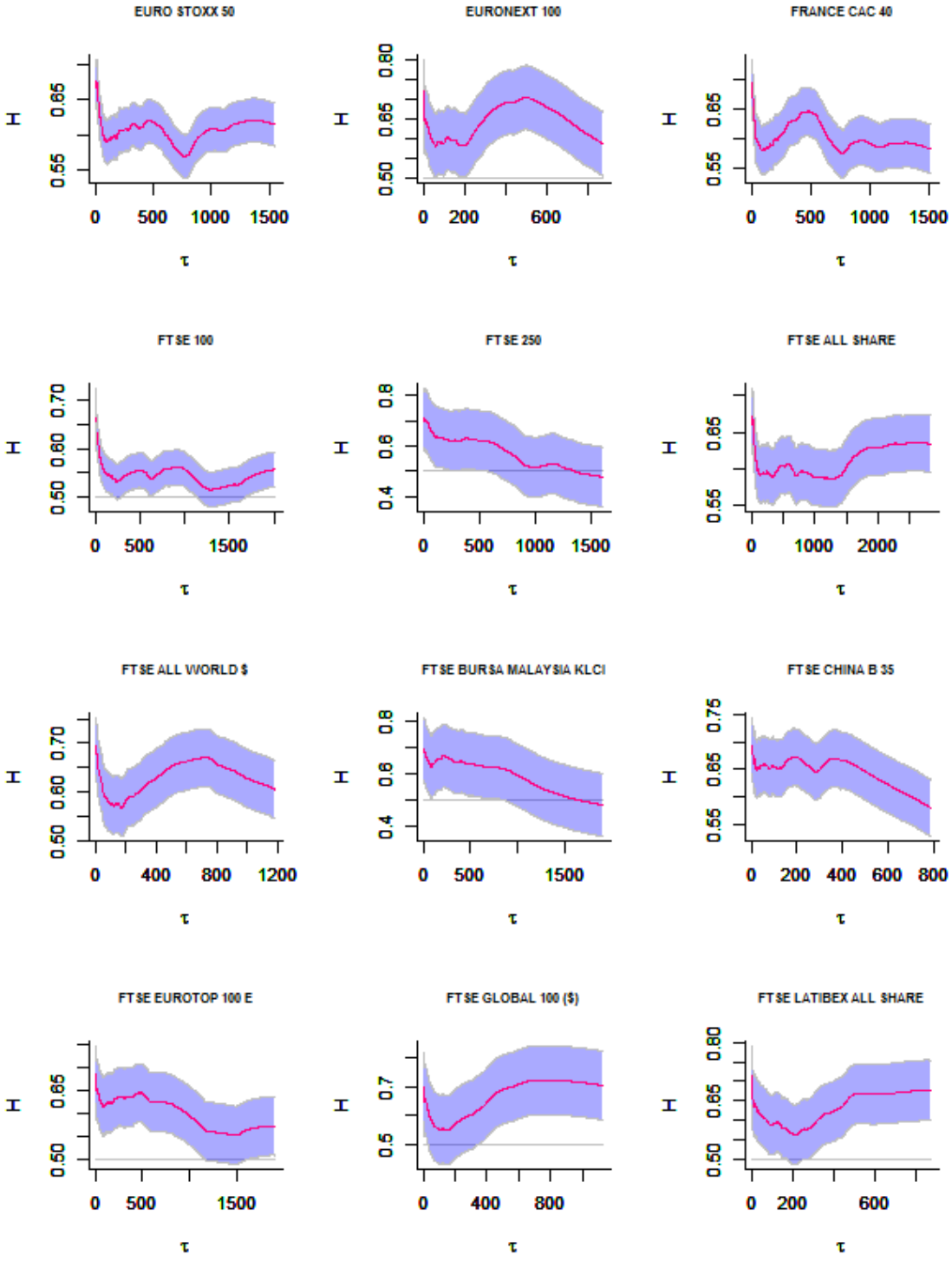


Figure 1: Sequential estimation of H

Note: 'Lower' and 'Upper' denote bootstrapped 95% confidence intervals.



Note: Shaded area denotes 95% confidence interval for H.



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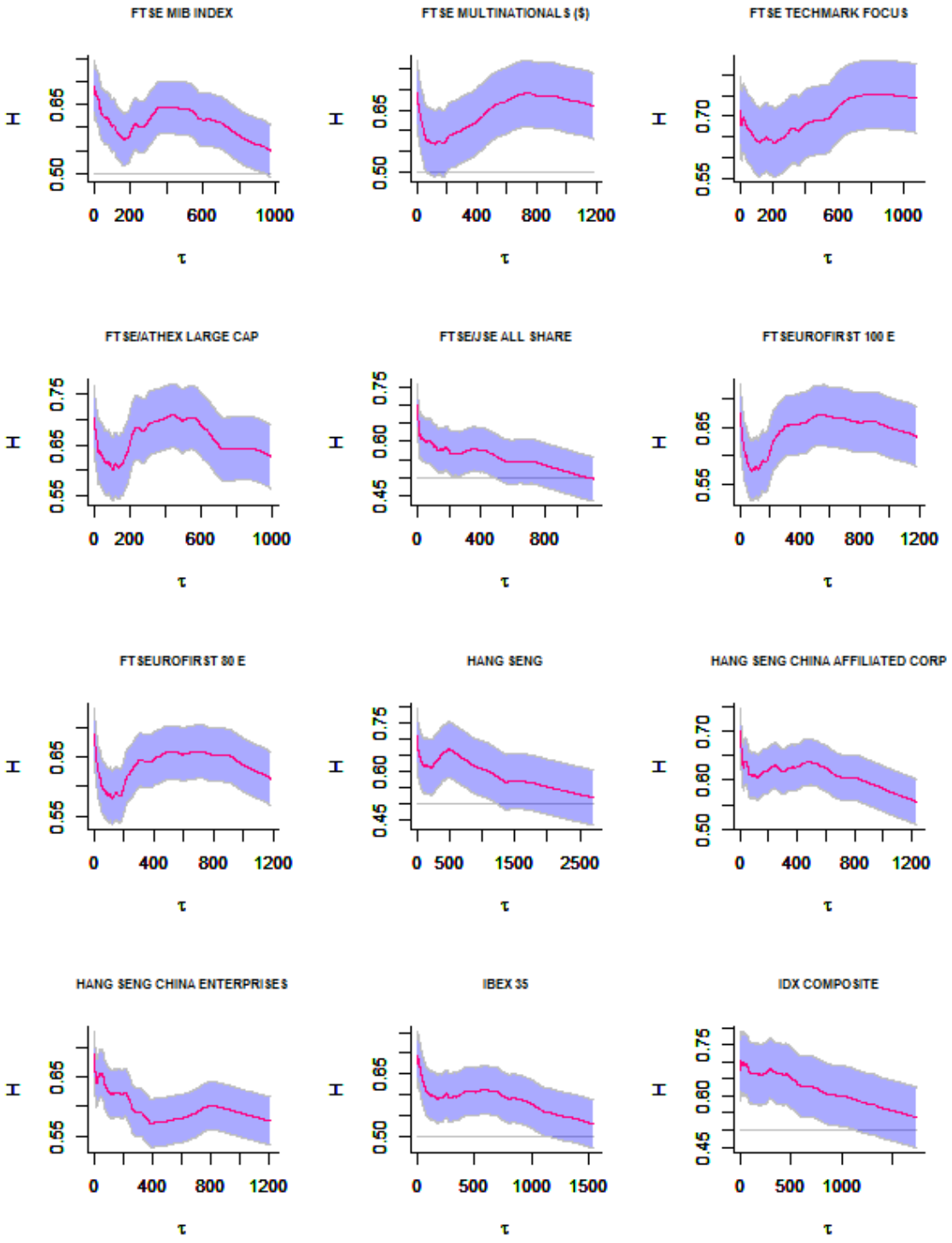

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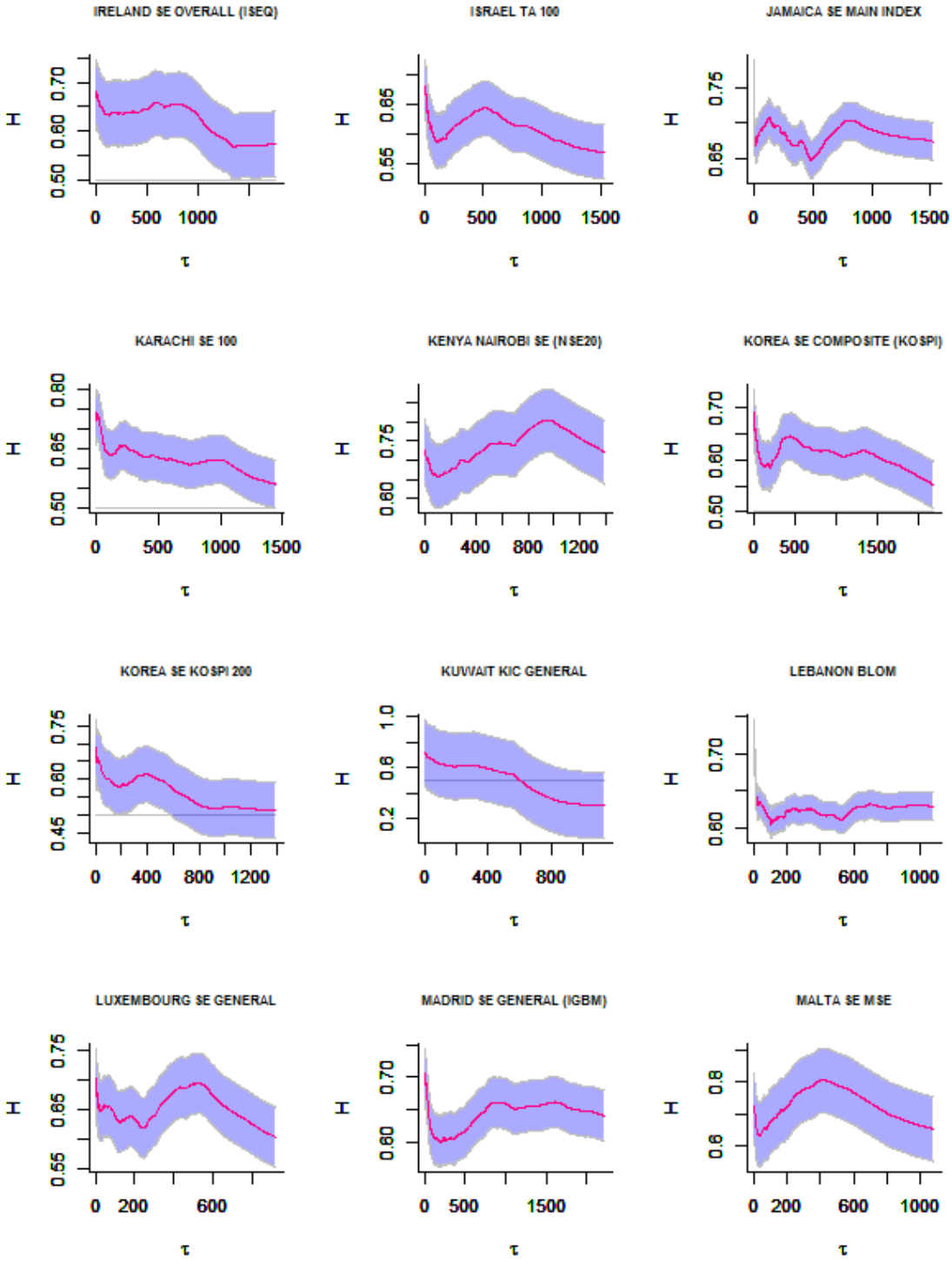

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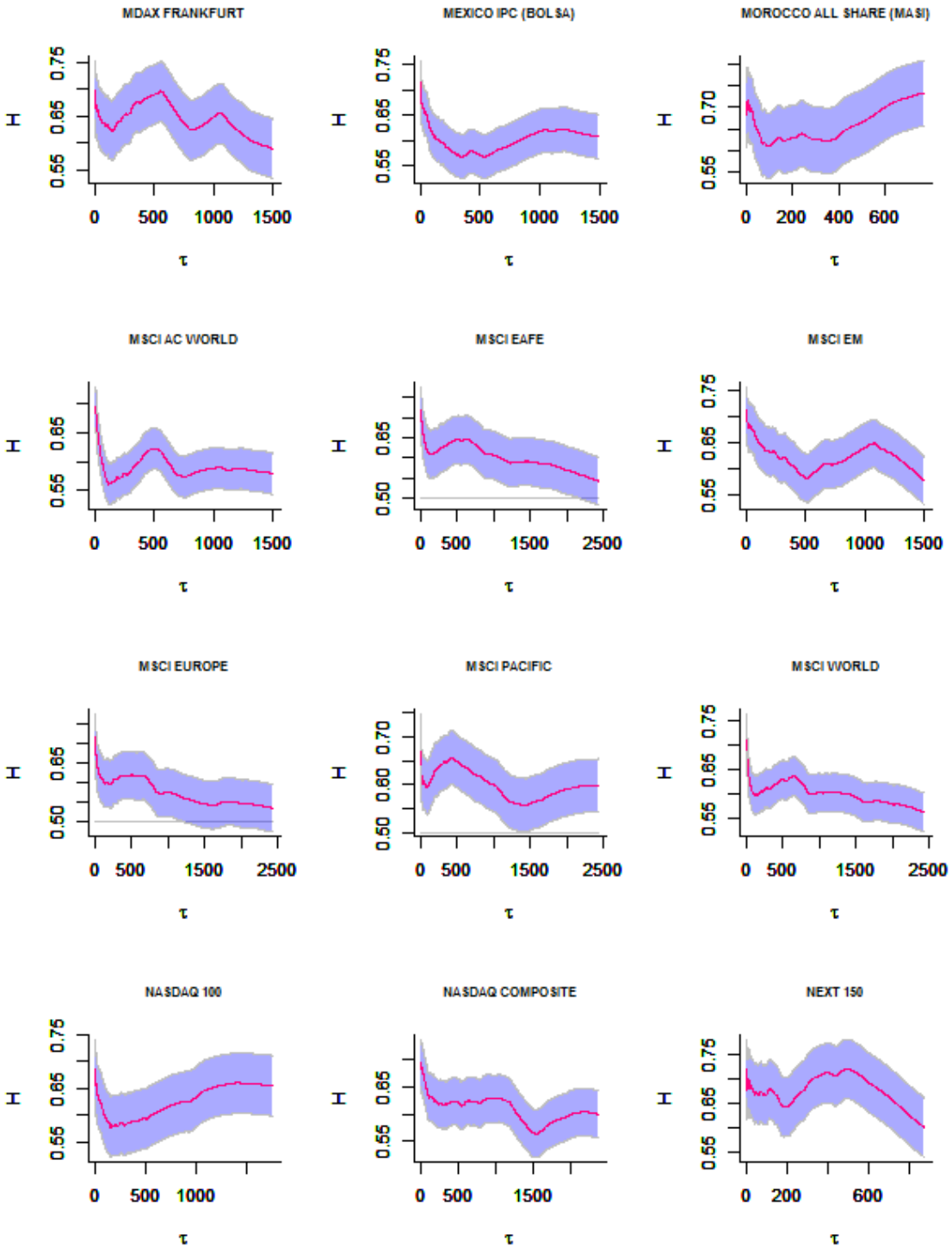

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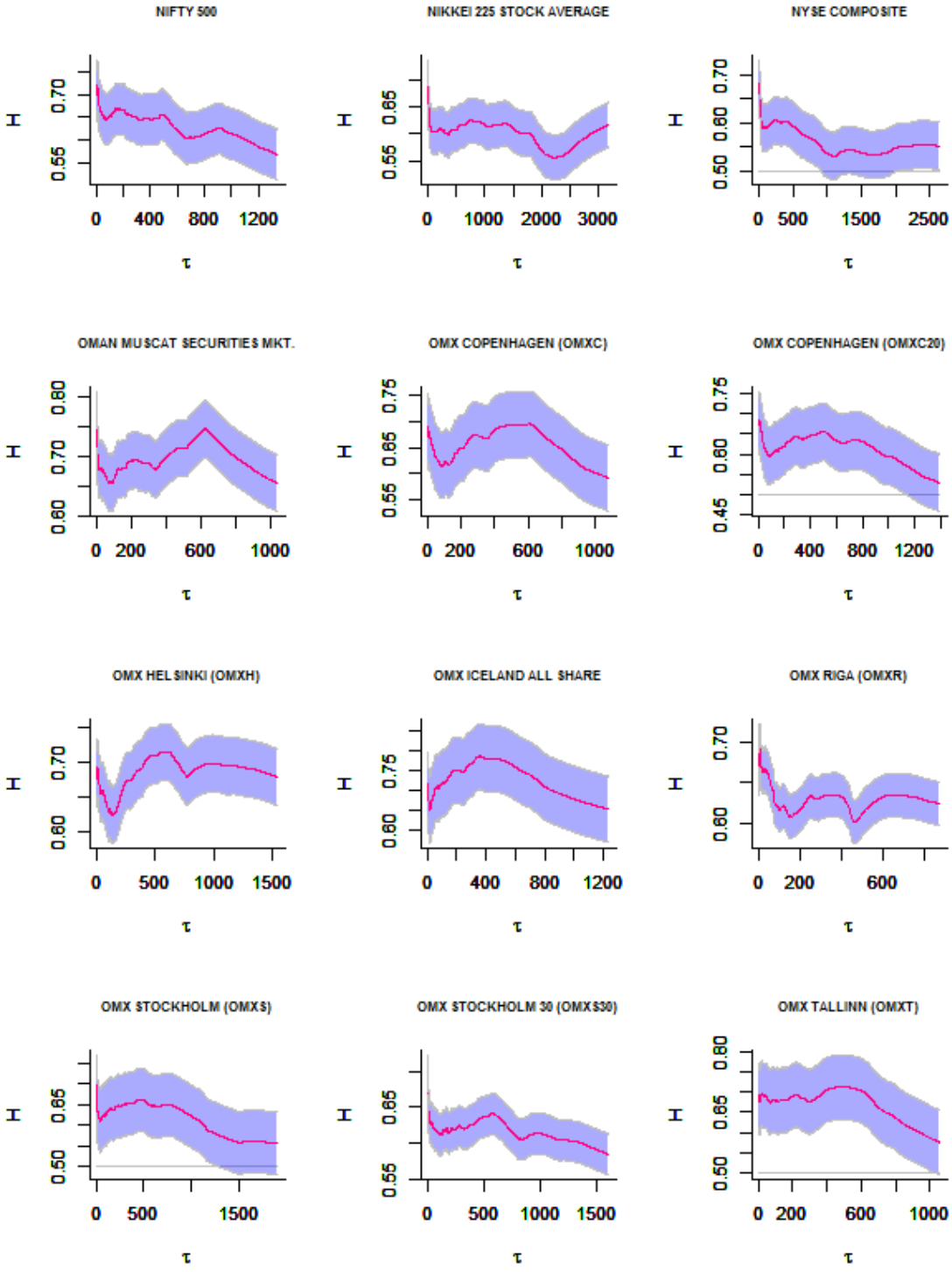

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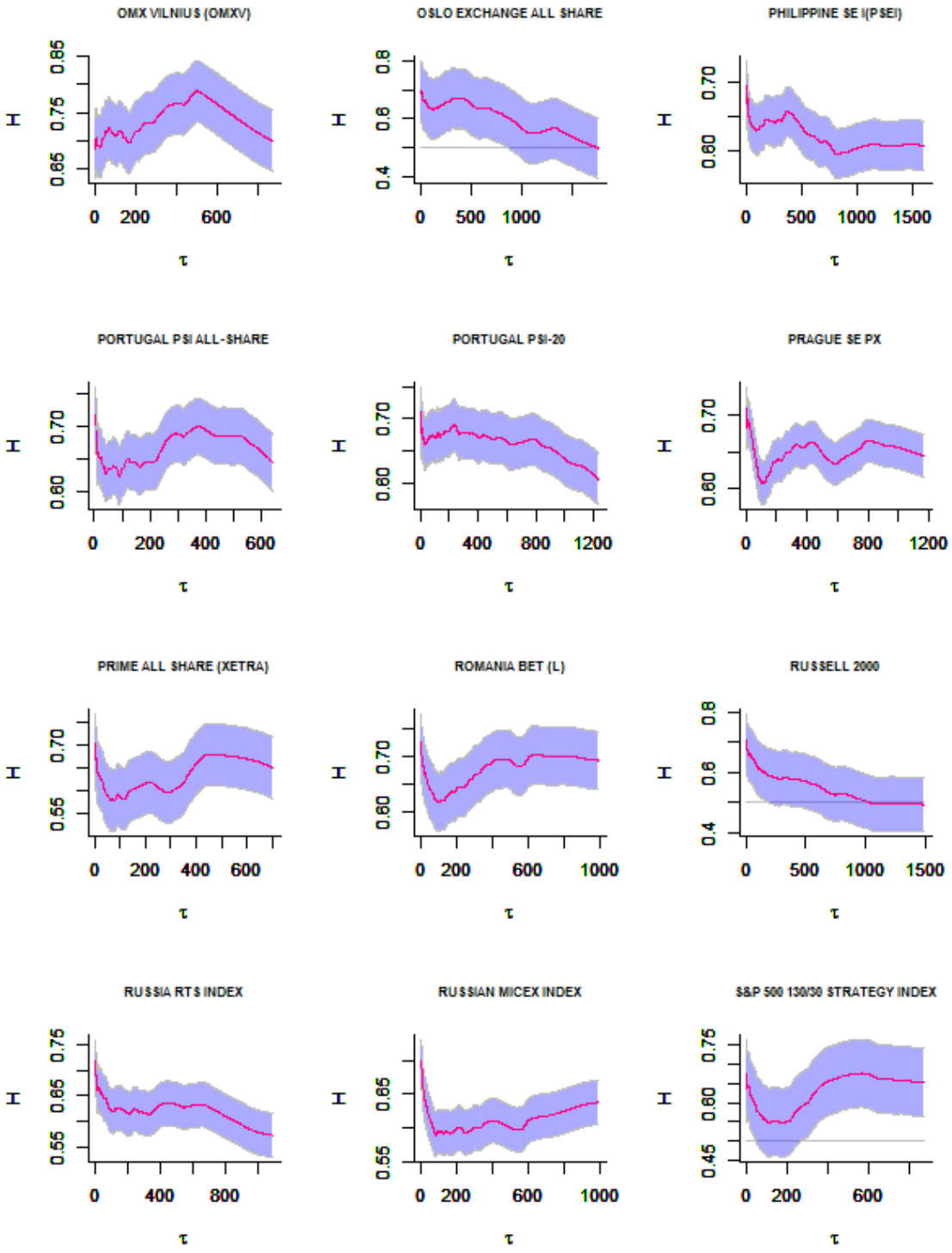

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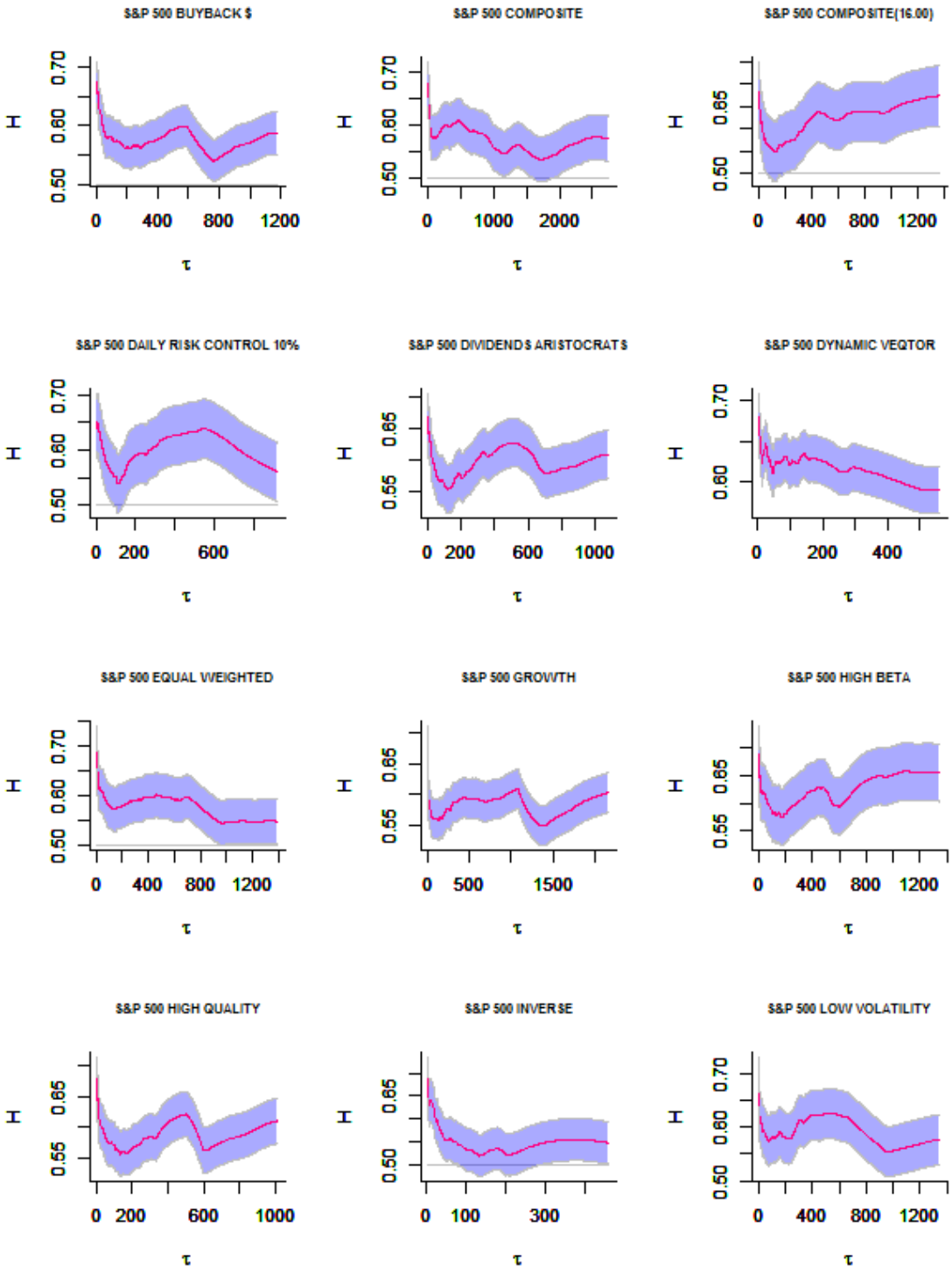

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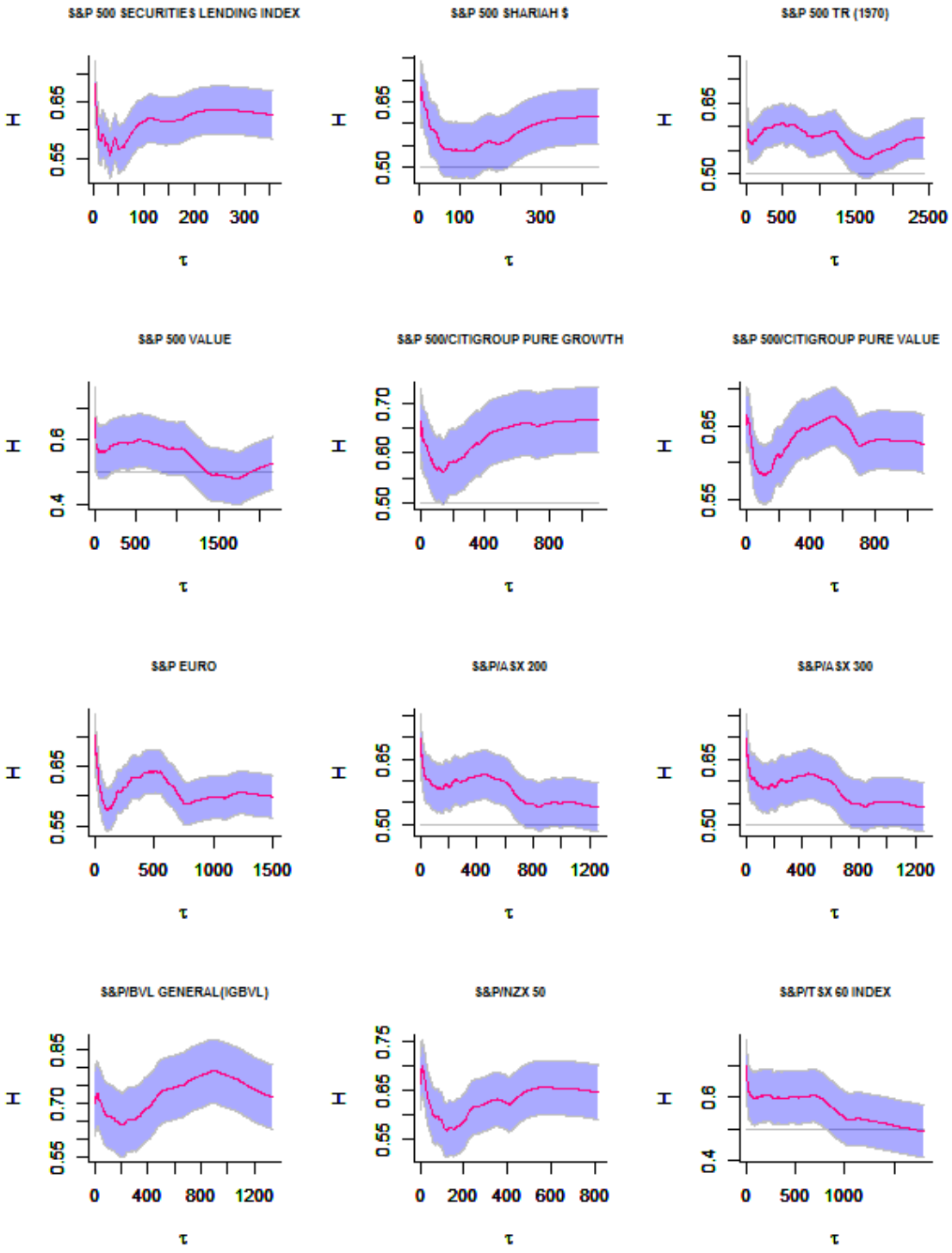

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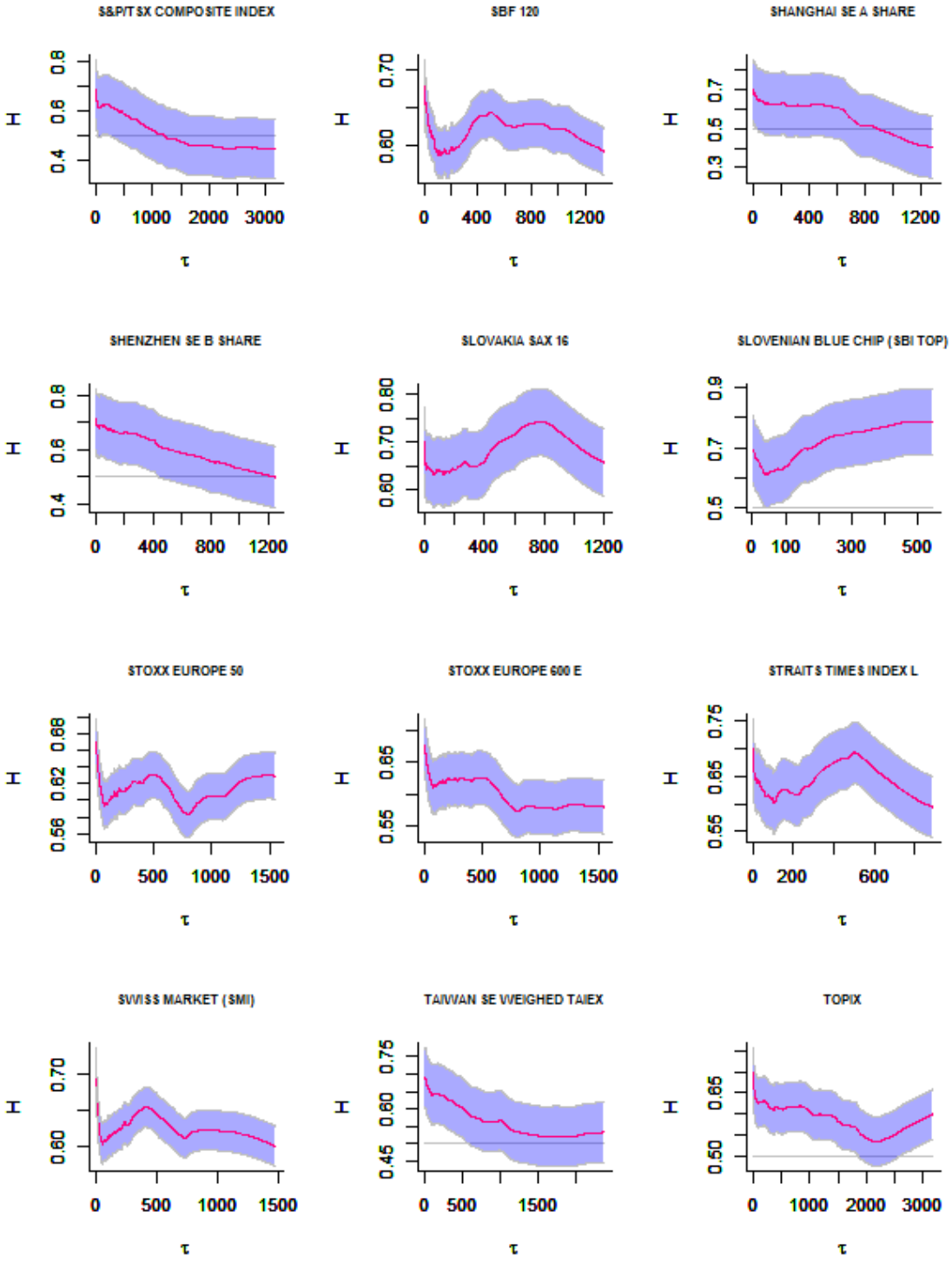

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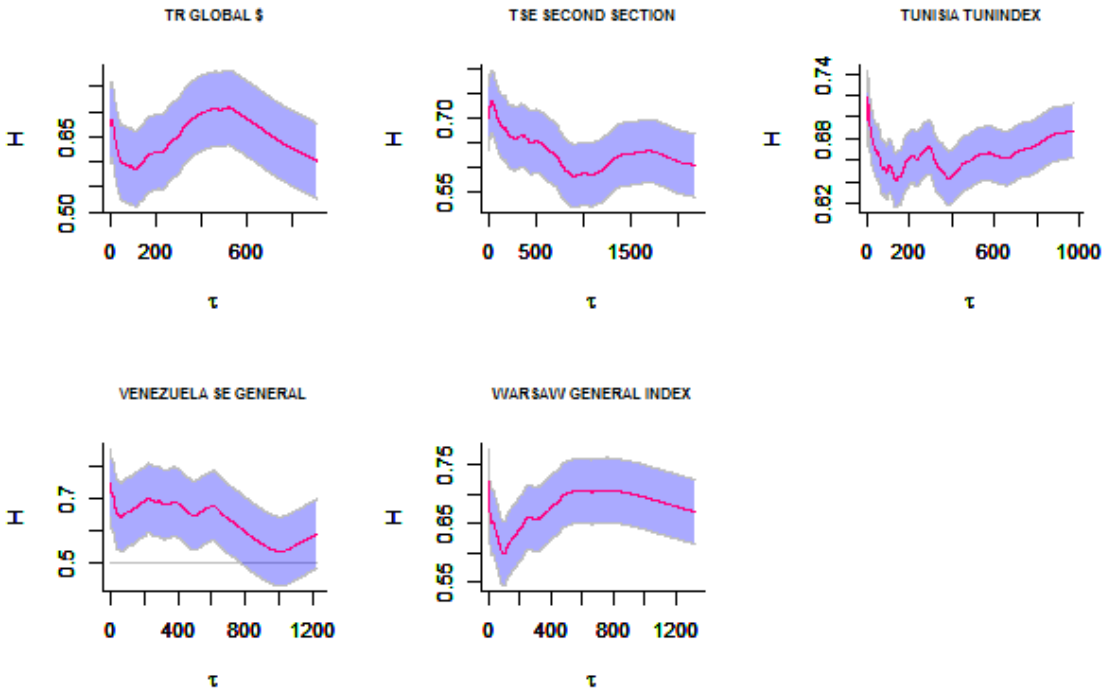

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