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A note on the stochastic portfolio optimization

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Abstract

We reduce the continuous-time dynamic (portfolio) optimiza tion problem to a simple, one-period optimization model. Our method is far simpler than the existing methods in the sense that it avoids the complex ities associated with the Hamilton-Jacobi-Bellman partial di□erential equa tion HJB PDE or the duality methods.

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Previous literature on the stochastic portfolio optimization adopted two methods: the HJB PDE (see, Liu et al (2003), among many others), the duality method (see, for example, Cvitanic and Karatzas (1992)), or a combination of both (see, for example, Castaneda-Leyva and Hernandez-Hernandez (2006)).

In this note, we provide an alternative, simple approach to stochastic optimization in the portfolio model. In doing so, we obtain the same solutions of the previous literature without the complexities adopted by these previous studies, such as the Hamilton-Jacobi-Bellman partial differential equations HJB PDEs, stochastic integrals, stochastic differential equations, and the associated technical assumptions (see, for example, Cvitanic and Zapatero (2004), Alghalith (2009), and Detemple (2013)). Below is a description of the method.

The investor maximizes the expected utility of the terminal wealth

$$V(X(t)) = \underset{\pi(t)}{Max}\ EU(X(T)),$$

where U is a sufficiently smooth utility function, V is the indirect utility function, X is the wealth process, π is the value of the risky asset, r_T is its random rate of return at the terminal (future) time T, r^f is the rate of return of the risk-free asset, and t is the current time. The terminal wealth at time t is given by

$$X(T) = \pi(t)(1 + r_T) + (X(t) - \pi(t))\left(1 + r^f\right). \quad (1)$$

It is common to define $r_T = Er_T + \sigma\epsilon_T$, where σ is the standard deviation of r_T and ϵ is a random variable ($E\epsilon_T = 0; Var(\epsilon_T) = 1$). We also adopt the self-financing assumption. The first-order condition is

$$EU'(X(T))\left(r_T - r^f\right) = 0. \quad (2)$$

Thus

$$\left(Er_T - r_T^f\right) EU'(X(T)) + Cov(U'(X(T)), r_T) = 0. \quad (3)$$

Under the assumption of normality (the assumption used by the stochastic portfolio literature), $Cov(U'(X(T)), r_T) = Cov(X(T), r_T)\, EU''(X(T))$ (Stein's lemma); $Cov(X(T), r_T) = \pi(t)\sigma^2$, therefore $Cov(U'(X(T)), r_T) = \pi(t)\sigma^2 EU''(X(T))$ and the optimal risky asset is given by

$$\pi^*(t) = -\frac{\left(Er_T - r^f\right) EU'(X(T))}{\sigma^2 EU''(X(T))} = -\frac{\left(Er_T - r^f\right) V_X}{\sigma^2 V_{XX}}. \tag{4}$$

At each current time s ($t \leq s < T$), we define the maximization problem as

$$V(X(s)) = \underset{\pi(s)}{Max}\ EU(X(T)),$$

and thus the solution at every current time s is given by

$$\pi^*(s) = -\frac{\left(Er_T - r^f\right) V_X}{\sigma^2 V_{XX}}. \tag{5}$$

This result is identical to the formula obtained by the previous literature.

In sum, this approach will be very useful for many future studies; however, it still requires the assumption of a smooth value function. Possible future extensions include a model with consumption, stochastic factor, or stochastic volatility.

References

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