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Evolutionary Implementation of Efficient Networks

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Abstract

This paper considers an evolutionary implementation problem of efficient outcomes (Sandholm 2007) in the context of network formation. We assume that players interact with each other in the long run, facing stochastic mistakes. Under no constraints, resulting networks can be inefficient in general. Our main result shows that we can construct a Pigouvian-type taxation mechanism such that the resulting networks are efficient with probability one in the long run.

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1 Introduction

A network game is introduced by Jackson and Wolinsky (1996) to study what networks emerge through self-interested agents' interaction. One of the problems for the resulting networks is over/under-connection among players relative to the socially optimal networks due to externalities. In view of an efficient network design, how to deal with this problem and accomplish socially optimal networks is one of the main issues.

In this note, we provide a solution to this problem by considering the evolutionary implementation of efficient networks introduced by Sandholm (2007). Suppose that the payoff function of each player consists of two different components. One is a common term among players which can be interpreted as a source of externalities, while the other is a pairwise idiosyncratic term. As time goes, each pair of players is randomly chosen and receives an opportunity to add or delete the link. In this stage, the players' action choice follows the logit choice rule, which reflects the probability of mistake to take an action in a similar way of the stochastic evolutionary games.[1] The probability depends on the payoff in a way that an action yielding lower payoffs is less likely to be chosen. In the process, a social planner can impose a tax according to externalities caused by each player. We show that a simple Pigouvian pricing rule implements the efficient networks in the long run.

A notable feature of the evolutionary implementation is that the social planner does not need to elicit the true type from each individual, which is a main difficulty in constructing a mechanism to achieve a social optimal state. In contrast, our pricing rule depends only on the common payoff term which is the source of externalities. In this sense, it is easier to implement the rule than the standard mechanism like a VCG mechanism.

The rest of this note is organized as follows. Section 2 describes the model. Section 3 gives a motivating example to explain how a trade-off between efficient outcome and stable network arises by self-interested agents' behavior. Section 4 considers the stochastic dynamic of network formation and Section 5 gives our main result. Finally, we conclude the note in Section 6. Proofs and some discussions are relegated to Appendix.

[1] Stochastic evolutionary games, initiated by Kandori et al. (1993) and Young (1993), where players mistake uniformly provide an equilibrium selection criterion by examining the robustness of equilibria against stochastic shocks. Stochastic process with the logit choice rule is considered by Blume (1993). In the similar spirit, Jackson and Watts (2002) and Tercieux and Vannetelbosch (2006) consider the stochastic evolutionary approach in network games to obtain the stable network under the uniform mistake rule.

2 Preliminaries

2.1 Network and Stability

Let $N = \{1, \cdots, n\}$ be a (finite) set of players. A network is described by an undirected graph whose nodes are players. Let $g^N = \{ij | i, j \in N, i \neq j\}$ be a set of all possible links. Then, a network g is a subset of g^N. We denote the set of all networks by $\mathbb{G}^N = \{g | g \subset g^N\}$. For each network $g \in \mathbb{G}^N$ and player $i \in N$, let $N_i(g) = \{j \in N | i \neq j \text{ and } ij \in g\}$ be the set of i's neighborhood in g. For each $S \in 2^N$ and $g \in \mathbb{G}^N$, let $g|_S = \{ij \in g | i \in S \text{ and } j \in S\}$ be a restricted network whose nodes are in S. We denote by $\mathbb{G}^S$ the set of networks where the set of players is S. For each $ij \in g$, let $g - ij = g \backslash \{ij\}$ be the network which remains after removing a link ij from g. Similarly, for each $ij \notin g$, let $g + ij = g \cup \{ij\}$ be the network formed by adding a link ij to g. The payoff function for player $i \in N$ is denoted by $\phi_i : \mathbb{G}^N \to \mathbb{R}$. Following Jackson and Wolinsky (1996), we call $\phi = (\phi_i)_{i \in N}$ a *network game.*

A solution concept on network games is *pairwise stability* defined by Jackson and Wolinsky (1996). A network is pairwise stable if there is neither a player who wants to sever the link with his neighbor nor a pair of players who agree to make a new link.

Definition 1. A network g is pairwise stable if

(i) for all $ij \in g, \phi_i(g) \geq \phi_i(g - ij)$ and $\phi_j(g) \geq \phi_j(g - ij)$, and

(ii) for all $ij \notin g$, if $\phi_i(g) < \phi_i(g + ij)$ then $\phi_j(g) > \phi_j(g + ij)$.

2.2 Payoffs

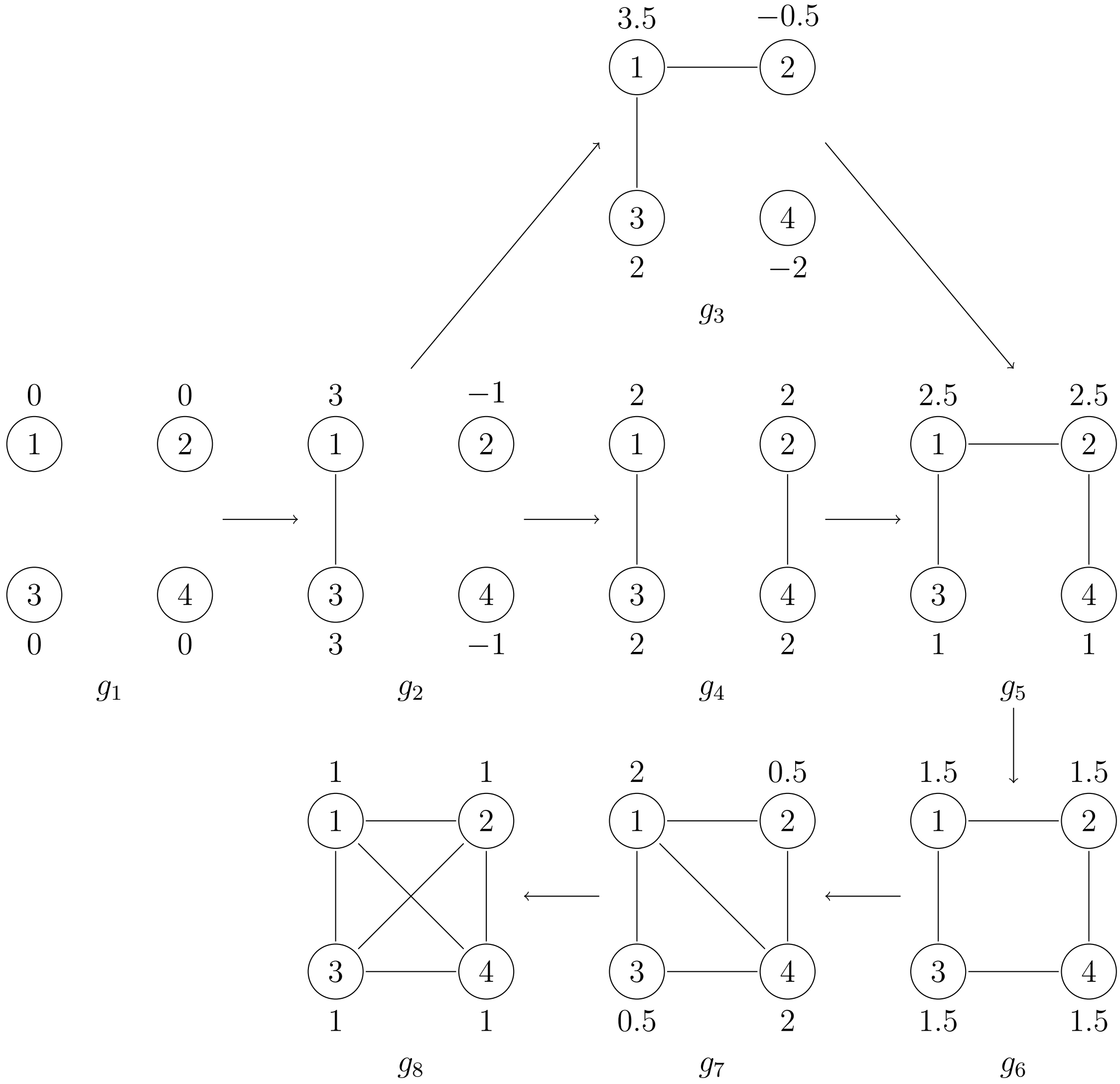
In our model, we assume that each player has two different components in the payoff: one is *common value* and the other is *pairwise idiosyncratic value.*[2] Let $u : \mathbb{G} \to \mathbb{R}$ be the common value and $\theta_{ij} : \{\emptyset, ij\} \to \mathbb{R}$ for each $i, j \in N$ be the pairwise idiosyncratic value. We assume the $\theta_{ij} = \theta_{ji}$, i.e., symmetric for each pair. Let Θ be the set of all profiles of pairwise idiosyncratic values and let $\theta = (\theta_{ij})_{i,j \in N} \in \Theta$ be a typical element. Summing the two components, player i's payoff function is given by

$$\phi_i(g) = u(g) + \sum_{j \neq i} \theta_{ij}(g|_{\{i,j\}}).$$

We discuss some examples of this model in Appendix C.

[2]For each network $g \in \mathbb{G}$, each player is affected by positive/negative externalities through the common component u in addition to the direct effects from idiosyncratic components. Due to this, there might be cases where over/under-connection among players relative to the socially efficient networks occurs.

Figure 1: Motivating Example



3 Motivating Example

The figure 1 based on Jackson (2008) illustrates over-connection problem. In the example, we see that the unique stable network is the complete network, but this network is Pareto dominated by the efficient network.

There are four players. Each number in the circle indicates each player and each number below or above the circle indicates each player's payoff. We assume that $\theta_{12} = \theta_{14} = \theta_{23} = \theta_{34} = 1.5$, $\theta_{13} = \theta_{24} = 4$ if such pair has link in the network and 0 otherwise. We also assume that $u(g_1) = 0$, $u(g_2) = -1$, $u(g_3) = u(g_4) = -2$, $u(g_5) = -3$, $u(g_6) = -4$, $u(g_7) = -5$, $u(g_8) = -6$, and $u(g)$ is sufficiently small for networks other than these eight networks. In the figure, an arrow represents an *improving path*, that is, the sequences of networks that can emerge as players add and delete links in a way that makes them better off. It is easy to see that the rest points of such a process are stable networks. In this example, the unique stable network g_8 is Pareto dominated by the efficient network g_4. The reason for the inefficiency is that having more links increases the idiosyncratic payoff although it has negative externalities, i.e., it decreases the common payoff.

4 Stochastic Evolutionary Dynamics

Suppose that each player has common and idiosyncratic components and these payoff relevant information does not change in the long-run behavior. Consider an infinite horizon discrete time process $(g_t)_{t\in\mathbb{Z}_+}$. In each period, one pair is chosen and they can change the current situation by removing or making a link. Following Hsieh et al. (2017), in each period, one of the following events happens:

Link Formation: At rate $\chi > 0$, a pair of agents ij which is not already connected receives an opportunity to form a link. The formation of a link depends on the marginal payoff the agents receive from the link plus an additive pairwise independently and identically distributed exogenous stochastic error term η_{ij}.

Link Removal: At rate $\xi > 0$, a pair of connected agents ij receives an opportunity to remove their link. The link is removed if at least one agent finds this profitable. The marginal payoffs from removing the link ij are perturbed by an additive pairwise independently and identically distributed exogenous stochastic error term η_{ij}.

We consider the logit-choice with noise level $\varepsilon > 0$.[3] Under this specification, our dynamic process $(g_t)_{t\in\mathbb{Z}_+}$ becomes a discrete time Markov chain by the following transition probabilities:

$$P^\varepsilon\left(g_t, g_{t+1} = g_t + ij\right) = \chi P\left(\left\{\phi_i\left(g_t + ij\right) - \phi_i\left(g_t\right) + \eta_{ij} > 0\right\} \cap \left\{\phi_j\left(g_t + ij\right) - \phi_j\left(g_t\right) + \eta_{ij} > 0\right\}\right),$$
$$P^\varepsilon\left(g_t, g_{t+1} = g_t - ij\right) = \xi P\left(\left\{\phi_i\left(g_t - ij\right) - \phi_i\left(g_t\right) + \eta_{ij} > 0\right\} \cup \left\{\phi_j\left(g_t - ij\right) - \phi_j\left(g_t\right) + \eta_{ij} > 0\right\}\right).$$

This Markov chain $(g_t)_{t\in\mathbb{Z}_+}$ is irreducible and aperiodic. Hence, there is a unique stationary distribution $\mu^\varepsilon(g)$, which captures the long-run behavior among players.

In particular, we are interestied in the long-run behavior of the Markov chain where noise is very small. Let

$$LSN(u, \theta) = \left\{ g \in \mathbb{G}^N \mid \lim_{\varepsilon\to 0} \mu^\varepsilon(g) > 0 \right\}$$

[3] This means that each error term η_{ij} follows i.i.d. logistic distribution with mean 0 and scale parameter ε^{-1}, i.e. $F(x) = \frac{\exp[\varepsilon^{-1}x]}{1+\exp[\varepsilon^{-1}x]}$.

be the set of *logit stochastically stable* networks.

4.1 Potentials and Stochastic Stability

Under some properties, the above set of networks can be analytically tractable. To show it, we introduce the concept of *network potentials.* Network potentials are introduced by Chakrabarti and Gilles (2007) and a characterization of the class of network games which admit network potentials is studied by Nakada (2016). The function is analogous to the potential function defined by Monderer and Shapley (1996) in noncooperative games.

Definition 2. A network game $\phi = (\phi_i)_{i \in N}$ admits a *network potential* if there is a function $\omega : \mathbb{G}^N \to \mathbb{R}$ such that, for any $g \in \mathbb{G}^N$ and $ij \in g$,

$$\phi_i(g) - \phi_i(g - ij) = \omega(g) - \omega(g - ij).$$

If there exists a network potential, we can have the following form of the stationary distribution.[4]

Proposition 1. *Suppose that a network game ϕ admits a network potential ω. Then, for each $\varepsilon > 0$, the unique stationary distribution of logit-response dynamics $\mu^\varepsilon(g)$ is given by*

$$\mu^\varepsilon(g) = \frac{\exp\left[\varepsilon^{-1}\omega(g) - m\ln\left(\frac{\xi}{\chi}\right)\right]}{\sum_{g' \in \mathbb{G}^N} \exp\left[\varepsilon^{-1}\omega(g') - m'\ln\left(\frac{\xi}{\chi}\right)\right]}$$

where m and m' are the number of links in g and g' respectively.

As $\varepsilon \to 0$, we obtain

$$\lim_{\varepsilon \to 0} \mu^\varepsilon(g) \begin{cases} > 0 & \text{if } \omega(g) \geq \omega(g') \text{ for all } g' \in \mathbb{G}^N, \\ = 0 & \text{otherwise.} \end{cases}$$

Therefore, we immediately obtain the following result.

Theorem 1. *Suppose that a network game ϕ admits a network potential ω. Then, a network g is stochastically stable if and only if g maximizes ω.*

Note that our model has a network potential function

$$\omega(g) = u(g) + \sum_{i<j} \theta_{ij}(g|_{\{i,j\}}).$$

[4] This result is a reformulated and slightly generalized version of Theorem 1 in Hsieh et al. (2017) where both actions and networks are changed according to the dynamics. We discuss a relation between our model and their model in Appendix B.

Hence, as a collorary, if the potential maximizer is not efficient, then such an inefficient network lasts in the long run. This is also the case for the example in Section 3.

Corollary 1. *Suppose that a network game ϕ admits a network potential ω and maximizers of it are not efficient. Then, stochastically stable networks are always inefficient.*

5 Implementation of the Efficient Networks

For each $i \in N$, let $p_i : \mathbb{G}^N \to \mathbb{R}$ be a pricing function for player i and $p = (p_i)_{i\in N}$ be a payment scheme. If a price scheme p is imposed, the common payoff from a network $g \in \mathbb{G}^N$ changes from $u(g)$ to $u(g) - p_i(g)$. Then the payoff is described by the pair $(u - p, \theta)$.

The social planer's objective is summarized by a *social choice correspondence*, which is a mapping $F : \Theta \rightrightarrows \mathbb{G}^N$. We say that a price scheme $p = (p_i)_{i\in N}$ *stochastically implements* the social choice correspondence F if for each type profile $\theta \in \Theta$,

$$LSN(u - p, \theta) = F(\theta).$$

In particular, the efficient social choice correspondence $F^* : \Theta \rightrightarrows \mathbb{G}^N$ is defined by

$$F^*(\theta) = \arg\max_{g\in\mathbb{G}^N} W(g, \theta)$$

where

$$W(g, \theta) = \sum_{i\in N} \phi_i(g) = nu(g) + 2\sum_{i<j} \theta_{ij}(g|_{\{i,j\}}).$$

To accomplish this objective, we consider the following payment scheme: for each $i \in N$ and each $g \in \mathbb{G}^N$,

$$p_i^*(g) = -\frac{1}{2}(n-2)\left(u(g) - u(g|_{N\setminus\{i\}})\right).$$

This payment scheme can be interpreted intuitively. As in Sandholm (2007), each player pays the marginal-externalities price for making links. If player i has a link with some player j, this link creates externalities $u(g) - u(g|_{N\setminus\{ij\}})$ to other $n-2$ players. Then, according to the externality pricing, player i has to pay the amount with j, which is described by multiplication of $-\frac{1}{2}(n-2)$. By considering the all possibilities, player i's total payment is $-\frac{1}{2}(n-2)\big(u(g) - u(g|_{N\setminus\{i\}})\big)$.

The following main result of this paper states that this payment can implement the efficient networks in the long run.

Theorem 2. *The payment scheme $p^* = (p_i^*)_{i\in N}$ stochastically implements the efficient social choice correspondence.*

Theorem 2 states that even individuals' selfish behavior can induce efficient outcome by using a suitable payment scheme. This result can be also interpreted as how a cost of selfish-behavior is internalized.

By Theorem 1, we have $LSN(u,\theta) = \mathrm{argmax}_{g\in\mathbb{G}^N}\omega(g') \neq \mathrm{argmax}_{g\in\mathbb{G}^N}W(g)$ in general. Then, the difference of welfare between the efficient outcomes and worst case outcomes is

$$PoA \equiv \frac{\max_{g\in\mathbb{G}^N}W(g)}{\min_{g\in LSN(u,\theta)}W(g)},$$

which is referred to as *price of anarchy* (e.g., Koutsoupias and Papadimitriou 1999; Papadimitriou 2001; Roughgarden 2005). In the motivating example discussed in section 3, the price of anarchy is equal to $W(g_4)/W(g_8) = 8/4 = 2$. In general, $PoA \geq 1$ and it is increased if there are bad outcomes. Thus, $PoA - 1$ is considered as a cost of individual's selfish-behavior. According to our payment scheme, $LSN(u - p^*, \theta) = \mathrm{argmax}_{g\in\mathbb{G}^N}W(g)$, so that there are no bad outcomes. That is, all the cost $PoA - 1$ is internalized and the price of anarchy is minimized to 1.

Corollary 2. *Whatever the price of anarchy PoA is in the original model, the payment scheme $p^* = (p_i^*)_{i\in N}$ can achieve the minimum price of anarchy.*

6 Concluding Remarks

In this note, we provide a solution to a trade-off between efficiency and inefficiency caused by individuals' selfish behavior. In contrast to the standard mechanism design approach, we consider the long-run behavior of each agent. We remark some points about our results.

In the model, we assume that the former term $u(\cdot)$ is common knowledge among players and the social planner but the latter term $(\theta_{ij})_{i,j\in N}$ is private information of players. A notable feature of this result is that social planner does not need to gather private information of players θ_{ij} because this pricing rule only depends on the common term $u(\cdot)$ but does not depend on the individual term θ_{ij}. Therefore, there is no incentive problem to induce a true individual type, which is a central problem in the usual mechanism like VCG mechanism.

We also mention that our payment scheme p^* is not a unique rule to implement efficient networks. More generally, one can show that payment scheme which satisfies $p_i(g) - p_i(g - ij) = -\frac{n-2}{2}(u(g) - u(g - ij))$ for each $i, j \in N$ can also implement efficient networks. One example is the following uniform rule: $p_i(g) = -\frac{n-2}{2}u(g)$ for each $i \in N$. Similar result holds for Sandholm (2007), which is not stated in the paper.

Our payment scheme is conditionally effective in the sense that specification of payoff function and assumption of mistake are heavily used to obtain the result. Nonetheless, since the rule itself is easy to implement, we believe that this approach is useful in some applications like Furusawa and Konishi (2007) once we can confirm the game admits a network potential.

A Proof

Proof of Proposition 1. Note that if there is a network potential, then our transition probabilities becomes

$$\begin{aligned}
P^{\varepsilon}\left(g_t, g_{t+1} = g_t + ij\right) &= \chi P\left(\{\phi_i\left(g_t + ij\right) - \phi_i\left(g_t\right) + \eta_{ij} > 0\} \cap \{\phi_j\left(g_t + ij\right) - \phi_j\left(g_t\right) + \eta_{ij} > 0\}\right) \\
&= \chi P\left(\omega\left(g_t + ij\right) - \omega\left(g_t\right) + \eta_{ij} > 0\right) \\
&= \chi P\left(-\eta_{ij} < \omega\left(g_t + ij\right) - \omega\left(g_t\right)\right) \\
&= \chi \frac{\exp\left[\varepsilon^{-1}\omega\left(g_t + ij\right)\right]}{\sum_{g' \in \{g_t, g_t + ij\}} \exp\left[\varepsilon^{-1}\omega(g')\right]}, \\
P^{\varepsilon}\left(g_t, g_{t+1} = g_t - ij\right) &= \xi P\left(\{\phi_i\left(g_t - ij\right) - \phi_i\left(g_t\right) + \eta_{ij} > 0\} \cup \{\phi_j\left(g_t - ij\right) - \phi_j\left(g_t\right) + \eta_{ij} > 0\}\right) \\
&= \xi P\left(\omega\left(g_t - ij\right) - \omega\left(g_t\right) + \eta_{ij} > 0\right) \\
&= \xi P\left(-\eta_{ij} < \omega\left(g_t - ij\right) - \omega\left(g_t\right)\right) \\
&= \xi \frac{\exp\left[\varepsilon^{-1}\omega\left(g_t - ij\right)\right]}{\sum_{g' \in \{g_t, g_t - ij\}} \exp\left[\varepsilon^{-1}\omega\left(g'\right)\right]}.
\end{aligned}$$

Then, it is enough to show that the distribution $\mu^{\varepsilon}(g)$ satisfies the *detailed balance condition*: for all $g, g' \in \mathbb{G}^N$, $\mu^{\varepsilon}(g)P^{\varepsilon}(g, g') = \mu^{\varepsilon}(g')P^{\varepsilon}(g', g)$.

Observe that the detailed balance condition is trivially satisfied if g' and g differ in more than one link since the transition probability is zero. Hence, we consider only the case of link creation $g' = g + ij$ and removal $g' = g - ij$. For the case of link creation with a transition from g to $g + ij$, the detailed balance condition is

$$\exp\left[\varepsilon^{-1}\omega(g) - m\ln\left(\tfrac{\xi}{\chi}\right)\right] \frac{\exp\left[\varepsilon^{-1}\omega(g+ij)\right]}{\sum_{g' \in \{g, g+ij\}} \exp\left[\varepsilon^{-1}\omega(g')\right]} \chi = \exp\left[\varepsilon^{-1}\omega(g+ij) - (m-1)\ln\left(\tfrac{\xi}{\chi}\right)\right] \frac{\exp\left[\varepsilon^{-1}\omega(g)\right]}{\sum_{g' \in \{g, g+ij\}} \exp\left[\varepsilon^{-1}\omega(g')\right]} \xi,$$

which is clearly satisfied. A similar argument holds for the removal of a link with a transition from g to $g - ij$ where the detailed balance condition leads

$$\exp\left[\varepsilon^{-1}\omega(g) - m\ln\left(\tfrac{\xi}{\chi}\right)\right] \frac{\exp\left[\varepsilon^{-1}\omega(g-ij)\right]}{\sum_{g' \in \{g, g-ij\}} \exp\left[\varepsilon^{-1}\omega(g')\right]} \chi = \exp\left[\varepsilon^{-1}\omega(g-ij) - (m-1)\ln\left(\tfrac{\xi}{\chi}\right)\right] \frac{\exp\left[\varepsilon^{-1}\omega(g)\right]}{\sum_{g' \in \{g, g-ij\}} \exp\left[\varepsilon^{-1}\omega(g')\right]} \xi.$$

Hence, $\mu^\varepsilon(g)$ is the stationary distribution of the Markov chain. $\square$

Proof of Theorem 2. Let $\hat{\phi}_i(g) = \phi_i(g) - p_i^*(g) = \frac{n}{2}u(g) - \frac{1}{2}(n-2)u\left(g|_{N\setminus\{i\}}\right) + \sum_{j\neq i}\theta_{ij}\left(g|_{\{i,j\}}\right)$ be the new network payoff function under the payment scheme. Note that for each $ij \in g$,

$$\begin{aligned}\hat{\phi}_i(g) - \hat{\phi}_i(g-ij) &= \frac{n}{2}u(g)\left(u(g) - u(g-ij)\right) + \theta_{ij}\left(g|_{\{i,j\}}\right) - \theta_{ij}\left((g-ij)|_{\{i,j\}}\right) \\ &= \frac{1}{2}\left(W(g,\theta) - W(g-ij,\theta)\right).\end{aligned}$$

This means that $\frac{1}{2}W(g,\theta)$ is a network potential for $\hat{\phi}$. Moreover, we can see that

$$F^*(\theta) = \arg\max_{g\in\mathbb{G}^N} W(g,\theta) = \arg\max_{g\in\mathbb{G}^N} \frac{1}{2}W(g,\theta).$$

Therefore, by Theorem 1, we can say that $LSN(u-p^*,\theta) = F^*(\theta)$. $\square$

B Relation with Potential Games

We explain how network games and non-cooperative games are different, and so are network potentials and potentials by Monderer and Shapley (1996). Let $N = \{1,\cdots,n\}$ be a set of players, A_i be a set of actions of player i and $\pi_i : A \to \mathbb{R}$ be a payoff function of player i. We call the tuple $(N,(A_i,\pi_i)_{i\in N})$ a non-cooperative game, or a game for short. Hereafter, we just call $\pi = (\pi_i)_{i\in N}$ as a game when there is no confusion. According to Monderer and Shapley (1996), we say that a function P is a *potential function* of the game $\pi_i = (\pi_i)_{i\in N}$ if for any $i \in N, a \in A$ and $a_i' \in A_i$,

$$\pi_i(a_i', a_{-i}) - \pi_i(a_i, a_{-i}) = P_i(a_i', a_{-i}) - P_i(a_i, a_{-i}).$$

Recall that the primitive of a network game is (N,ϕ) where $\phi_i : \mathbb{G}^N \to \mathbb{R}$, which is not itself a non-cooperative game.

To see the relationship more clearly, let us consider the difference between our model and that of Hsieh et al. (2017). They consider the following hybrid-model of noncooperative games and network games. [5] Let $A_i = \mathbb{R}$ and payoff function be such that

$$\pi_i(a,g) = n\eta a_i - n\nu a_i^2 - ba_i\sum_{j\neq i}a_j + \rho a_i\sum_{j\neq i}g_{ij}a_j - \zeta d_i,$$

where $b,\eta,\nu,\rho \in \mathbb{R}$ and $d_i = |N_i(g)|$. Consider the following function $P : A\times\mathbb{G}^N \to \mathbb{R}$

[5] This type of game is called *game on networks* in the literature (Jackson and Zenou 2014).

defined as

$$P(a,g) = \sum_{i\in N}(n\eta a_i - n\nu a_i^2) - \frac{b}{2}\sum_{i\in N}\sum_{j\neq i} a_i a_j + \frac{\rho}{2}\sum_{i\in N}\sum_{j\neq i} g_{ij}a_i a_j - \zeta m.$$

By fixing g, they shows that the function $P(\cdot, g) : A \to \mathbb{R}$ is a potential function in the sense of Monderer and Shapley (1996).[6]

Fix $a \in A$ and let $u(g) \equiv 0$ and $\theta_{ij} = (\rho a_i a_j - \zeta)g_{ij}$. Then, the model of Hsieh et al. (2017) is a special class of our model as a network game because $\phi_i(g) = \phi_i(a,g) = \sum_{j\neq i}(\rho a_i a_j - \zeta)g_{ij} + C_i$ where $C_i \in \mathbb{R}$ is a constant which does not depend on g. In this sense, our Proposition 1 is a generalization of Theorem 1 of Hsieh et al. (2017) as a stochastic evolutionary dynamic on network games. Also, our Proposition 1 is not implied by the result of Blume (1993) because two models are different as we discussed above.

C Examples of the Model

We consider some examples of our model to explain how our model works in the specific application.

1. Transportation network

An example is a transportation network design among cities. There is a representative driver in each region and a transportation commissioner who wants to construct a beneficial transportation network for her city. We assume that a direct effect of the construction of a load between city i and j is measured by a bilateral term θ_{ij}. In addition to the direct effect, the drivers' decision on which load to use depends on the whole transportation network, which is a source of externalities. In particular, when a new load is constructed between two cities, the total congestion time can be increased, which is known as Braess' paradox. In this example, such externalities from the congestion correspond to $u(g)$.

We provide a formal micro-foundation to this story. Let $N = \{1, \cdots, n\}$ be a set of cities and we also denote representative driver in the city by i. Given a transportation network g, let $A_i = 2^g$ be the available routes for each driver. For each $ij \in a \in A$, let $n_{ij}(a) = |\{k \in N | ij \in a_k\}|$ be the number of drivers who use the route ij. Let us define each representative driver's payoff function as

$$\pi_i(a,g) = \sum_{jk\in a_i; jk\in N_i(g)} C_{jk}(n_{jk}(a)) + \sum_{jk\in a_i; jk\notin N_i(g)} D_{jk}(n_{jk}(a))$$

[6]They also show that $P(a,\cdot) : \mathbb{G}^N \to \mathbb{R}$ satisfies the property of network potential when we see the game as a network game by fixing the action profile $a \in A$ (i.e, $\phi_i(\cdot) = \pi_i(a,\cdot)$). For their Theorem 1, they use this property to obtain the stationary distribution of action profiles and networks. However, they do not mention the relationship between this function and a network potential.

where $C_{jk}, D_{jk} : \mathbb{G}^N \to \mathbb{R}_-$ are decreasing functions. Here, it costs a driver only a congestion cost to go from her town to a neighbor town. On the other hand, she suffers an additional cost (such as tolls) when she goes to a distant city. In this sense, we consider two types of congestion costs, C_{ij} and D_{ij} ($ij \in g$). [7]

Suppose that the transportation commissioner in each city seeks to maximize the net payoff of the representative driver. The commissioner benefits the driver in her region by constructing loads between her town and other towns. However, it takes a cost (e.g. maintenance cost) to construct a load, and we assume that the cost is identical to the congestion cost. Formally, let us define the payoff function of each transportation commissioner as

$$W_i(g,a) = \underbrace{\pi_i(a,g)}_{\text{payoff of the driver}} + \underbrace{\sum_{k \neq i} \sum_{ik \in a_k; ik \in N_i(g)} C_{ki}(n_{ik}(a))}_{\text{additional cost}}.$$

Then, by the direct calculation, we can show that,

$$W_i(g,a) = \sum_{l \in N} \sum_{jk \in a_l; jk \in N_i(g)} \big(C_{jk}(n_{jk}(a)) - D_{jk}(n_{jk}(a))\big) + \sum_{l \in N} \sum_{jk \in a_l} D_{jk}(n_{jk}(a))$$

plus constant λ_i which only depends on $g|_{N \setminus \{i\}}$. Let $u(g) \equiv \sum_{l \in N} \sum_{jk \in a_l} D_{jk}(n_{jk}(a))$ and $\theta_{ij}(g|_{ij}) \equiv \sum_{l \in N; ij \in a_l} \big(C_{ij}(n_{ij}(a)) - D_{ij}(n_{ij}(a))\big)$ if $ij \in g$ and 0 otherwise. Then, the payoff function in the network formation by each transportation commissioner is represented as

$$\phi_i(g) \equiv W_i(g,a) = u(g) + \sum_{j \neq i} \theta_{ij}(g|_{ij}).$$

2. R&D network

Another example is Cournot competition with R&D networks studied by Hsieh et al. (2017), which is discussed in Appendix B. Each firm engages in the R&D collaboration with another firm to reduce production cost, whose benefit is measured by a bilateral term θ_{ij}. In the original model of Hsieh et al. (2017), they assume that there is no externalities from the R&D collaboration. However, it might be the case that research output induces not only bilateral benefit but also industry-wide externality from knowledge spread, which corresponds to $u(g)$.

[7] This is a generalization of the standard congestion game (e.g., Beckmann et al. 1956; Rosenthal 1973) where congestion cost in any route is assumed same one i.e., $C(\cdot) = D(\cdot)$.

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