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Hyperopic Topologies Once Again

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Abstract

We construct a hyperopic topology on l^{infty} whose dual coincides with the set of bounded, purely finitely additive measures. This topology is strictly coarser than the norm topology and its dual is strictly contained into the hyperopic dual obtained by Monteiro et al. (2018).

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1 Introduction

The use of topologies in infinite dimensional spaces has been a frequent practise of many authors to capture economic behaviours of individuals. The topologies that stand out most are the Mackey topology (see Bewley 1972) and the strict topology which coincides with the Mackey in l^∞, see for example Brown and Lewis (1981); Conway (1992). Preferences, be they continuous, upper or lower with respect to the earlier topologies capture impatience; myopia; wariness, etc; see for example Araujo, Novinski and Pascoa (2011).

Recently Monteiro et al. (2018) have formalised hyperopic tastes via hyperopic topologies. They guaranteed the existence of the largest locally convex hyperopic topology. However, to have a topology with many open sets it would not seem to be very advantageous, since the number of compact sets would be reduced which could undermine the maximisation process of the utility functions representing hyperopic tastes of economic agents. Agents with hyperopic tastes are called hyperopic economic agents. They only see the distant future by completely neglecting any short run consumption stream.

The main objective of this short paper is to produce a locally convex hyperopic topology with less open sets than the topology defined in Monteiro et al. (2018). We do that by choosing in a convenient manner a sub-family of seminorms from hyperopic seminorms defined by Monteiro et al. (2018). We prove that the dual of l^∞ with this new topology equals the set of bounded, purely finitely additive measures. Applications of these kind of measures can be found in Gilles (1989) and Gilles and LeRoy (1992).

This new topology is important not only as a mathematical object[1] but because of its technical practicality in terms of its applications. See for example, Bastianello (2017). More precisely, this author used the strict hyperopic topology[2] generated by a special family of seminorms indexed by summable sequences whose elements are null at the most for a finite set of indexes.

The paper is organised as follows: Section 2 contains the terminology and the main definitions about hyperopic preferences and topologies. Section 3 deals with the strict hyperopic topology and in it is established the main results of this paper. Finally, the paper ends with a brief conclusion.

[1]Which is important in its own right.

[2]Denoted by β_{sh} and defined in Section 3 below.

2 Notation and basic definitions

Our analysis is carried out in the same setting used in Monteiro et al. (2018). Let l^{∞} be the set of bounded real valued sequences $x = (x_n)_{n \geqq 1}$. The sup-norm is denoted $|x|_{\infty} = \sup\{|x_n| : n \geqq 1\}$. The sup-norm topology is denoted τ_{∞}. We denote by l^1 the set of summable sequences. That is $(x_n)_{n \geqq 1} \in l^1$ if $\sum_{n=1}^{\infty} |x_n| < \infty$. Let $e_n \in l^{\infty}$ be the sequence such that $e_n(m) = 0$ if $m \neq n$ and $e_n(n) = 1$. The vector space generated by $\{e_n : n \geqq 1\}$ is denoted by F. Let $\mathcal{P}(\mathbb{N})$ be the set of subsets of the set of natural numbers, $\mathbb{N} = \{1, 2, \ldots\}$.

Definition 1 *Let* ba$(\mathbb{N})$ *be the set of bounded finitely additive measures. Then,* $\mu \in$ba$(\mathbb{N})$ *is purely finitely additive if* $\mu(A) = 0$ *whenever* $A \subset \mathbb{N}$ *is finite.*

This is equivalent to requiring $\mu(\{n\}) = 0$ for every natural number n. We denote by $pa(\mathbb{N})$ the set of bounded, purely finitely additive measures.

2.1 Preferences

A preference relation on l^{∞} is a complete and transitive binary relation on l^{∞}. Formally:

Definition 2 *A binary relation (on l^{∞}), $\succeq \subset l^{\infty} \times l^{\infty}$, is complete if for every x, y in l^{∞}, $x \succeq y$ or $y \succeq x$. It is transitive if $x \succeq y$ and $y \succeq z$ implies $x \succeq z$.*

By $x \succ y$ we mean that $x \succeq y$ and $\neg(y \succeq x)$, and $x \sim y$ means $x \succeq y$ and $y \succeq x$. For any $x \in l^{\infty}$, we define its $n-$head denoted by x_{hn} to be

$$x_{hn}(k) = \begin{cases} x_k & \text{if} \quad 1 \leqq k \leqq n \\ 0 & \text{if} \quad k > n, \end{cases}$$

and its $n-$tail as $x_n^t = x - x_{hn}$.

Suppose τ is a topology on l^{∞} and $\succeq$ is a preference relation on l^{∞}.

Definition 3 *The preference relation $\succeq$ is τ continuous if for all $x \in l^{\infty}$ the sets $\{y \in l^{\infty} : y \succeq x\}$ and $\{y \in l^{\infty} : x \succeq y\}$ are τ closed.*

2.2 Hyperopic preferences and topologies

Definition 4 *Let $\succeq$ be a preference relation on l^{∞}. Then, $\succeq$ is hyperopic if for all $x, y, z \in l^{\infty}$, $x \succ y$ implies $x \succ y + z$, for every $z \in F$.*

The following definition follows in the same spirit of Brown and Lewis (1981) or Raut(1987) who defined myopic topologies.

Definition 5 *A topology τ on l^∞ is said to be hyperopic if every τ continuous preference relation on l^∞ is hyperopic.*

According to Monteiro el al. (2018) the largest hyperopic topology β_h is generated by Γ_h which consists of all hyperopic seminorms. A seminorm q is hyperopic if and only if $q(z) = 0, \forall z \in F$.

3 The strict hyperopic topology

Let $\mathcal{A}$ be a subset of the set of all sequences $a \in l^1$ such that $a_n \neq 0$ for all but finitely many n. Thus $\sum_{n\geq N} |a_n|$ is always non zero. For each $a \in \mathcal{A}$, define on l^∞ the following function $p_a : l^\infty \to R_+$ as

$$p_a(x) = \lim \sup_{N\to\infty} \frac{\sum_{n\geq N} |a_n x_n|}{\sum_{n\geq N} |a_n|} \tag{1}$$

Clearly p_a satisfies all properties required to be a seminorm. The family of seminorms indexed by $\mathcal{A}$ generates a strict topology[3] which is hyperopic, see Example 3 in Monteiro et al. (2018).

Let us denote by Γ_{sh} the set of seminorms indexed by $\mathcal{A}$ and by $\beta_{sh} := \tau(\Gamma_{sh})$ the locally convex topology generated.

Now, define $\Gamma_s = \{q \in \Gamma_h : q \text{ is norm-continuous}\}$. Clearly Γ_s is nonempty as it contains Γ_{sh}. Since Γ_s contains all norm-continuous seminorms, the locally convex topology that it generates is the largest strict hyperopic topology[4]. Let us denote this topology by β_s. Notice that β_s is not generated by the seminorms p_a with $a \in \mathcal{A}$. The only role that $\{p_a : a \in \mathcal{A}\}$ plays is to prove that Γ_s is nonempty.

Remark 1 *It is useful noting that the normalisation of seminorms defined by (1) is purely technical as shown in the appendix. If $\sum_{n\geq N} |a_n|$ is removed from the denominator in (1) above, this new family of seminorms belong to Γ_s since each element of this new family will go on being a norm-continuous seminorm.*

[3]See Buck (1958) to justify the adjective "strict".

[4]This topology can also be obtained by using Zorn's Lemma applied to the partial ordering relation "to be finer than " on the set of all hyperopic topologies which are coarser than the norm topology.

Remark 2 *It is also important to point out that the strict hyperopic topology β_s, like hyperopic topology β_h introduced by Monteiro et al. (2018), is not Hausdorff. This is due to hyperopic seminorms are not separating by definition, since they vanish on elements in l^∞ of finite support.*

3.1 The strict hyperopic dual of l^∞

On one hand, $\beta_s \subset \tau_\infty$ since by definition all seminorms generate β_s are norm continuous. On the other hand, since $\Gamma_s \subset \Gamma_h, \beta_s \subset \beta_h$. Moreover, from the fact that all elements belonging to Γ_s are norm-continuous, it immediately follows that $\beta_s \subset \tau_\infty$.

Proposition 1 $(l^\infty, \beta_s)' = \mathrm{pa}\,(\mathbb{N})$.

Proof. From the above paragraph it follows that $(l^\infty, \beta_s)' \subset (l^\infty, \beta_h)' \cap (l^\infty, \tau_\infty)'$. Thus, $(l^\infty, \beta_s)' \subset pa\,(\mathbb{N})$ immediately follows from Theorem 4.6 in Monteiro et al. (2018).

To prove the converse, let μ be an element belonging to $pa\,(\mathbb{N})$. We must prove that the linear functional $f_\mu(x) := \int_N x(n)d\mu(n), \forall x \in l^\infty$ associated to μ is β_h- continuous and norm-continuous. The β_h- continuity follows from Lemma 4.4 in Monteiro et al.(2018). In fact, $f_{mu}(e_m) = \int_N e_m(n)d\mu(n) = \mu(\{m\}) = 0$. The norm-continuity follows that μ is zero only in finite subsets of N. In fact, $|f_\mu(x)| \leq \mu(N)||x||_\infty$.

■

Remark 3 *In economic terms, characterizing the dual of l^∞ with respect to β_s as being $pa(N)$ has an important implication in terms of pricing or valuation. If we priced with a price belonging to $pa(N)$ any consumption plan, modelled by a sequence of l^∞ containing infinitely many zeros, the price of such a consumption plan would be zero. This is compatible with hyperopic agents who neglect any short run consumption stream.*

4 Concluding remarks

We have constructed a new topology on l^∞ called strict hyperopic topology. This topology is coarser than the norm topology and was obtained by refining the set of all hyperopic seminorms defined by Monteiro et al. (2018). Moreover, we are able to show that the strict hyperopic dual of l^∞ coincides with $pa\,(\mathbb{N})$, the set of bounded, purely finitely additive measures. If we had larger quantity compact sets[5] the chance of finding maximizers of hyperopic

[5]Although this fact make us have less continuous functions.

preferences in compact budget sets would increase. It is useful pointing out that this trade-off between compactness of sets and continuity of preferences is not being dealt with in this paper. The economic implications of the topology β_s of not having the Hausdorff property also have not been addressed in this paper. The two matters above will be subjects of future research.

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