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Fundamental comparative statics of the canonical agricultural household model

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Abstract

Despite the genesis of the agricultural household model being more than four decades ago, a complete characterization of its fundamental comparative statics has eluded researchers. By employing a differential approach based on generalized compensated derivatives, this gap is closed for a generic form of the agricultural household model. Extensions of the model and the concomitant results are contemplated too. Because the fundamental comparative statics are heretofore unknown as well as observable, they yield a host of new empirically testable implications of the model, and as such permit a complete empirical scrutiny of it for the first time.

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1. Introduction

More than 40 years have passed since Kuroda and Yotopoulos (1978) developed the first agricultural household (hereafter, AH) model. In spite of a number of extensions and generalizations of the basic model, many of which are documented by Taylor and Adelman (2003), the fundamental comparative statics of the canonical AH model developed by Strauss (1986) have yet to be derived in their full generality. The goal, therefore, is to correct this shortcoming. This is of value not only for completeness, but for empirical reasons too, as the resulting comparative statics are observable and refutable, thereby permitting a complete statistical test of the theory for the first time.

Note that fundamental, or intrinsic, or basic, comparative statics are defined as the refutable comparative statics of a static optimization problem that follow from the (i) technical assumption that an interior, locally differentiable solution exists, and (ii) basic assumptions that make up the underlying economic theory. For example, the negative semidefiniteness of the Slutsky matrix is the fundamental comparative statics of the neoclassical utility maximization model, inasmuch as it follows from the technical assumption that an interior, locally differentiable solution exists, and the economic assumptions that preferences are monotonic in goods and independent of prices and income, and that agents are price-takers. Importantly, the main result herein is as basic to the canonical AH model as the Slutsky matrix is to the neoclassical utility maximization model.

As suggested by the neoclassical utility maximization model, the key to deriving the fundamental comparative statics of the AH model is that they should be formulated in a compensated form, i.e., in the form of a linear combination of the partial derivatives of the decision variables with respect to the parameters. Strauss (1986) appears to be the first to recognize the importance of compensated comparative statics in the AH model. Despite this recognition, Strauss (1986) was not successful in uncovering the fundamental comparative statics of the AH model.

The most successful attempt at deriving the fundamental comparative statics of an AH model is that by Saha (1994). Saha (1994) contemplated a special case of the AH model in which an AH was assumed to (i) produce one good by way of a concave production function, (ii) consume two goods, namely, the output of the farm and a composite commodity, and (iii) have no fixed factors of production. In the language of Paris (1989), the model studied by Saha (1994) contains a broken symmetry, and is thus less general than the AH model developed by Strauss (1986) and studied herein. Furthermore, because of the assumed concavity of the production function, the results of Saha (1994) are less general than they could be for this reason too.

The method employed herein to derive the fundamental comparative statics was developed by Partovi and Caputo (2006, 2007). The basic idea lies in the observation that the given parameters of an optimization problem are not, in general, the natural ones for formulating basic comparative statics results. That this is true can be seen in the neoclassical utility maximization model, where a linear combination of partial derivatives in the form of a compensated derivative must be employed in order to obtain the semidefiniteness of the Slutsky matrix. Crucially, Partovi and Caputo (2006) showed that in order to derive the constraint-free semidefiniteness property of a comparative statics matrix without requiring any restriction on the structure of an optimization problem, the compensated derivatives must be chosen along directions that are tangent to the level surfaces of all the constraint functions at each point of the parameter space. Equivalently, acting on the constraint functions, the compensated derivatives are required to return zero at all points of parameter space. This null property is the defining feature of generalized compensated derivatives. Indeed, it is the property responsible for the form of the compensated derivatives, and concomitantly, the form of the fundamental comparative statics. Crucially, by employing compensated derivatives that possess the null property, the process of deriving comparative statics differentially accounts for the constraints of the optimization problem, thereby leading to a semidefinite comparative statics matrix free of constraint.

2. The Canonical Agricultural Household Model and Assumptions

Let $\mathbf{C} \in \mathbb{R}^{\ell}_{++}$ denote the vector of goods consumed by a price-taking AH at prices $\mathbf{p} \in \mathbb{R}^{\ell}_{++}$, where $C_\ell \in \mathbb{R}_{++}$ is the consumption of leisure time and $p_\ell \in \mathbb{R}_{++}$ is its price, i.e., the wage rate. Preferences are represented by a felicity function $U(\cdot)$, the value of which is $U(\mathbf{C})$.

An AH has $T \in \mathbb{R}_{++}$ units of time available, which it divides between leisure $C_\ell \in \mathbb{R}_{++}$ and household labor $H \in \mathbb{R}_{++}$. As a result the time constraint is $T = H + C_\ell$. An AH also produces a vector of outputs $\mathbf{Q} \in \mathbb{R}^{M}_{++}$, which it sells at prices $\mathbf{q} \in \mathbb{R}^{M}_{++}$. The outputs are produced by combining a vector of variable inputs distinct from labor, say $\mathbf{V} \in \mathbb{R}^{N}_{++}$, a fixed input $K \in \mathbb{R}_{++}$, as well as the total amount of labor used in production $L \in \mathbb{R}_{++}$, the latter being purchased at the wage rate $p_\ell \in \mathbb{R}_{++}$. Observe that if $H < L$, then an AH is a net buyer of labor, whereas if $H > L$, then it is a net seller. In view of the multiproduct nature of an AH, the production function $F(\cdot)$ is given in implicit form as $F(\mathbf{Q},\mathbf{V},L;K) = 0$, and is the second constraint faced by an AH.

The third and final constraint faced by an AH is its budget constraint. Total revenue consists of the revenue from the sale of its outputs $\mathbf{q}'\mathbf{Q}$, where "′" denotes transposition, labor earnings $p_\ell H$, and exogenous income $E \in \mathbb{R}_{++}$. An AH has variable input costs $\mathbf{r}'\mathbf{V}$, where $\mathbf{r} \in \mathbb{R}^{N}_{++}$ is the variable input price vector, total labor costs $p_\ell L$, and expenditures on consumption excluding leisure, namely $\sum_{i=1}^{\ell-1} p_i C_i$. The budget constraint of an AH therefore takes the form $\mathbf{q}'\mathbf{Q} + p_\ell H + E = \mathbf{r}'\mathbf{V} + p_\ell L + \sum_{i=1}^{\ell-1} p_i C_i$. But by solving the time constraint for household labor, i.e., $H = T - C_\ell$, and substituting it in the budget constraint, one arrives at the so-called full-income version of the budget constraint, viz., $\mathbf{q}'\mathbf{Q} + p_\ell T + E - \mathbf{r}'\mathbf{V} - p_\ell L - \mathbf{p}'\mathbf{C} = 0$.

Pulling all of the above information together, an AH is asserted to behave as if solving the constrained optimization problem

$$\max_{\mathbf{C},\mathbf{Q},\mathbf{V},L} \left\{ U(\mathbf{C}) \text{ s.t. } \mathbf{q}'\mathbf{Q} + p_\ell T + E - \mathbf{r}'\mathbf{V} - p_\ell L - \mathbf{p}'\mathbf{C} = 0,\ F(\mathbf{Q},\mathbf{V},L;K) = 0 \right\}. \tag{1}$$

Problem (1) is essentially the canonical AH model put forth by Strauss (1986, pp. 71–73). In order to ease the notation, define $\mathbf{a} \stackrel{\text{def}}{=} (\mathbf{p},\mathbf{q},\mathbf{r},K,E) \in \mathbb{R}^{\ell+M+N+2}_{++}$ as the parameter vector, i.e.,

$$a_\gamma \stackrel{\text{def}}{=} \begin{cases} p_\gamma, & \gamma = 1,2,\ldots,\ell, \\ q_{\gamma-\ell}, & \gamma = \ell+1,\ell+2,\ldots,\ell+M, \\ r_{\gamma-\ell-M}, & \gamma = \ell+M+1,\ell+M=2,\ldots,\ell+M+N, \\ K, & \gamma = \ell+M+N+1, \\ E, & \gamma = \ell+M+N+2. \end{cases} \tag{2}$$

Definition (2) is also helpful in deriving the basic comparative statics of problem (1). Note that the parameter T is suppressed as an element of $\mathbf{a}$ henceforth, seeing as it is a fixed constant.

The following assumptions are imposed on problem (1) and discussed subsequently.

(A1) For all $\mathbf{C} \in \mathbb{R}^{\ell}_{++}$, $U(\cdot) \in C^{(2)}$ and $U_{C_l}(\mathbf{C}) > 0$, $l = 1,2,\ldots,\ell$.

(A2) For all $(\mathbf{Q},\mathbf{V},L;K) \in \mathbb{R}^{M+N+2}_{++}$, $F(\cdot) \in C^{(2)}$, $F_{Q_m}(\mathbf{Q},\mathbf{V},L;K) > 0$, $m = 1,2,\ldots,M$, $F_{V_n}(\mathbf{Q},\mathbf{V},L;K) < 0$, $n = 1,2,\ldots,N$, $F_L(\mathbf{Q},\mathbf{V},L;K) < 0$, and $F_K(\mathbf{Q},\mathbf{V},L;K) < 0$.

(A3) For all $\mathbf{a} \in A$, A an open set, there exists a $C^{(1)}$ interior solution to problem (1), denoted by $\left(\mathbf{C}^*(\mathbf{a}),\mathbf{Q}^*(\mathbf{a}),\mathbf{V}^*(\mathbf{a}),L^*(\mathbf{a})\right)$, with Lagrange multipliers $\left(\lambda^*(\mathbf{a}),\mu^*(\mathbf{a})\right)$.

Assumption (A1) says that preferences are representable by a twice continuously differentiable function that is strictly monotonic. Supposition (A2) asserts that the implicit production function is twice continuously differentiable, strictly increasing in the outputs, and strictly decreasing in the inputs—the usual assumptions. Note that no global curvature assumptions have been placed on the felicity or production functions, as none are required for the derivation of the fundamental comparative statics. And finally, because the focus is on differential comparative statics, supposition (A3) is essential, as the main result relies only on local necessary conditions of optimality and local differentiability.

3. Preliminary Results

This section is devoted to establishing four results that are useful in proving the central result of the paper. To begin, define $\mathbf{x} \stackrel{\text{def}}{=} (\mathbf{C},\mathbf{Q},\mathbf{V},L) \in \mathbb{R}^{\ell+M+N+1}_{++}$ as the decision vector, that is,

$$x_k \stackrel{\text{def}}{=} \begin{cases} C_k, & k=1,2,\ldots,\ell, \\ Q_{k-\ell}, & k=\ell+1,\ell+2,\ldots,\ell+M, \\ V_{k-\ell-M}, & k=\ell+M+1,\ell+M=2,\ldots,\ell+M+N, \\ L, & k=\ell+M+N+1. \end{cases} \tag{3}$$

As was the case for Eq. (2), Eq. (3) is helpful in deriving the fundamental comparative statics of problem (1). The Lagrangian for problem (1) is given by

$$\Lambda(\mathbf{x},\lambda,\mu;\mathbf{a}) \stackrel{\text{def}}{=} U(\mathbf{C}) + \lambda[\mathbf{q}'\mathbf{Q} + p_\ell T + E - \mathbf{r}'\mathbf{V} - p_\ell L - \mathbf{p}'\mathbf{C}] + \mu F(\mathbf{Q},\mathbf{V},L;K), \tag{4}$$

while the first-order necessary conditions are

$$\Lambda_{x_k}(\mathbf{x},\lambda,\mu;\mathbf{a}) = U_{C_k}(\mathbf{C}) - \lambda p_k = 0,\ k=1,2,\ldots,\ell, \tag{5}$$

$$\Lambda_{x_k}(\mathbf{x},\lambda,\mu;\mathbf{a}) = \lambda q_{k-\ell} + \mu F_{Q_{k-\ell}}(\mathbf{Q},\mathbf{V},L;K) = 0,\ k=\ell+1,\ell+2,\ldots,\ell+M, \tag{6}$$

$$\Lambda_{x_k}(\mathbf{x},\lambda,\mu;\mathbf{a}) = -\lambda r_{k-\ell-M} + \mu F_{V_{k-\ell-M}}(\mathbf{Q},\mathbf{V},L;K) = 0,\ k=\ell+M+1,\ell+M+2,\ldots,\ell+M+N, \tag{7}$$

$$\Lambda_{x_k}(\mathbf{x},\lambda,\mu;\mathbf{a}) = -\lambda p_\ell + \mu F_L(\mathbf{Q},\mathbf{V},L;K) = 0,\ k=\ell+M+M+1, \tag{8}$$

$$\Lambda_\lambda(\mathbf{x},\lambda,\mu;\mathbf{a}) = \mathbf{q}'\mathbf{Q} + p_\ell T + E - \mathbf{r}'\mathbf{V} - p_\ell L - \mathbf{p}'\mathbf{C} = 0, \tag{9}$$

$$\Lambda_\mu(\mathbf{x},\lambda,\mu;\mathbf{a}) = F(\mathbf{Q},\mathbf{V},L;K) = 0. \tag{10}$$

Being a solution to problem (1) for all $\mathbf{a} \in A$, $\left(\mathbf{C}^*(\mathbf{a}),\mathbf{Q}^*(\mathbf{a}),\mathbf{V}^*(\mathbf{a}),L^*(\mathbf{a})\right)$ and $\left(\lambda^*(\mathbf{a}),\mu^*(\mathbf{a})\right)$ are necessarily the solution to Eqs. (5)–(10).

The arrive at the first two results, rearrange Eq. (8) to get $\lambda/\mu = F_L(\mathbf{Q},\mathbf{V},L;K)/p_\ell < 0$, the inequality following from assumption (A2). Then substitute $\lambda/\mu = F_L(\mathbf{Q},\mathbf{V},L;K)/p_\ell$ in Eqs. (6) and (7) and combine them with Eq. (10) to get a system of $M+N+1$ equations in the $M+N+1$ variables $(\mathbf{Q},\mathbf{V},L)$, to wit,

$$\begin{gathered} \frac{F_L(\mathbf{Q},\mathbf{V},L;K)}{p_\ell} = -\frac{F_{Q_{k-\ell}}(\mathbf{Q},\mathbf{V},L;K)}{q_{k-\ell}},\ k=\ell+1,\ell+2,\ldots,\ell+M, \\ \frac{F_L(\mathbf{Q},\mathbf{V},L;K)}{p_\ell} = \frac{F_{V_{k-\ell-M}}(\mathbf{Q},\mathbf{V},L;K)}{r_{k-\ell-M}},\ k=\ell+M+1,\ell+M+2,\ldots,\ell+M+N, \\ F(\mathbf{Q},\mathbf{V},L;K) = 0, \end{gathered} \tag{11}$$

the solution of which is $\left(\mathbf{Q}^*(\mathbf{a}),\mathbf{V}^*(\mathbf{a}),L^*(\mathbf{a})\right)$. Seeing as $(p_1,p_2,\ldots,p_{\ell-1},E)$ do not appear in Eq. (11), it follows that $\left(\mathbf{Q}^*(\mathbf{a}),\mathbf{V}^*(\mathbf{a}),L^*(\mathbf{a})\right)$ are not functions of $(p_1,p_2,\ldots,p_{\ell-1},E)$. To find the solution for $\mathbf{C}$ and λ, substitute $\left(\mathbf{Q}^*(\mathbf{a}),\mathbf{V}^*(\mathbf{a}),L^*(\mathbf{a})\right)$ in Eq. (9) and solve it and Eq. (5) for $\mathbf{C}$ and λ, yielding $\mathbf{C}^*(\mathbf{a})$ and $\lambda^*(\mathbf{a})$. Finally, substitute $\left(\mathbf{Q}^*(\mathbf{a}),\mathbf{V}^*(\mathbf{a}),L^*(\mathbf{a})\right)$ and $\lambda^*(\mathbf{a})$ in, say, Eq. (8) and solve it for μ to get $\mu^*(\mathbf{a})$. The preceding shows that Eqs. (5)–(10) are recursive

Now observe that by assumption (A1) and Eq. (5) that $\lambda^*(\mathbf{a}) \equiv U_{C_k}(\mathbf{C}^*(\mathbf{a}))/p_k > 0$, $k = 1,2,\ldots,\ell$. And finally, recall that $\lambda^*(\mathbf{a})/\mu^*(\mathbf{a}) \equiv F_L(\mathbf{Q}^*(\mathbf{a}),\mathbf{V}^*(\mathbf{a}),L^*(\mathbf{a});K)/p_\ell < 0$. As $\lambda^*(\mathbf{a}) > 0$, it follows that $\mu^*(\mathbf{a}) < 0$. The deductions in this and the preceding paragraph are summarized in the following lemma.

Lemma 1. *Under assumptions (A1)–(A3), Eqs. (5)–(10) are recursive,* $(\mathbf{Q}^*(\mathbf{a}),\mathbf{V}^*(\mathbf{a}),L^*(\mathbf{a}))$ *are not functions of* $(p_1,p_2,\ldots,p_{\ell-1},E)$, $\lambda^*(\mathbf{a}) > 0$, *and* $\mu^*(\mathbf{a}) < 0$.

4. Fundamental Comparative Statics

The process of constructing the fundamental comparative statics of problem (1) begins by recalling the definition of the parameter vector $\mathbf{a}$ given in Eq. (2). The gradient operator with respect $\mathbf{a}$ is then defined as

$$\nabla_{\mathbf{a}} \stackrel{\text{def}}{=} \left(\frac{\partial}{\partial p_1},\frac{\partial}{\partial p_2},\ldots,\frac{\partial}{\partial p_\ell},\frac{\partial}{\partial q_1},\frac{\partial}{\partial q_2},\ldots,\frac{\partial}{\partial q_M},\frac{\partial}{\partial r_1},\frac{\partial}{\partial r_2},\ldots,\frac{\partial}{\partial r_N},\frac{\partial}{\partial K},\frac{\partial}{\partial E}\right). \tag{12}$$

Next, define $g^1(\mathbf{x};\mathbf{a}) \stackrel{\text{def}}{=} \mathbf{q}'\mathbf{Q} + p_\ell T + E - \mathbf{r}'\mathbf{V} - p_\ell L - \mathbf{p}'\mathbf{C}$ and $g^2(\mathbf{x};\mathbf{a}) \stackrel{\text{def}}{=} F(\mathbf{Q},\mathbf{V},L;K)$. The normal directions to the level set of the constraint functions in parameter space are thus

$$\nabla_{\mathbf{a}} g^1(\mathbf{x};\mathbf{a}) = (-C_1,-C_2,\ldots,-C_{\ell-1},T-L-C_\ell,Q_1,Q_2,\ldots,Q_M,-V_1,-V_2,\ldots,-V_N,0,1) \in \mathbb{R}^{\ell+M+N+2}, \tag{13}$$

$$\nabla_{\mathbf{a}} g^2(\mathbf{x};\mathbf{a}) = (\mathbf{0}'_\ell,\mathbf{0}'_M,\mathbf{0}'_N,F_K(\mathbf{Q},\mathbf{V},L;K),0) \in \mathbb{R}^{\ell+M+N+2}, \tag{14}$$

where, say, $\mathbf{0}'_M$ is the null row vector in $\mathbb{R}^M$. As $\nabla_{\mathbf{a}} g^1(\mathbf{x};\mathbf{a}) \cdot \nabla_{\mathbf{a}} g^2(\mathbf{x};\mathbf{a}) = 0$, the normal vectors are orthogonal and thus linearly independent. By the implicit function theorem, the pair of constraints $g^1(\mathbf{x};\mathbf{a}) = 0$ and $g^2(\mathbf{x};\mathbf{a}) = 0$ define an $(\ell+M+N)$-dimensional manifold in $\mathbb{R}^{\ell+M+N+2}$. This implies that the dimension of the tangent hyperplane to the level set of the constraint functions is $\ell+M+N$, and hence that $\ell+M+N$ basis vectors are required for its complete description.

What is now required is a set of $\ell+M+N$ basis vectors in $\mathbb{R}^{\ell+M+N+2}$ for the tangent hyperplane to the level set of the constraint functions in parameter space. A set of basis vectors for the aforesaid hyperplane is given by

$$\mathbf{t}^\alpha \stackrel{\text{def}}{=} (\mathbf{e}^\alpha_\ell,\mathbf{0}'_M,\mathbf{0}'_N,0,C_\alpha + \delta_{\ell\alpha}[L-T]),\ \alpha = 1,2,\ldots,\ell, \tag{15}$$

$$\mathbf{t}^\alpha \stackrel{\text{def}}{=} (\mathbf{0}'_\ell,\mathbf{e}^{\alpha-\ell}_M,\mathbf{0}'_N,0,-Q_{\alpha-\ell}),\ \alpha = \ell+1,\ell+2,\ldots,\ell+M, \tag{16}$$

$$\mathbf{t}^\alpha \stackrel{\text{def}}{=} (\mathbf{0}'_\ell,\mathbf{0}'_M,\mathbf{e}^{\alpha-\ell-M}_N,0,V_{\alpha-\ell-M}),\ \alpha = \ell+M+1,\ell+M+2,\ldots,\ell+M+N, \tag{17}$$

where, say, $\mathbf{e}^\alpha_\ell$ is the standard basis vector in $\mathbb{R}^\ell$, i.e., $\mathbf{e}^\alpha_\ell \stackrel{\text{def}}{=} (0_1,0_2,\ldots,0_{\alpha-1},1_\alpha,0_{\alpha+1},\ldots,0_\ell)$, with unity in the αth position and zeroes elsewhere, and $\delta_{\ell\alpha}$ is the Kronecker delta function. As $\nabla_{\mathbf{a}} g^\kappa(\mathbf{x};\mathbf{a}) \cdot \mathbf{t}^\alpha = 0$ for $\kappa = 1,2$ and $\alpha = 1,2,\ldots,\ell+M+N$, these vectors lie in the tangent hyperplane to the level set of the constraint functions. Moreover, in view of the fact that the only solution to the linear system $\sum_{\alpha=1}^{\ell+M+N} c_\alpha \mathbf{t}^\alpha = \mathbf{0}_{\ell+M+N+2}$ is the null vector $\mathbf{c} = \mathbf{0}_{\ell+M+N}$, the $\ell+M+N$ vectors $\mathbf{t}^\alpha$ are linearly independent and thus form a basis for the said hyperplane, as claimed.

Equipped with the $\ell+M+N$ basis vectors $\mathbf{t}^\alpha$, it is now a simple matter to derive the complete set of generalized compensated derivatives (GCDs) for problem (1). Following Partovi and Caputo (2006), they are defined, in general, as

$$D_\alpha(\mathbf{x};\mathbf{a}) \stackrel{\text{def}}{=} \mathbf{t}^\alpha \cdot \nabla_{\mathbf{a}},\ \alpha = 1,2,\ldots,\ell+M+N. \tag{18}$$

In particular, using Eq. (12) and Eqs. (15)–(17), the CGDs are given by

$$D_\alpha(\mathbf{x};\mathbf{a}) = \frac{\partial}{\partial p_\alpha} + \left[C_\alpha + \delta_{\ell\alpha}[L-T]\right]\frac{\partial}{\partial E}\ ,\ \alpha = 1,2,\ldots,\ell\ , \tag{19}$$

$$D_\alpha(\mathbf{x};\mathbf{a}) = \frac{\partial}{\partial q_{\alpha-\ell}} - Q_{\alpha-\ell}\frac{\partial}{\partial E}\ ,\ \alpha = \ell+1,\ell+2,\ldots,\ell+M\ , \tag{20}$$

$$D_\alpha(\mathbf{x};\mathbf{a}) = \frac{\partial}{\partial r_{\alpha-\ell-M}} + V_{\alpha-\ell-M}\frac{\partial}{\partial E}\ ,\ \alpha = \ell+M+1,\ell+M+2,\ldots,\ell+M+N\ . \tag{21}$$

The $\ell+M+N$ GCDs in Eqs. (19)–(21) possess the null property discussed in §1, seeing as $D_\alpha(\mathbf{x};\mathbf{a}) \circ g^\kappa(\mathbf{x};\mathbf{a}) \equiv 0$ for $\kappa = 1,2$ and $\alpha = 1,2,\ldots,\ell+M+N$.

Given the $\ell+M+N$ GCDs in Eqs. (19)–(21), one can apply Theorem 1 of Partovi and Caputo (2006, 2007) to derive the fundamental, constraint-free, negative semidefinite $(\ell+M+N)\times(\ell+M+N)$ comparative statics matrix (CSM) for problem (1), say $\Omega(\mathbf{a})$. Using the heretofore established notation, Theorem 1 of Partovi and Caputo (2006, 2007) asserts that the typical element of $\Omega(\mathbf{a})$ takes the form

$$\Omega_{\alpha\beta}(\mathbf{a}) = -\sum_{k=1}^{\ell+M+N+1}\left[D_\alpha\left(\mathbf{x}^*(\mathbf{a});\mathbf{a}\right)\circ \Lambda_{x_k}\left(\mathbf{x}^*(\mathbf{a}),\lambda^*(\mathbf{a}),\mu^*(\mathbf{a});\mathbf{a}\right)\right]\left[D_\beta\left(\mathbf{x}^*(\mathbf{a});\mathbf{a}\right)\circ x_k^*(\mathbf{a})\right], \tag{22}$$

for $\alpha,\beta = 1,2,\ldots,\ell+M+N$. The proof of the ensuing proposition is given in Appendix I.

Proposition 1. *Let assumptions (A1)–(A3) hold. The fundamental comparative statics matrix* $\Omega^*(\mathbf{a})$ *of problem (1) is negative semidefinite with a maximum rank of* $\ell+M+N-1$, *where*

$$\Omega^*(\mathbf{a}) \stackrel{\text{def}}{=} \left[\begin{array}{c|c|c}
\begin{gathered}\frac{\partial C_\alpha^*(\mathbf{a})}{\partial p_\beta} + C_\beta^*(\mathbf{a})\frac{\partial C_\alpha^*(\mathbf{a})}{\partial E}\\ +\delta_{\ell\beta}[L^*(\mathbf{a})-T]\frac{\partial C_\alpha^*(\mathbf{a})}{\partial E}\\ +\delta_{\alpha\ell}\delta_{\ell\beta}\frac{\partial L^*(\mathbf{a})}{\partial p_\beta}\\ {\scriptstyle \alpha=1,2,\ldots,\ell}\\ {\scriptstyle \beta=1,2,\ldots,\ell}\end{gathered} &
\begin{gathered}\frac{\partial C_\alpha^*(\mathbf{a})}{\partial q_{\beta-\ell}} - Q_{\beta-\ell}^*(\mathbf{a})\frac{\partial C_\alpha^*(\mathbf{a})}{\partial E}\\ +\delta_{\alpha\ell}\frac{\partial L^*(\mathbf{a})}{\partial q_{\beta-\ell}}\\ {\scriptstyle \alpha=1,2,\ldots,\ell}\\ {\scriptstyle \beta=\ell+1,\ell+2,\ldots,\ell+M}\end{gathered} &
\begin{gathered}\frac{\partial C_\alpha^*(\mathbf{a})}{\partial r_{\beta-\ell-M}} + V_{\beta-\ell-M}^*(\mathbf{a})\frac{\partial C_\alpha^*(\mathbf{a})}{\partial E}\\ +\delta_{\alpha\ell}\frac{\partial L^*(\mathbf{a})}{\partial r_{\beta-\ell-M}}\\ {\scriptstyle \alpha=1,2,\ldots,\ell}\\ {\scriptstyle \beta=\ell+M+1,\ell+M+2,\ldots,\ell+M+N}\end{gathered} \\ \hline
\begin{gathered}-\delta_{\ell\beta}\frac{\partial Q_{\alpha-\ell}^*(\mathbf{a})}{\partial p_\beta}\\ {\scriptstyle \alpha=\ell+1,\ell+2,\ldots,\ell+M}\\ {\scriptstyle \beta=1,2,\ldots,\ell}\end{gathered} &
\begin{gathered}-\frac{\partial Q_{\alpha-\ell}^*(\mathbf{a})}{\partial q_{\beta-\ell}}\\ {\scriptstyle \alpha=\ell+1,\ell+2,\ldots,\ell+M}\\ {\scriptstyle \beta=\ell+1,\ell+2,\ldots,\ell+M}\end{gathered} &
\begin{gathered}-\frac{\partial Q_{\alpha-\ell}^*(\mathbf{a})}{\partial r_{\beta-\ell-M}}\\ {\scriptstyle \alpha=\ell+1,\ell+2,\ldots,\ell+M}\\ {\scriptstyle \beta=\ell+M+1,\ell+M+2,\ldots,\ell+M+N}\end{gathered} \\ \hline
\begin{gathered}\delta_{\ell\beta}\frac{\partial V_{\alpha-\ell-M}^*(\mathbf{a})}{\partial p_\beta}\\ {\scriptstyle \alpha=\ell+M+1,\ell+M+2,\ldots,\ell+M+N}\\ {\scriptstyle \beta=1,2,\ldots,\ell}\end{gathered} &
\begin{gathered}\frac{\partial V_{\alpha-\ell-M}^*(\mathbf{a})}{\partial q_{\beta-\ell}}\\ {\scriptstyle \alpha=\ell+M+1,\ell+M+2,\ldots,\ell+M+N}\\ {\scriptstyle \beta=\ell+1,\ell+2,\ldots,\ell+M}\end{gathered} &
\begin{gathered}\frac{\partial V_{\alpha-\ell-M}^*(\mathbf{a})}{\partial r_{\beta-\ell-M}}\\ {\scriptstyle \alpha=\ell+M+1,\ell+M+2,\ldots,\ell+M+N}\\ {\scriptstyle \beta=\ell+M+1,\ell+M+2,\ldots,\ell+M+N}\end{gathered}
\end{array}\right]. \tag{23}$$

Proposition 1 gives the heretofore unknown fundamental comparative statics of the canonical AH model. The result only relies on the maximization assertion and the basic economic assumptions defining the model. It does not rely on global curvature assumptions on preferences or technology, or on functional form stipulations. Nor does it rely on second-order sufficient conditions as does the derivation of comparative statics based on the implicit function theorem. Also new is the upper bound on the rank of $\Omega^*(\mathbf{a})$, which implies that it is singular, as its order exceeds its rank by at least one. It is also worth noting that Proposition 1 provides a generalization of the results of Saha (1994, p 114 and Lemma 1), because, as mentioned in §1, the AH model he contemplated is considerably less general than that contemplated here. Moreover, in contrast to the claim of Saha (1994, p. 114), the properties of $\Omega^*(\mathbf{a})$ do not depend on the concavity of the production function, seeing as it was not needed to prove Proposition 1.

Crucially, the CSM is observable, as it consists of the levels or partial derivatives of the consumption rates, total labor employed, the rates of output, and the variable inputs, which are typically available or derivable from demand and supply functions that make use of such data. This is important because it permits, for the first time, full empirical scrutiny of the canonical model's basic empirically testable implications.

Because (i) the form of the CSM $\Omega^*(\mathbf{a})$ depends on the GCDs in Eqs. (19)–(21), (ii) the GCDs depend on the basis vectors in Eqs. (15)–(17), and (iii) the basis vectors are not unique, it follows that $\Omega^*(\mathbf{a})$ is not unique. Theorem 5 of Partovi and Caputo (2006) acknowledges this fact and provides the formal relationship between GCDs and CSMs when they are formed using different sets of basis vectors. A key deduction is that because two sets of basis vectors for the tangent hyperplane to the level set of the constraint functions in parameter space provide a description fully equivalent to each other, the two CSMs derived from them are congruent. Not only that, the CSMs are essentially equivalent in the sense that they are of equal rank and the semidefiniteness of one implies that of the other. Even so, it should be noted that congruency does not imply similarity of economic implications, because the two CSMs can be quite different with respect to such matters as observability and empirical verification, a point further elaborated upon below.

It is important to understand that no problems arise because a CSM is not unique. Indeed, the fact that a CSM is not unique is a strength of the Partovi and Caputo (2006) methodology, as is now explained. If, for example, theoretical matters are the main focus, the freedom in the choice of basis vectors allows one to choose those that yield a form of the CSM that has the most intuitive, or natural, economic interpretation. If, on the other hand, interest lies in empirical testing of the negative semidefiniteness of a CSM, then one could choose a set of basis vectors that yield a form of the CSM that includes only those endogenous and exogenous variables for which one has the data to carry out such an empirical test.

The southeast $(M+N)\times(M+N)$ block along the diagonal of $\Omega^*(\mathbf{a})$ gives the basic comparative statics of the production side of the AH model, as it involves the uncompensated (or partial) derivatives of the output supply and variable input demand functions with respect to their prices The negative semidefiniteness of the block implies that the laws of output supply and factor demand hold, and its symmetry yields a generalization of the prototypical reciprocity relations of the price-taking, profit-maximizing model of a firm. The proof of Proposition 1 shows that because the output supply and variable input demand functions are independent of exogenous income, the block does not involve compensated derivatives. But this is simply an implication of the separability of the production and consumption decisions indicated in Lemma 1.

The northwest $\ell\times\ell$ block along the diagonal of $\Omega^*(\mathbf{a})$ gives the fundamental comparative statics of consumption and leisure. It shows that for the goods and leisure, the form of the compensated comparative statics with respect to the prices of the goods is identical to that of the neoclassical Slutsky matrix. This is to be expected, as the prices of the goods enter the canonical AH and neoclassical budget constraints identically. For the wage rate, on the other hand, the compensation must account for the fact that the wage rate does not enter in a prototypical fashion, owing to the fact that an AH uses its own labor to produce its outputs. This complication can be seen in the ℓth column of the submatrix. Indeed, the compensation takes an even more complicated form when it pertains to leisure, as shown by the additional term in the ℓth row and column of the submatrix. For these reasons, the reciprocity relations are more complicated than their neoclassical counterparts as well.

The last four off-diagonal submatrices give the remaining reciprocity relations of the canonical AH model, and they too provide a more general, and thus richer, set of compensated comparative statics than those extant, for the reasons already mentioned. To sharpen these reciprocity results, assume that $\ell = 2$, $M = 1$, and $N = 1$. It then follows from the symmetry of $\Omega^*(\mathbf{a})$ that $\partial C_1^*(\mathbf{a})/\partial q_1 = Q_1^*(\mathbf{a})\partial C_1^*(\mathbf{a})/\partial E$. This asserts that the market good consumed by an AH is normal if and only if the consumption of the market good increases as the market price of the good produced by the AH increases. Another such reciprocity relation follows from the aforesaid symmetry as well, and is given by $\partial C_1^*(\mathbf{a})/\partial r_1 = -V_1^*(\mathbf{a})\partial C_1^*(\mathbf{a})/\partial E$. It asserts that the market good consumed by an AH is normal if and only if the consumption of the market good decreases as the market price of the non-labor input used by the AH increases. Taken together, the two results show that an increase in the market price of the good produced by the AH has the opposite effect on the consumption of the market good as does an increase in market price of the non-labor input used by the AH, i.e., $\operatorname{sign}\left[\partial C_1^*(\mathbf{a})/\partial q_1\right] = -\operatorname{sign}\left[\partial C_1^*(\mathbf{a})/\partial r_1\right]$.

Several, but not all, of the properties of $\Omega^*(\mathbf{a})$ can be derived by way of more traditional approaches. Take the southeast $(M+N)\times(M+N)$ block along the diagonal of $\Omega^*(\mathbf{a})$. Its negative semidefiniteness can be derived by appealing to the (i) recursive nature of the AH model given in Lemma 1, (ii) convexity of the implied AH indirect profit function in the output and input prices, and (iii) envelope theorem. Similarly, one can derive the negative semidefiniteness of the northwest $\ell \times \ell$ block along the diagonal of $\Omega^*(\mathbf{a})$ by (i) forming identities between the AH consumption functions and the Marshallian consumption functions using the AH indirect profit function, (ii) using the chain rule and the Slutsky equation on said identities, and (iii) invoking the negative semidefiniteness of the Slutsky matrix. That said, the symmetry between the AH consumption functions and the output supply and input demand functions contained in the remaining four off-diagonal submatrices of $\Omega^*(\mathbf{a})$ does not appear to be readily deduced using the preceding approaches. This is because they work on but a portion of the AH model, in contrast to the methodology of Partovi and Caputo (2006). For this reason, the negative semidefiniteness of $\Omega^*(\mathbf{a})$ cannot be deduced using the above approaches either.

In closing, note that Appendix II briefly considers the preceding results for a perturbation of the AH model that includes a staple good, defined as a good which is produced and consumed.

5. Conclusion

By employing the differential comparative statics method of Partovi and Caputo (2006), the fundamental comparative statics of the canonical AH model have been derived. In doing so, the underlying reason for the compensation scheme has been illuminated, and because the method of derivation is general, it may be readily applied to extensions of the canonical model. What is more, because the fundamental comparative statics are measurable with the usual types of data employed by empiricists, they form the basic, empirically testable properties of the AH model. Therefore, for the first time, a full empirical test of the basic behavioral properties of the canonical AH model can be carried out.

6. Appendix I

Proof of Proposition 1. First note that the fundamental comparative statics matrix $\Omega(\mathbf{a})$, the typical element of which is given by Eq. (22), is negative semidefinite by Theorem 1 of Partovi and Caputo (2006). Then apply Eq. (22) to Eqs. (5)–(8) using the GCDs defined in Eqs. (19)–(21) to derive the elements of $\Omega(\mathbf{a})$:

$$\begin{aligned}\underset{\substack{\alpha=1,2,\ldots,\ell\\ \beta=1,2,\ldots,\ell}}{\Omega_{\alpha\beta}(\mathbf{a})} = \lambda^*(\mathbf{a})\Bigg\{&\left[\frac{\partial C^*_\alpha(\mathbf{a})}{\partial p_\beta}+\left[C^*_\beta(\mathbf{a})+\delta_{\ell\beta}[L^*(\mathbf{a})-T]\right]\frac{\partial C^*_\alpha(\mathbf{a})}{\partial E}\right]\\ &+\delta_{\alpha\ell}\left[\frac{\partial L^*(\mathbf{a})}{\partial p_\beta}+\left[C^*_\beta(\mathbf{a})+\delta_{\ell\beta}[L^*(\mathbf{a})-T]\right]\frac{\partial L^*(\mathbf{a})}{\partial E}\right]\Bigg\},\end{aligned}\tag{24}$$

$$\underset{\substack{\alpha=1,2,\ldots,\ell\\ \beta=\ell+1,\ell+2,\ldots,\ell+M}}{\Omega_{\ell\beta}(\mathbf{a})} = \lambda^*(\mathbf{a})\left\{\left[\frac{\partial C^*_\alpha(\mathbf{a})}{\partial q_{\beta-\ell}}-Q^*_{\beta-\ell}(\mathbf{a})\frac{\partial C^*_\alpha(\mathbf{a})}{\partial E}\right]+\delta_{\alpha\ell}\left[\frac{\partial L^*(\mathbf{a})}{\partial q_{\beta-\ell}}-Q^*_{\beta-\ell}(\mathbf{a})\frac{\partial L^*(\mathbf{a})}{\partial E}\right]\right\},\tag{25}$$

$$\underset{\substack{\alpha=1,2,\ldots,\ell\\ \beta=\ell+M+1,\ell+M+2,\ldots,\ell+M+N}}{\Omega_{\ell\beta}(\mathbf{a})} = \lambda^*(\mathbf{a})\left\{\left[\frac{\partial C^*_\alpha(\mathbf{a})}{\partial r_{\beta-\ell-M}}+V^*_{\beta-\ell-M}(\mathbf{a})\frac{\partial C^*_\alpha(\mathbf{a})}{\partial E}\right]+\delta_{\alpha\ell}\left[\frac{\partial L^*(\mathbf{a})}{\partial r_{\beta-\ell-M}}+V^*_{\beta-\ell-M}(\mathbf{a})\frac{\partial L^*(\mathbf{a})}{\partial E}\right]\right\},\tag{26}$$

$$\underset{\substack{\alpha=\ell+1,\ell+2,\ldots,\ell+M\\ \beta=1,2,\ldots,\ell}}{\Omega_{\alpha\beta}(\mathbf{a})} = -\lambda^*(\mathbf{a})\left[\frac{\partial Q^*_{\alpha-\ell}(\mathbf{a})}{\partial p_\beta}+\left[C^*_\beta(\mathbf{a})+\delta_{\ell\beta}[L^*(\mathbf{a})-T]\right]\frac{\partial Q^*_{\alpha-\ell}(\mathbf{a})}{\partial E}\right],\tag{27}$$

$$\underset{\substack{\alpha=\ell+1,\ell+2,\ldots,\ell+M\\ \beta=\ell+1,\ell+2,\ldots,\ell+M}}{\Omega_{\alpha\beta}(\mathbf{a})} = -\lambda^*(\mathbf{a})\left[\frac{\partial Q^*_{\alpha-\ell}(\mathbf{a})}{\partial q_{\beta-\ell}}-Q^*_{\beta-\ell}(\mathbf{a})\frac{\partial Q^*_{\alpha-\ell}(\mathbf{a})}{\partial E}\right],\tag{28}$$

$$\underset{\substack{\alpha=\ell+1,\ell+2,\ldots,\ell+M\\ \beta=\ell+M+1,\ell+M+2,\ldots,\ell+M+N}}{\Omega_{\alpha\beta}(\mathbf{a})} = -\lambda^*(\mathbf{a})\left[\frac{\partial Q^*_{\alpha-\ell}(\mathbf{a})}{\partial r_{\beta-\ell-M}}+V^*_{\beta-\ell-M}(\mathbf{a})\frac{\partial Q^*_{\alpha-\ell}(\mathbf{a})}{\partial E}\right],\tag{29}$$

$$\underset{\substack{\alpha=\ell+M+1,\ell+M+2,\ldots,\ell+M+N\\ \beta=1,2,\ldots,\ell}}{\Omega_{\alpha\beta}(\mathbf{a})} = \lambda^*(\mathbf{a})\left[\frac{\partial V^*_{\alpha-\ell-M}(\mathbf{a})}{\partial p_\beta}+\left[C^*_\beta(\mathbf{a})+\delta_{\ell\beta}[L^*(\mathbf{a})-T]\right]\frac{\partial V^*_{\alpha-\ell-M}(\mathbf{a})}{\partial E}\right],\tag{30}$$

$$\underset{\substack{\alpha=\ell+M+1,\ell+M+2,\ldots,\ell+M+N\\ \beta=\ell+1,\ell+2,\ldots,\ell+M}}{\Omega_{\alpha\beta}(\mathbf{a})} = \lambda^*(\mathbf{a})\left[\frac{\partial V^*_{\alpha-\ell-M}(\mathbf{a})}{\partial q_{\beta-\ell}}-Q^*_{\beta-\ell}(\mathbf{a})\frac{\partial V^*_{\alpha-\ell-M}(\mathbf{a})}{\partial E}\right],\tag{31}$$

$$\underset{\substack{\alpha=\ell+M+1,\ell+M+2,\ldots,\ell+M+N\\ \beta=\ell+M+1,\ell+M+2,\ldots,\ell+M+N}}{\Omega_{\alpha\beta}(\mathbf{a})} = \lambda^*(\mathbf{a})\left[\frac{\partial V^*_{\alpha-\ell-M}(\mathbf{a})}{\partial r_{\beta-\ell-M}}+V^*_{\beta-\ell-M}(\mathbf{a})\frac{\partial V^*_{\alpha-\ell-M}(\mathbf{a})}{\partial E}\right].\tag{32}$$

By Lemma 1, $\left(\mathbf{Q}^*(\mathbf{a}),\mathbf{V}^*(\mathbf{a}),L^*(\mathbf{a})\right)$ are not functions of $(p_1,p_2,\ldots,p_{\ell-1},E)$, hence the partial derivatives of $\left(\mathbf{Q}^*(\mathbf{a}),\mathbf{V}^*(\mathbf{a}),L^*(\mathbf{a})\right)$ with respect to $(p_1,p_2,\ldots,p_{\ell-1},E)$ vanish identically, thereby resulting in a considerable simplification of Eqs. (24)–(32), and furthermore, implying that, e.g., $\partial L^*(\mathbf{a})/\partial p_\beta=\delta_{\ell\beta}\,\partial L^*(\mathbf{a})/\partial p_\beta$, $\beta=1,2,\ldots,\ell$. Moreover, as $\lambda^*(\mathbf{a})>0$ by Lemma 1, it may be divided out of Eqs. (24)–(32), thereby yielding the negative semidefinite matrix $\Omega^*(\mathbf{a})$. The rank conclusion follows from Theorem 4 of Partovi and Caputo (2006). *Q.E.D.*

7. Appendix II

A worthwhile perturbation of the AH model defined by Eq. (1) would be to include a set of so-called staple goods, defined as goods which are produced and consumed by an AH. In order to keep matters relatively simple, and in an effort not to become bogged down with too many formalities akin to that leading up to Proposition 1 and Appendix I, assume that the first good consumed by an AH is a staple. This assumption implies the existence of a third constraint, namely, $g^3(\mathbf{x};\mathbf{a}) \stackrel{\text{def}}{=} p_1 - q_1 = 0$, which asserts the price of consuming the staple good is the market price of the good produced by an AH. Because this constraint does not involve a decision variable, it does not affect the first-order necessary conditions given in Eqs. (5)–(10). But for the purpose of comparative statics, the third constraint changes a few things, as is now demonstrated.

In addition to the two normal directions to the level set of the constraint functions in parameter space given in Eqs. (13) and (14), there is now a third normal direction, namely,

$$\nabla_{\mathbf{a}} g^3(\mathbf{x};\mathbf{a}) = \left(\mathbf{e}_\ell^1, -\mathbf{e}_M^1, \mathbf{0}_N', 0, 0\right) \in \mathbb{R}^{\ell+M+N+2}. \tag{33}$$

Because the only solution to $\sum_{s=1}^{3} k_s \nabla_{\mathbf{a}} g^s(\mathbf{x};\mathbf{a}) = 0$ is $k_1 = k_2 = k_3 = 0$, the normal vectors are linearly independent. By the implicit function theorem, the constraints $g^s(\mathbf{x};\mathbf{a}) = 0$, $s = 1,2,3$, define an $(\ell + M + N - 1)$-dimensional manifold in $\mathbb{R}^{\ell+M+N+2}$. This implies that the dimension of the tangent hyperplane to the level set of the three constraint functions is $\ell + M + N - 1$, and hence that $\ell + M + N - 1$ basis vectors are required for its complete description. In contrast to the AH model defined by Eq. (1), one fewer basis vector is required in the present version of the model, seeing as the same parameters appear in both but the present version has one more constraint involving the parameters, namely, $g^3(\mathbf{x};\mathbf{a}) \stackrel{\text{def}}{=} p_1 - q_1 = 0$.

The following $\ell + M + N - 1$ vectors lie in the tangent hyperplane to the level set of the constraint functions in parameter space

$$\mathbf{t}^\alpha \stackrel{\text{def}}{=} (\mathbf{e}_\ell^\alpha, \delta_{1\alpha}\mathbf{e}_M^1, \mathbf{0}_N', 0, C_\alpha - \delta_{1\alpha}Q_\alpha + \delta_{\ell\alpha}[L-T]),\ \alpha = 1,2,\ldots,\ell, \tag{34}$$

$$\mathbf{t}^\alpha \stackrel{\text{def}}{=} (\mathbf{0}_\ell', \mathbf{e}_M^{\alpha-\ell}, \mathbf{0}_N', 0, -Q_{\alpha-\ell}),\ \alpha = \ell+2,\ldots,\ell+M, \tag{35}$$

$$\mathbf{t}^\alpha \stackrel{\text{def}}{=} (\mathbf{0}_\ell', \mathbf{0}_M', \mathbf{e}_N^{\alpha-\ell-M}, 0, V_{\alpha-\ell-M}),\ \alpha = \ell+M+1, \ell+M+2,\ldots,\ell+M+N, \tag{36}$$

seeing as $\nabla_{\mathbf{a}} g^s(\mathbf{x};\mathbf{a}) \cdot \mathbf{t}^\alpha = 0$ for $s = 1,2,3$ and $\alpha \in \mathrm{I} \stackrel{\text{def}}{=} \{1,2,\ldots,\ell+M+N\} \setminus \{\ell+1\}$. Because the only solution to the linear system $\sum_{\alpha\in\mathrm{I}} c_\alpha \mathbf{t}^\alpha = \mathbf{0}_{\ell+M+N+2}$ is the null vector $\mathbf{c} = \mathbf{0}_{\ell+M+N-1}$, the $\ell + M + N - 1$ vectors $\mathbf{t}^\alpha$ are linearly independent and thus form a basis for the said hyperplane.

Using Eq. (12) and Eqs. (34)–(36) in the definition of a GCD given in Eq. (18), the CGDs in the present case are given by

$$D_\alpha(\mathbf{x};\mathbf{a}) = \frac{\partial}{\partial p_\alpha} + \delta_{1\alpha}\frac{\partial}{\partial q_1} + \left[C_\alpha - \delta_{1\alpha}Q_\alpha + \delta_{\ell\alpha}[L-T]\right]\frac{\partial}{\partial E},\ \alpha = 1,2,\ldots,\ell, \tag{37}$$

$$D_\alpha(\mathbf{x};\mathbf{a}) = \frac{\partial}{\partial q_{\alpha-\ell}} - Q_{\alpha-\ell}\frac{\partial}{\partial E},\ \alpha = \ell+2,\ldots,\ell+M, \tag{38}$$

$$D_\alpha(\mathbf{x};\mathbf{a}) = \frac{\partial}{\partial r_{\alpha-\ell-M}} + V_{\alpha-\ell-M}\frac{\partial}{\partial E},\ \alpha = \ell+M+1, \ell+M+2,\ldots,\ell+M+N. \tag{39}$$

The $\ell + M + N - 1$ GCDs in Eqs. (37)–(39) possess the null property in view of the fact that $D_\alpha(\mathbf{x};\mathbf{a}) \circ g^s(\mathbf{x};\mathbf{a}) \equiv 0$ for $s = 1,2,3$ and $\alpha \in \mathrm{I}$. Comparison of the GCDs in Eqs. (19)–(21) and Eqs. (37)–(39) shows that there are two small differences, as is now explained.

The only meaningful difference between the two sets of GCDs occurs in $D_1(\mathbf{x};\mathbf{a})$. Comparing Eq. (37) to Eq. (19) shows the assumption that the first output produced is also consumed manifests itself in differential form by the addition of $\frac{\partial}{\partial q_1} - Q_1 \frac{\partial}{\partial E}$. This is not unexpected. The constraint $p_1 = q_1$ resulting from the staple good assumption implies that a change in p_1 requires a concomitant change in q_1 that must be fully analogous to the compensated change given in Eq. (38) in order to leave the budget constraint unaffected. The other difference between the two sets of GCDs is that the index in Eq. (38) omits the term $\alpha = \ell + 1$ and thus the GCD $D_{\ell+1}(\mathbf{x};\mathbf{a}) = \frac{\partial}{\partial q_1} - Q_1 \frac{\partial}{\partial E}$. But this is precisely the GCD that was added in forming $D_1(\mathbf{x};\mathbf{a})$ in the present context, as noted above. Thus the change in the set of GCDs resulting from the assumption that output one is a staple good amounts to removing $D_{\ell+1}(\mathbf{x};\mathbf{a}) = \frac{\partial}{\partial q_1} - Q_1 \frac{\partial}{\partial E}$ from the original set of GCDs in Eqs. (19)–(21) and inserting it into $D_1(\mathbf{x};\mathbf{a})$ in Eq. (37). This change also reveals the form of the GCDs if it were instead assumed that a subset of the goods produced by an AH were staple goods. Furthermore, because the change in the GCDs under the present stipulation is limited to $D_1(\mathbf{x};\mathbf{a})$, it is left as an exercise for the reader to derive the CSM in this case.

8. References

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