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A simple model of natural oligopoly with an unlimited number of firms

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Abstract

In vertical models of product differentiation, consumers agree on their ranking of product quality but differ in their willingness to pay for it. When product quality is bounded and the range of willingness to pay is narrow, there are ``natural oligopoly'' equilibria in which a finite number of firms enter the market regardless of market size. I relax both of these assumptions and consider a simple vertical model in which the number of entering firms increases with market size. I derive analytical expressions for equilibrium prices, markups, and shares for any number of entering firms and limiting expressions for market shares and concentration as the number of firms grows large. The limiting market structure is highly concentrated. The market share of the largest firm converges from above to .58, the combined share of the four largest firms to .99, and the HHI to .44 (4,400). I conclude that vertical models can give rise to natural oligopoly when the range of quality is unbounded and the number of entering firms is unlimited.

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1 Introduction

The relationship of product differentiation to market structure has been extensively studied in economics. In models of horizontal product differentiation (Hotelling, 1929), consumers have different ideal points in a space of products, so there is no uniform ranking of product features across consumers. The equilibrium of horizontal models converges to the competitive outcome as the market becomes large, relative to the fixed cost of entry (Shaked and Sutton, 1987). In vertical models, consumers agree on their ranking of product attributes, but differ in their willingness to pay for them. Such models can exhibit a "finiteness" property (Jaskold Gabszewicz and Thisse, 1980; Shaked and Sutton, 1982, 1983; Curtis Eaton and Lipsey, 1989; Sutton, 1991). When the product (quality) space is bounded, there are equilibria where only a finite number of firms can enter—independent of market size—and there is no convergence to the competitive outcome.

I consider a vertical model of price competition where the "finiteness" property does not hold. The quality space is unbounded and consumers are heterogeneous enough to permit entry by an unlimited number of firms. In the first stage of the game, N firms sequentially pay an identical fixed cost of entry, choose a level of quality, and choose to enter a market. In the second stage, firms with identical marginal costs compete by choosing prices. I study equilibria in which each consumer purchases one unit of a good of variable quality, and the market size is large enough so that all N firms can enter, occupy N different quality levels, and each earn positive profits. I obtain analytical expressions for equilibrium prices, markups, and shares for a market with N firms. Limiting market shares, k-firm concentration ratios and HHI are obtained for the case where N grows large.

Despite unlimited entry, the equilibrium market structure is a natural oligopoly in which prices, market shares, and profits increase with quality. As the number of entering firms increases, the market share of the largest firm converges from above to .58, the 4-firm concentration ratio to .99, and the HHI to .44 (4,400).

The main contribution of the paper is to obtain closed form and limiting solutions to a simple, vertical product differentiation model in which the finiteness property does not hold and to demonstrate that the market structure is nevertheless highly concentrated. The result underscores how quality competition can be a powerful source of market concentration in many industries (Sutton, 1991).

This paper is close in spirit to the canonical contributions of Jaskold Gabszewicz and Thisse (1980); Shaked and Sutton (1982, 1983, 1987) and explores a variation on their original assumptions. Models of vertical product differentiation have since been extended to consider, inter alia, uncertainty (Bergemann and Valimaki, 1997), entry deterrence and dynamics (Lutz, 1997; Bergemann and Valimaki, 2002; Noh and Moschini, 2006; Auer and Sauré, 2017), multi-product and multi-quality firms (Constantatos and Perrakis, 1997; Tauman et al., 1997; Johnson and Myatt, 2003; Barigozzi and Ma, 2018), innovation (Greenstein and Ramey, 1998; Baron, 2021), price discrimination (Liu and Serfes, 2005; Herweg, 2012), quality standards (Garella, 2006), network effects (Kuhn, 2007; Argenziano, 2008), heterogeneous costs (Brécard, 2010), capacity (Boccard and Wauthy, 2010), income inequality (Yurko, 2011), international trade (Fajgelbaum et al., 2011), quantity competition (Lambertini and Tampieri, 2012), endogenous market structures (Gayer, 2017), cartel stability (Bos and Marini, 2019), overlapping ownership (Li et al., 2023), and empirical estimation (Gandhi and Nevo, 2021).

2 Model

I define a simple, two-stage model of quality and price competition. In the first stage, N firms sequentially decide whether to enter the market. In order to enter, a firm must pay a fixed cost F and choose its product quality $q \in (1, 2, \ldots, N)$. In the second stage, entering firms with identical marginal cost c compete a la Bertrand on prices given their quality.

The quality assumption is central to the model and deserves comment. The quality space depends on an exogenous number of potential entrants (N). Because I consider equilibria in which all N firms enter, equilibrium quality is unbounded as N grows large. If the quality space was instead a closed interval, the market shares would grow arbitrarily small with N and the equilibrium would converge to the competitive outcome. Unbounded quality is thus necessary for the natural oligopoly result. The assumption that adjacent firms are always a fixed distance apart in quality contributes to market concentration, but this distance falls relative to the support of quality (N) so it is not clear *a priori* which force will predominate as N grows large.[1] In addition, because of the fixed distance assumption, concentration does not result by allowing the leading firms to pull away in quality.

Given the entry, quality, and price decisions of firms, consumers choose whether or not to consume a single unit of the good. The utility of a consumer that buys from a firm with quality q and price p_q is given by:

$$u(\theta, q) = 1 + \theta q - p_q \tag{1}$$

where θ is uniformly distributed on the unit interval $[\underline{\theta}, \overline{\theta}]$.

2.1 Equilibrium

2.1.1 The Two-Firm Case

First consider the case of "finiteness" presented by Shaked and Sutton (1982). Suppose that two firms enter the market with quality $q_1 = 1$ and $q_2 = 2$ and that $p_2 > p_1$. A consumer θ will be indifferent between the two firms if $\theta = p_2 - p_1$. The 2-firm equilibrium is unique and requires that $\overline{\theta} - 2\underline{\theta} > 0$, that is, consumers' willingness to pay must be sufficiently far apart for two firms to coexist (Tirole, 1988).[2] In contrast, if willingness to pay is too close—i.e. consumers are relatively homogeneous—then there will be intense price competition and the market can only support one firm, as in the case of Bertrand competition with an undifferentiated good. Even if the lower quality firm was to sell at marginal cost, there would be no demand for their product. The assumption that θ is uniformly distributed on the unit interval [0,1] implies that the willingness to pay assumption is satisfied ($\overline{\theta} - 2\underline{\theta} > 0$). This is a reasonable assumption whenever low-quality, low-price firms can enter the market and capture at least some market share, and it is necessary to avoid "finiteness."

[1] In a horizontal model, if the range of location grows with N but firms are a fixed distance apart, then market shares will grow arbitrarily small with N.

[2] For the lowest valuing consumer to purchase from firm 1, it is further required that $\frac{2}{3}(\overline{\theta} + \underline{\theta}) > c$

2.1.2 The N-Firm Case

Now consider an equilibrium in which N firms sequentially enter and occupy N different quality levels so that firms can be indexed by their quality. By the logic of Bertrand competition, no two firms will choose the same quality, otherwise they will earn zero profits in the second stage. The market size is assumed to be large enough, relative to F, so that it is profitable for all N firms to enter.

Assume that all consumers purchase a good, so the consumers who are indifferent between firms n and $n-1$ are:

$$\theta_n = p_n - p_{n-1}, \ \ 2 \leq n \leq N \tag{2}$$

These indifference conditions yield shares (s_n) for each firm. Each firm then solves:

$$\max_{p_j} (p_n - c) M s_n(\mathbf{p}, \mathbf{q}) \tag{3}$$

where M represents the market size and market shares depend on N-dimensional vectors of prices, and qualities. Each profit function is strictly concave in the firm's own price. The first-order conditions yield the following system of difference equations:

$$p_n = \begin{cases} \frac{1}{2}p_2 + \frac{1}{2}c, \ \ n = 1 \\ \frac{1}{4}p_{n-1} + \frac{1}{4}p_{n+1} + \frac{1}{2}c, \ \ 2 \leq n \leq N-1 \\ \frac{1}{2}p_{n-1} + \frac{1}{2}c + \frac{1}{2}, \ \ n = N \end{cases} \tag{4}$$

Prices, shares, and profits are increasing in quality. The first entrant will therefore choose the highest quality, the second entrant will choose the second highest quality, and so on.

Proposition 1. If an equilibrium exists, then for all integers $N \geq j > k \geq 1$ the equilibrium price (p_j^*), share (s_j^*), and profit are strictly higher for j than k.

Proof. See Appendix. □

The system of difference equations formed by the firm reaction functions imply both the existence of an equilibrium and exact solutions for equilibrium prices, shares, and profits.

Proposition 2. For large enough market size M, there exists an equilibrium in which N firms enter at N different quality levels with equilibrium prices:

$$p_n^*(c, n, N) = c + \left(\frac{\sqrt{3}}{3}\right) \frac{(2+\sqrt{3})^{n-1} + (2-\sqrt{3})^{n-1}}{(2+\sqrt{3})^{N-1} - (2-\sqrt{3})^{N-1}}$$

Proof. See Appendix. □

The equilibrium prices in Proposition 2 are a markup over marginal cost that increases with quality $(n = q)$. Proposition 2 can be verified for $N = 2$ ($\mathbf{p}^* = [c + \frac{1}{3}, c + \frac{2}{3}]$) and $N = 3$ ($\mathbf{p}^* = [c + \frac{1}{12}, c + \frac{2}{12}, c + \frac{7}{12}]$). These two cases illustrate a general feature of Proposition 2. The price of the highest quality firm decreases with entry ($\frac{2}{3} > \frac{7}{12}$ and $p_N^*(c, N, N) > p_{N+1}^*(c, N+1, N+1)$).

Proposition 2 implies the following market shares:

$$s_n^*(n,N) = \begin{cases} \frac{\sqrt{3}}{3} \cdot \frac{1}{(2+\sqrt{3})^{N-1} - (2-\sqrt{3})^{N-1}}, & n=1 \\ \frac{2\sqrt{3}}{3} \cdot \frac{(2+\sqrt{3})^{n-1} + (2-\sqrt{3})^{n-1}}{(2+\sqrt{3})^{N-1} - (2-\sqrt{3})^{N-1}}, & 2 \leq n \leq N-1 \\ \frac{\sqrt{3}}{3} \cdot \frac{(2+\sqrt{3})^{N-1} + (2-\sqrt{3})^{N-1}}{(2+\sqrt{3})^{N-1} - (2-\sqrt{3})^{N-1}}, & n=N \end{cases} \tag{5}$$

These expressions for shares can be verified for N=2 ($\mathbf{s}^* = [\frac{1}{3}, \frac{2}{3}]$) and N=3 ($\mathbf{s}^* = [\frac{1}{12}, \frac{4}{12}, \frac{7}{12}]$). Entry by a third firm occupying the highest quality rung only succeeds in capturing $\frac{7}{12}$ of the market, while the share of the 2-firm quality leader was $\frac{2}{3}$. In general, the share of the top firm decreases with the number of entrants. Using Equation 5, it is possible to calculate market concentration (k-firm ratios and HHI) for any equilibrium with N firms occupying N quality levels.

2.2 Limiting Results

The limit of the model is a natural oligopoly featuring high concentration despite an unlimited number of firms.[3]

Proposition 3. As the number of firms (N) and market size (M(N)) grows large, equilibrium market shares converge to:

$$\lim_{N\to\infty} s_n^* = \begin{cases} 0, & n = 1,2,3,\dots \\ \frac{2\sqrt{3}}{3}\left(\frac{1}{(2+\sqrt{3})^{N-n}}\right), & n = N-1, N-2, \dots \\ \frac{\sqrt{3}}{3}, & n = N \end{cases} \tag{6}$$

Proof. See Appendix □

The limiting market shares of the four largest firms are [.58, .31, .08, .02]. Market concentration is typically defined using statistics such as the k-firm concentration ratio or the HHI.

Proposition 4. The limiting k-firm concentration ratio is:

$$\lim_{N\to\infty} k_N = \frac{\sqrt{3}}{3} + \left(1 - \frac{\sqrt{3}}{3}\right)\left(1 - \frac{1}{(2+\sqrt{3})^{k-1}}\right) \tag{7}$$

Proof. See Appendix □

The limiting concentration ratios for k=1 to 4 are [.58, .89, .97, .99], which indicate a highly concentrated market.

Proposition 5. The limiting HHI is:

$$\lim_{N\to\infty} HHI_N = \frac{4\sqrt{3}-3}{9} \tag{8}$$

[3] In taking the limit as N grows large, I do not consider how firms behave in transitioning from one equilibrium to another.

Proof. See Appendix □

The limiting HHI is approximately equal to .44, which is only slightly smaller than the HHI associated with a duopoly (.5).

3 Conclusion

In the classic models of quality competition (Shaked and Sutton, 1982, 1983), a finite number of firms enter the market regardless of market size. In the model in this paper, an unlimited number of firms enter the market, but the market nevertheless remains a natural oligopoly. The limiting market structure is highly concentrated and dominated by the four largest firms.

A number of assumptions are required for the limiting result that is the paper's main contribution. First, consumer willingness to pay is heterogeneous enough to allow unlimited entry. This assumption implies that low-quality, low-price firms can capture market share, and it is required to avoid the finiteness result of Shaked and Sutton (1982, 1983).

Second, the quality space is unbounded. Unbounded quality is a natural assumption if firms can improve on any given quality level and the quality space grows with market size. If the quality space were instead a closed interval, then the equilibrium market shares would grow arbitrarily small as the market grows large. The fixed interval between quality levels contributes to market concentration, but the quality distance between firms also shrinks relative to the growing quality space.

Third, the assumption of a uniform distribution of willingness to pay and the chosen functional form for utility permit a closed-form solution for prices, markups, shares, and profits in the model, which enables the derivation of the limiting results. There is no strong empirical reason to believe the uniformity assumption (Benassi et al., 2019). However, as an analytical matter, uniformity does not achieve concentration by biasing the willingness to pay distribution in favor of high quality. The model predicts more (less) concentration than the uniform baseline if willingness to pay is skewed toward higher (lower) qualities.

Finally, it is assumed that marginal and fixed costs are equal for all firms. This is done so as not to advantage or disadvantage high-quality firms. It is also reasonable to assume instead that firms must pay a higher fixed cost to attain a higher quality. This has the natural interpretation of strategic competition in fixed costs. The model in this paper is relevant to industries—such as those with high advertising-to-sales ratios (Sutton, 1991)—where strategic competition in fixed costs is applicable. Allowing fixed costs to vary in this way does not change any of the Propositions of the model, except now producer surplus, rather than profit, is ordered by quality.

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Appendix

Proof of Proposition 1. Suppose $p_j^* \leq p_k^*$. Then demand for good k is 0. But then entry for firm k cannot be rational. So there is a contradiction and $p_j^* > p_k^*$. Suppose $s_j^* \leq s_k^*$. Then $p_{j+1}^* - 2p_j^* + p_{j-1}^* \leq p_{k+1}^* - 2p_k^* + p_{k-1}^*$. Note from the first-order conditions that $p_{j+1}^* + p_{j-1}^* = 4p_j^* + 2c$. The previous inequality simplifies to $p_j^* \leq p_k^*$, which is a contradiction, so $s_j^* > s_k^*$. Finally, $p_j^* > p_k^*$ and $s_j^* > s_k^*$ imply that profits are strictly higher for firm j. $\square$

The proof of Proposition 2 requires two preliminary Lemmas. First, define the $N \times N$, tri-diagonal symmetric matrix:

$$A(\lambda) = \begin{pmatrix} -\lambda & 1 & 0 & \dots & \dots & \dots \\ 1 & -2\lambda & 1 & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & 0 & 1 & -2\lambda & 1 \\ \dots & \dots & \dots & 0 & 1 & -\lambda \end{pmatrix} \tag{9}$$

Lemma 1. The matrix $A(\lambda)$ has inverse:

$$a_{jk}^{-1}(\lambda) = \left(\frac{1}{1-\lambda^2}\right) \frac{T_{j-1}(\lambda) T_{N-k}(\lambda)}{U_{N-2}(\lambda)},\ 1 \leq j \leq k \leq N \tag{10}$$

where $T_j(\lambda)$ and $U_j(\lambda)$ are Chebyshev polynomials of the first and second type (Rivlin, 2020).

Proof. See Dow (2003). $\square$

Lemma 2. The row elements of matrix $A(\lambda)$ satisfy the equation:

$$-2\sum_{k=1}^{N} a_{jk}^{-1}(\lambda) + a_{j1}^{-1}(\lambda) + a_{jN}^{-1}(\lambda) = \frac{1+\lambda}{\lambda^2 - 1} \tag{11}$$

Proof. Define the variable $b_{jk}^{-1}(\lambda)$:

$$b_{jk}^{-1}(\lambda) = T_{j-1} T_{N-k}(\lambda),\ 1 \leq j \leq k \leq N \tag{12}$$

The row sum of $b_{jk}^{-1}(\lambda)$ can be expressed as:

$$\sum_{k=1}^{N} b_{jk}^{-1}(\lambda) = \sum_{k=j}^{N} b_{jk}^{-1}(\lambda) + \sum_{k=1}^{j-1} b_{kj}^{-1}(\lambda) \tag{13}$$

Taking the right side of Equation 13 and substituting from the definition of $b^{-1}_{jk}(\lambda)$ gives:

$$\begin{aligned}\sum_{k=j}^{N} b^{-1}_{jk}(\lambda) &= \sum_{k=j}^{N} T_{j-1}(\lambda)T_{N-k}(\lambda) \\ &= \frac{1}{2}\sum_{k=j}^{N} T_{N-1+j-k}(\lambda) + \frac{1}{2}\sum_{k=j}^{N} T_{|j-1-N+k|}(\lambda)\end{aligned} \tag{14}$$

$$\begin{aligned}\sum_{k=1}^{j-1} b^{-1}_{kj}(\lambda) &= \sum_{k=1}^{j-1} T_{k-1}(\lambda)T_{N-j}(\lambda) \\ &= \frac{1}{2}\sum_{k=1}^{j-1} T_{N-1-j+k}(\lambda) + \frac{1}{2}\sum_{k=1}^{j-1} T_{|j-1-N+k|}(\lambda)\end{aligned} \tag{15}$$

Equations 14 and 15 make use of the product rule for Chebyshev polynomials of the first type:

$$2T_j(\lambda)T_k(\lambda) = T_{j+k}(\lambda) + T_{|j-k|}(\lambda) \tag{16}$$

Adding Equation 14 and 15 and multiplying by 2:

$$2\sum_{k=1}^{N} b^{-1}_{jk}(\lambda) = \sum_{k=j}^{N} T_{N-1+j-k}(\lambda) + \sum_{k=1}^{j-1} T_{N-1-j+k}(\lambda) + \sum_{k=1}^{N} T_{|j-1-N+k|}(\lambda) \tag{17}$$

Each of the right-hand side terms can be further simplified:

$$\sum_{k=j}^{N} T_{N-1+j-k}(\lambda) = \sum_{k=j-1}^{N-1} T_k(\lambda) \tag{18}$$

$$\sum_{k=1}^{j-1} T_{N-1-j+k}(\lambda) = \sum_{k=N-j}^{N-2} T_k(\lambda) \tag{19}$$

$$\sum_{k=1}^{N} T_{|j-1-N+k|}(\lambda) = \sum_{k=0}^{N-j} T_k(\lambda) + \sum_{k=1}^{j-1} T_k(\lambda) \tag{20}$$

Equation 17 is then:

$$2\sum_{k=1}^{N} b^{-1}_{jk}(\lambda) = \sum_{k=0}^{N-2} T_k(\lambda) + \sum_{k=1}^{N-1} T_k(\lambda) + T_{N-j}(\lambda) + +T_{j-1}(\lambda) \tag{21}$$

The product rules for Chebyshev polynomials in Equation 16 imply that $b_{j1}^{-1}(\lambda) = T_{N-j}$ and $b_{jN}^{-1}(\lambda) = T_{j-1}$. It follows that:

$$2\sum_{k=1}^{N} b_{jk}^{-1}(\lambda) - b_{j1}^{-1}(\lambda) - b_{jN}^{-1} = \sum_{k=0}^{N-2} T_k(\lambda) + \sum_{k=1}^{N-1} T_k(\lambda) \tag{22}$$

Equation 22 can further simplified by converting to Chebyshev polynomials of the second type. The conversion formula is:

$$2T_k(\lambda) = U_k(\lambda) - U_{k-2}(\lambda) \tag{23}$$

Applying this to Equation 22 yields:

$$\begin{aligned}\sum_{k=0}^{N-2} T_k(\lambda) + \sum_{k=1}^{N-1} T_k(\lambda) &= 1 + 2\lambda + \frac{1}{2}\sum_{k=2}^{N-2}(U_k(\lambda) - U_{k-2}(\lambda)) + \frac{1}{2}\sum_{k=2}^{N-1}(U_k(\lambda) - U_{k-2}(\lambda)) \\ &= \frac{1}{2}U_{N-1}(\lambda) + U_{N-2}(\lambda) + \frac{1}{2}U_{N-3}(\lambda)\end{aligned} \tag{24}$$

Returning to Equation 11, it has been shown that:

$$\begin{aligned}-2\sum_{k=1}^{N} a_{jk}^{-1}(\lambda) + a_{j1}^{-1}(\lambda) + a_{jN}^{-1}(\lambda) &= \left(\frac{1}{1-\lambda^2}\right)\frac{1}{U_{N-2}(\lambda)}\left(-2\sum_{k=1}^{N} b_{jk}^{-1}(\lambda) + b_{j1}^{-1}(\lambda) + b_{jN}^{-1}\right) \\ &= \left(\frac{1}{\lambda^2-1}\right)\frac{1}{U_{N-2}(\lambda)}\left(\sum_{k=0}^{N-2} T_k(\lambda) + \sum_{k=1}^{N-1} T_k(\lambda)\right) \\ &= \left(\frac{1}{\lambda^2-1}\right)\frac{1}{U_{N-2}(\lambda)}\left(\frac{1}{2}U_{N-1}(\lambda) + U_{N-2}(\lambda) + \frac{1}{2}U_{N-3}(\lambda)\right) \\ &= \frac{1+\lambda}{\lambda^2-1}\end{aligned} \tag{25}$$

The final equality makes use of the recurrence relationship for Chebyshev polynomials of the second type:

$$U_k(\lambda) = 2\lambda U_{k-1}(\lambda) - U_{k-2}(\lambda) \tag{26}$$

$\square$

Proof of Proposition 2. The equilibrium price vector $\mathbf{p}^*$ solves the following matrix equation:

$$\begin{pmatrix} -2 & 1 & 0 & \dots & \dots & \dots \\ 1 & -4 & 1 & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & 0 & 1 & -4 & 1 \\ \dots & \dots & \dots & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} p_1^* \\ p_2^* \\ \vdots \\ p_{N-1}^* \\ p_N^* \end{pmatrix} = \begin{pmatrix} -2c+c \\ -2c \\ \vdots \\ -2c \\ -2c+c-1 \end{pmatrix} \tag{27}$$

where $\mathbf{p}^*$ has been pre-multiplied by $A(2)$. The existence of the inverse of $A(2)$ proves the existence of a price equilibrium. Pre-multiplication of Equation 27 by $A^{-1}(2)$ implies:

$$\begin{aligned} p_n^*(c,n,N) &= c\left(-2\sum_{k=1}^{N} a_{jk}^{-1}(2) + a_{j1}^{-1}(2) + a_{jN}^{-1}(2)\right) - a_{nN}^{-1}(2) \\ &= c\left(\frac{1+2}{2^2-1}\right) - a_{nN}^{-1}(2) \\ &= c + \frac{1}{3}\left(\frac{T_{n-1}(2)}{U_{N-2}(2)}\right) \\ &= c + \frac{\sqrt{3}}{3} \cdot \frac{(2+\sqrt{3})^{n-1} + (2-\sqrt{3})^{n-1}}{(2+\sqrt{3})^{N-1} - (2-\sqrt{3})^{N-1}} \end{aligned} \tag{28}$$

where Equation 28 uses Lemma 1, Lemma 2, and the following expressions for Chebyshev polynomials:

$$T_n(\lambda) = \frac{1}{2}(\lambda + \sqrt{\lambda^2-1}))^n + \frac{1}{2}(\lambda - \sqrt{\lambda^2-1}))^n \tag{29}$$

$$U_n(\lambda) = \frac{(\lambda + \sqrt{\lambda^2-1})^{n+1} - (\lambda - \sqrt{\lambda^2-1})^{n+1}}{2\sqrt{\lambda^2-1}} \tag{30}$$

$\square$

Proof of Proposition 3. First consider firms starting from the smallest market shares:

$$\begin{aligned}
\lim_{N\to\infty} s_n^* &= \lim_{N\to\infty} \frac{2\sqrt{3}}{3}\left(\frac{(2+\sqrt{3})^{n-1}+(2-\sqrt{3})^{n-1}}{(2+\sqrt{3})^{N-1}-(2-\sqrt{3})^{N-1}}\right),\ n=1,2,3,\ldots \\
&= \lim_{N\to\infty} \frac{2\sqrt{3}}{3}\left(\frac{(2+\sqrt{3})^{n-1}+(2-\sqrt{3})^{n-1}}{(2+\sqrt{3})^{N-1}}\right)\left(\frac{1}{1-\left(\frac{2-\sqrt{3}}{2+\sqrt{3}}\right)^{N-1}}\right),\ n=1,2,3,\ldots \\
&= 0
\end{aligned} \tag{31}$$

Next consider the largest firms, starting from N-j, where j=1.

$$\begin{aligned}
\lim_{N\to\infty} s_n^* &= \lim_{N\to\infty} \frac{2\sqrt{3}}{3}\left(\frac{(2+\sqrt{3})^{N-j-1}+(2-\sqrt{3})^{N-j-1}}{(2+\sqrt{3})^{N-1}-(2-\sqrt{3})^{N-1}}\right),\ j=1,2,3,\ldots \\
&= \lim_{N\to\infty} \frac{2\sqrt{3}}{3}\left(\frac{(2+\sqrt{3})^{N-j-1}+(2-\sqrt{3})^{N-j-1}}{(2+\sqrt{3})^{N-1}}\right),\ j=1,2,3,\ldots \\
&= \frac{2\sqrt{3}}{3}\left(\frac{1}{(2+\sqrt{3})^{j}}\right)+\frac{2\sqrt{3}}{3}\left(\frac{1}{(2+\sqrt{3})^{j}}\right)\lim_{N\to\infty}\left(\frac{2-\sqrt{3}}{2+\sqrt{3}}\right)^{N-1},\ j=1,2,3,\ldots \\
&= \frac{2\sqrt{3}}{3}\left(\frac{1}{(2+\sqrt{3})^{j}}\right),\ j=1,2,3,\ldots
\end{aligned} \tag{32}$$

Finally, consider the largest firm N:

$$\begin{aligned}
\lim_{N\to\infty} s_n^* &= \lim_{N\to\infty} \frac{\sqrt{3}}{3}\left(\frac{(2+\sqrt{3})^{N-1}+(2-\sqrt{3})^{N-1}}{(2+\sqrt{3})^{N-1}-(2-\sqrt{3})^{N-1}}\right), \\
&= \frac{\sqrt{3}}{3}
\end{aligned} \tag{33}$$

Note also that the sum of the limiting market shares is equal to one.

$$
\begin{aligned}
\sum_{j=0}^{\infty} \lim_{N\to\infty} s^*_{N-j} &= \frac{\sqrt{3}}{3} + \sum_{j=1}^{\infty} \frac{2\sqrt{3}}{3}\left(\frac{1}{(2+\sqrt{3})^j}\right) \\
&= \frac{\sqrt{3}}{3} + \frac{2\sqrt{3}}{3}\left(\frac{1}{(1+\sqrt{3})}\right) \\
&= \frac{\sqrt{3}}{3} + \frac{2\sqrt{3}}{3}\left(\frac{\sqrt{3}-1}{2}\right) \\
&= \frac{\sqrt{3}}{3} + 1 - \frac{\sqrt{3}}{3} \\
&= 1
\end{aligned} \tag{34}
$$

$\square$

Proof of Proposition 4.

$$
\begin{aligned}
\lim_{N\to\infty} k_N &= \frac{\sqrt{3}}{3} + \frac{2\sqrt{3}}{3}\sum_{i=1}^{k-1} \frac{1}{(2+\sqrt{3})^i} \\
&= \frac{\sqrt{3}}{3} + \left(1 - \frac{\sqrt{3}}{3}\right)\left(1 - \frac{1}{(2+\sqrt{3})^{k-1}}\right)
\end{aligned} \tag{35}
$$

$\square$

Proof of Proposition 5.

$$
\begin{aligned}
\lim_{N\to\infty} HHI_N &= \left(\frac{\sqrt{3}}{3}\right)^2 + \left(\frac{2\sqrt{3}}{3}\right)^2 \sum_{i=1}^{\infty} \frac{1}{(2+\sqrt{3})^{2i}} \\
&= \frac{1}{3} + \frac{4}{3}\left(\frac{1}{(2+\sqrt{3})^2 - 1}\right) \\
&= \frac{4\sqrt{3}-3}{9}
\end{aligned} \tag{36}
$$

$\square$