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Singularity of input-output matrices: Structural linkages and spectral properties

Mohit Arora

*Department of Economics, Union College, Schenectady
NY*

Deepankar Basu

*Department of Economics, University of Massachusetts
Amherst*

Abstract

Analyzing 5 historical and 670 contemporary input-output (IO) tables, we show that all of these IO matrices are singular. We map this phenomenon onto underlying production network topologies. While singularity trivially emerges from sectors lacking intermediate forward or backward linkages, it could frequently persist due to deeper economic properties: proportional cost structures across sectors, isolated economic sub-networks, and supply-chain hierarchies lacking intra-industry transactions. As a subset of these matrices is non-diagonalizable, we leverage this singularity to derive simple, rank and eigenvalue based diagnostic criteria for diagonalizability. This provides an efficient computational test for network shock propagation models using eigendecomposition.

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Contact: Mohit Arora - marorajnu@gmail.com, Deepankar Basu - dbasu@econs.umass.edu.

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Minimax causal inference under tail heterogeneity

Omid M Ardakani
Georgia Southern University

Abstract

Standard causal estimators deteriorate when treatment modifies the tail index of outcomes. I generalize the maximum domain of attraction to nonstationary triangular arrays, establishing uniform convergence of block maxima to time-varying generalized extreme value distributions. Under such shifts, the supports of the limiting distributions separate, likelihood ratios are unbounded, and inverse probability weighting loses root- n consistency and asymptotic normality once the heavier tail index exceeds one half. I derive minimax lower bounds for the causal estimation rate; a tail-adaptive doubly robust estimator attains them up to logarithms. In simulations, tail adaptation cuts integrated mean absolute error by 20 to 41 percent at crisis-scale samples and stabilizes performance under strong tail heterogeneity. An application to bank stock returns around the 2020 Federal Reserve emergency facilities illustrates the framework.



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Diagonalizability of singular matrices: some theory and an application to input-output tables

Mohit Arora

*Department of Economics, Union College,
Schenectady NY*

Deepankar Basu

*Department of Economics, University of
Massachusetts Amherst*

Abstract

We provide necessary and sufficient conditions for diagonalizability of any singular matrix using its rank and eigenvalues. We apply our results to 5 historical (Germany, India, and Japan) and 670 contemporary input-output matrices (spanning 43 countries for 15 years from the 2016 World Input-Output Database and 25 years of the U.S. economy). Our analysis reveals that some of these matrices are not diagonalizable.

1. Introduction

In this paper, we present a fundamental empirical feature of input-output (IO) tables: the vast majority of IO matrices are singular.¹ We show that singularity is universal across both historical and contemporary IO matrices. This feature is often observed by IO matrices having a full row or column of zeros or both. However, this is also evident in cases where a row or column is linearly dependent on other rows/columns.

We explore the economic meaning of singularity in IO matrices, demonstrating how linear dependence maps directly onto underlying economic structures. Singularity trivially emerges when sectors lack intermediate forward or backward linkages, generating zero rows or columns. However, because zero vectors are a sufficient rather than necessary condition, singularity frequently persists in dense matrices too (where no all-zero rows or columns exist). In these cases, linear dependence signals deeper economic properties: proportional cost structures across sectors, isolated economic blocks trading exclusively in closed sub-networks, or supply-chain hierarchies with sectors that report zero intra-industry transactions.

Recent literature has utilized eigendecomposition of the IO matrix to study the dynamic propagation of shocks through production networks (Liu and Tsyvinski, 2020, 2024). Eigendecomposition relies on the requirement that the matrix of eigenvectors associated with the IO matrix is invertible—that is, the matrix must be *diagonalizable*. Matrices that cannot be diagonalized are known as *defective*.

While standard linear algebra texts establish that distinct eigenvalues ensure diagonalizability (Strang, 2006, chapter 5), empirical IO matrices frequently feature repeated eigenvalues. Determining diagonalizability by computing eigenspace dimensions can be computationally cumbersome. We exploit the empirically observed pervasive singularity in IO matrices to provide simple, alternative diagnostic criteria for diagonalizability based on matrix rank and distinct non-zero eigenvalues.

2. Empirical Evidence on Singularity of IO Matrices

For any economy, let U denote the inter-industry transaction matrix and $\langle X \rangle$ denote the diagonal matrix of industry-level gross outputs.² The IO expenditure share matrix is defined as:

$$A = U \langle X \rangle^{-1}. \quad (1)$$

We evaluate 675 empirical IO matrices across multiple databases and historical periods to test for singularity ($\det(A) = 0$) and diagonalizability. While an IO matrix is trivially singular if it contains an all-zero row or column, this is merely a sufficient condition. To determine whether the pervasive singularity we observe is driven simply by these missing linkages or by deeper structural redundancy, we first evaluate the frequency of zero-vectors across our datasets.

¹Throughout this paper, the IO matrix refers to the expenditure share matrix defined in Section 2.

²We drop industries with zero gross output to ensure that $\langle X \rangle$ is invertible.

2.1. Contemporary IO Tables: WIOD and U.S. BLS

1. **WIOD (2016 Release):** Analyzing 56-industry annual tables for 43 countries from 2000 to 2014 (645 tables total) from the World Input-Output Database (WIOD), we find that *every single IO matrix is singular*.
2. **U.S. Economy (1997–2021):** Using highly disaggregated 192-industry nominal tables from the U.S. Bureau of Labor Statistics (BLS), we find that the IO matrices are singular for *all 25 years*.

To unpack the drivers of this pervasive singularity, we analyze the frequency of zero-linkage vectors (zero rows representing a lack of intermediate forward linkages, and zero columns representing a lack of intermediate backward linkages). As reported in Table 1, 50.54% of the WIOD matrices contain at least one zero row, and 65.89% contain at least one zero column. Furthermore, when these zero vectors occur, they represent deeply entrenched structural features rather than temporary measurement gaps. Table 2 demonstrates that the year-to-year persistence of the zero rows and columns in the WIOD dataset exceeds 99%.

Crucially, however, Table 1 also reveals that zero-linkages do not fully explain this pervasive singularity: A substantial number of WIOD matrices contain absolutely no zero rows or columns. The fact that these fully dense matrices remain singular indicates that deeper, structural redundancies are at play—an economic phenomenon we unpack in Section 3.

Table 1: Frequency of Zero-Entries in WIOD IO Tables

Metric	Count	Percentage
Total IO Matrices Analyzed	645	100%
Matrices with at least one Zero Row	326	50.54%
Matrices with at least one Zero Column	425	65.89%
Matrices where Zero Rows and Columns are exactly the same sectors	303	46.98%

Table 2: Sector-Level Persistence of Zero-Entries (WIOD)

Structural Feature	Year-to-Year Persistence
Zero Rows	99.10%
Zero Columns	99.25%
Zero Both (Symmetric Zeros)	99.67%

The U.S. BLS matrices exhibit a distinct network topology. As shown in Table 3, 100% of the highly disaggregated U.S. tables possess at least one zero row, yet none contain a single zero column. Consistent with the WIOD data, Table 4 confirms that these zero rows are highly persistent across the 25-year sample.

Table 3: Frequency of Zero-Entries in U.S. BLS IO Tables

Metric	Count	Percentage
Total IO Matrices Analyzed	25	100%
Matrices with at least one Zero Row	25	100%
Matrices with at least one Zero Column	0	0%
Matrices where Zero Rows and Cols are exactly the same sectors	0	0%

Table 4: Sector-Level Persistence of Zero-Entries (U.S. BLS)

Structural Feature	Year-to-Year Persistence
Zero Rows	100%
Zero Columns	N/A
Zero Both (Symmetric Zeros)	N/A

2.2. Historical IO Tables

Table 5 reports frequency of zero vectors in 5 historical IO tables – IO matrices for Japan (1935), Germany (1936), and three Indian IO tables from the 1950s (1951–52, 1953–54, and 1955–56).³ All of these historical matrices feature both zero rows and zero columns.⁴

Table 5: Frequency of Zero-Entries in Historical IO Tables

Metric	Count	Percentage
Total IO Matrices Analyzed	5	100%
Matrices with at least one Zero Row	5	100%
Matrices with at least one Zero Column	5	100%
Matrices where Zero Rows and Cols are exactly the same sectors	2	40%

3. Economic Intuition: What Does Singularity Tell Us?

The universal presence of singularity in empirical IO tables is not merely a mathematical curiosity; it contains direct economic intuition regarding the structure of production networks.

³The input-output table for Japan in 1935 was made available by Liu and Tsyvinski (2020) in Appendix C of their paper while we found the German 1936 input-output table from Fremdling and Staeglin (2014) following Liu and Tsyvinski (2020)’s references. The Indian input-output tables for 1951-52, 1953-54 and 1955-56 were found in IARNIW (1960), UN (1961) and Chakraverti (1968) respectively. These input-output tables are available upon request.

⁴Persistence analysis is not done on the historical IO tables because of a lack of availability of continuous time series data.

Singularity of a square matrix ($\det(A) = 0$) is equivalent to linear dependence among the rows or columns of A . Economically, this can arise from several structural sources:

3.1. Zero Linkages

A column of zeros in the matrix A indicates a sector that purchases zero intermediate inputs from domestic industries (lacking backward linkages). Conversely, a row of zeros represents a sector that sells zero intermediate output to other domestic industries (lacking forward linkages), supplying its goods or services exclusively to final demand.

Depending on how these zero-linkages align, they create specific network topologies that reduce the matrix rank and force singularity of the matrix A :

- **Enclave Sectors (Total Isolation):** When a sector exhibits both zero intermediate inputs and zero intermediate sales, it functions as an enclave, entirely disconnected from the domestic inter-industry supply chain. For example, in the WIOD database, “Activities of households as employers; undifferentiated goods- and services-producing activities of households for own use” frequently emerges as an enclave sector for certain countries and years.
- **Sinks (Zero Forward, Active Backward):** These sectors purchase intermediate inputs from the domestic economy but do not sell any output back into the intermediate network. For example, “Federal electric utilities” in the U.S. BLS tables function as terminal nodes that siphon inputs but route their entire output to final demand.
- **Sources (Active Forward, Zero Backward):** Conversely, some sectors supply intermediate goods to other industries but purchase zero intermediate inputs themselves, relying exclusively on primary inputs like labor or imported capital. For example, in the WIOD database, “Activities auxiliary to financial services and insurance activities” emerges as a source sector for Malta between 2000 and 2003.

3.2. Structural Redundancy

Even when every sector actively trades (i.e., the matrix A lacks zero-linkage vectors as is the case with a substantial number of IO matrices in Table 1), singularity can still arise due to deeper structural features of the economy. These redundancies are typically driven by the following economic phenomena:

- **Proportional Cost Structures:** If two sectors have identical supply chain structures but differ only in scale (e.g., they buy from the exact same suppliers in the exact same proportions), their columns in the IO matrix A will be linearly dependent multiples of one another (Hatanaka, 1952).
- **Hierarchical Supply Chains:** Perfectly one-way supply chains—where raw materials flow to intermediate goods, and then to final goods with no feedback loops—result in triangular matrices. Because the determinant of a triangular matrix is the product of its principal diagonal entries, such a supply chain is singular whenever at least one sector does not use its own output (yielding a zero on the principal diagonal).⁵

⁵There is a detailed discussion of hierarchical production networks in Liu (2019).

- **Isolated Economic Blocks:** If the economy fractures into disconnected sub-economies that trade exclusively internally, the matrix takes on a block-diagonal structure (Miller and Blair, 2009). While isolation alone does not guarantee singularity, it restricts the rank space; if any of these closed-loop blocks contain duplicate processes or internal linear dependencies, the entire economy’s IO matrix becomes singular.

4. Diagnostic Criteria for Diagonalizability of Singular Matrices

Given that empirical IO matrices are routinely singular, researchers can bypass standard eigenspace calculations by utilizing simplified conditions connecting matrix rank and non-zero eigenvalues. The mathematical proofs of these conditions are provided in Appendix A.

Theorem 1. (*Necessary Condition for Diagonalizability*). *Let A be an $n \times n$ matrix of real numbers. If A is diagonalizable, then $\text{rank}(A)$ equals the number of non-zero eigenvalues of A (taking account of the algebraic multiplicity of each of the nonzero eigenvalues).*

Theorem 2. (*Sufficient Condition for Singular Matrix Diagonalizability*). *Let A be an $n \times n$ matrix of real numbers with at least one zero eigenvalue. If*

1. $\text{rank}(A) = \text{number of non-zero eigenvalues}$, and
2. *all non-zero eigenvalues are distinct,*

then A is diagonalizable.

As demonstrated in Table 6, these conditions instantly reveal that the 1950s Indian IO tables are non-diagonalizable because $\text{rank}(A) \neq \text{number of non-zero eigenvalues}$.

Table 6: Structure and Diagonalizability of Historical IO Tables

Country-Year	Dimension (n)	Rank	Non-zero Eigenvalues	Status
Japan 1935	23	22	22	Singular, Diagonalizable
Germany 1936	40	39	39	Singular, Diagonalizable
India 1951-52	36	32	31	Singular, Non-Diagonalizable
India 1953-54	36	32	31	Singular, Non-Diagonalizable
India 1955-56	36	34	33	Singular, Non-Diagonalizable

Applying the same conditions to the contemporary WIOD and US IO tables, we find that 5 out of the 20 US IO tables are non-diagonalizable while 7 out of the 645 WIOD IO tables are non-diagonalizable.

5. Conclusion

This paper shows that singularity is an inherent feature of empirical input-output matrices reflecting the absence of linkages, proportional cost structures, isolated economic blocks,

and supply-chain hierarchies lacking intra-industry transactions. Because these structural realities lead to rank-deficiency, the spectral properties of production networks cannot be taken for granted. For researchers attempting eigendecomposition to study dynamic network shock analysis, verifying diagonalizability is essential. The simplified rank-eigenvalue criteria presented in this note offer an efficient diagnostic tool for analyzing the diagonalizability of singular production networks.

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A. Mathematical Proofs

Proof of Theorem 1. According to the supposition of the theorem, the matrix A is diagonalizable. Let D be the diagonal matrix of eigenvalues of A , and let P be the matrix of corresponding eigenvectors. Since A is diagonalizable, the matrix of eigenvectors, P , is nonsingular. Hence, we can write: $D = P^{-1}AP$. Thus, we have $\text{rank}(A) = \text{rank}(P^{-1}A) = \text{rank}(P^{-1}AP) = \text{rank}(D) = \text{number of nonzero eigenvalues (taking account of the algebraic multiplicity of each of the nonzero eigenvalues)}$, where we have used the well known result that multiplication by nonsingular matrices does not change the rank of a matrix (Meyer, 2000, p. 210). $\square$

Proof of Theorem 2. Suppose A has $n - k$ nonzero eigenvalues and 0 is repeated k times as an eigenvalue of A , where $k \geq 1$.

Let us first consider the zero eigenvalues. Since 0 is repeated k times as an eigenvalue of A , its algebraic multiplicity $AM_A(0) = k$. By the supposition of this theorem, $\text{rank}(A) = n - k$. By the rank-nullity theorem (Strang, 2006, section 2.4), we have $\text{rank}(A) + \text{nullity}(A) = n$. Thus, $\text{nullity}(A) = n - \text{rank}(A) = n - (n - k) = k$. Thus, the geometric multiplicity $GM_A(0) = \text{nullity}(A) = k = AM_A(0)$.

Let us now consider the nonzero eigenvalues. By the supposition of the theorem, each of the nonzero eigenvalues are distinct. Let $\lambda \neq 0$ be an eigenvalue of A . By the supposition of the theorem, its algebraic multiplicity $AM_A(\lambda) = 1$. Since the geometric multiplicity is weakly less than arithmetic multiplicity of any eigenvalue (Meyer, 2000, p. 512), we have $0 < GM_A(\lambda) \leq AM_A(\lambda) = 1$. This implies that $GM_A(\lambda) = AM_A(\lambda) = 1$, because $GM_A(\lambda)$ is an integer.

Thus, we have proved that for each eigenvalue of A , whether it is zero or nonzero, geometric and algebraic multiplicity is equal. This implies that A is diagonalizable (Meyer, 2000, p. 512). $\square$