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The existence of anonymous, stationary, monotone social welfare functions satisfying restricted dominance: an answer to an open question in intergenerational equity

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Abstract

In this note, we consider social welfare functions on infinite utility streams and answer an open question posed in Asheim et al (2021, Remark 1). There exists a real-valued representation of social welfare preferences that satisfies monotonicity, stationarity, restricted dominance, and strong anonymity, hence also finite anonymity. Our social welfare function satisfies a stronger finite-rank sensitivity condition, Liminf-Restricted Dominance, in the sense of Asheim et al. (2022) and Asheim et al. (2025). Our construction shows that limited discontinuity is not redundant in Asheim et al. (2021, Proposition 2): once that diagonal continuity condition is dropped, even arbitrary-permutation invariance can coexist with stationarity and non-trivial sensitivity.

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1 Introduction

In this short note we investigate an open question on the non-existence of social welfare function (SWF, henceforth) stated in Asheim et al. (2021, Remark 1). The authors establish impossibility of representation of social welfare preferences by a real-valued social welfare function when the preferences satisfy monotonicity, stationarity, restricted dominance, limited discontinuity and finite anonymity in Proposition 2 of the paper. After proving the result, they explore if the impossibility result persists when the limited discontinuity condition is dropped and frame the conjecture as follows:

"We are unaware of whether dropping Limited discontinuity allows for the existence of a monotone swf satisfying the remaining axioms of Proposition 2: Restricted dominance, Stationarity and Finite anonymity. We conjecture that there exists no such swf even if Limited discontinuity is dropped, but we have neither a proof of impossibility nor an example of a swf that satisfies the remaining axioms. This issue remains an open question."

We present a monotone SWF that satisfies restricted dominance, stationarity, and finite anonymity as a counterexample. This SWF shows that the conjecture is false. The social welfare function we construct satisfies stronger properties (than the ones in the conjecture) in two respects First, it is invariant under all permutations of $\mathbb{N}$, not only finite permutations. Thus it satisfies Strong Anonymity, the impartiality axiom requiring evaluation to be invariant when arbitrary permutations are applied to an infinite utility stream. Second, the example satisfies a stronger form of restricted dominance, namely Liminf-Restricted Dominance: an improvement in a component increases social welfare whenever the component improved upon lies below the limit inferior of the resulting stream. This is the sensitivity boundary emphasized by Asheim et al. (2022) and used in the characterization of Strongly Anonymous Utilitarianism in Asheim et al. (2025). Our construction is not a strongly anonymous utilitarian criterion in their sense; rather, it is a lexicographic threshold-count criterion. Its role is to show that the impossibility result in Asheim et al. (2021) comes from the continuity-type regularity assumption, not from stationarity and anonymity alone.

The rest of the paper is organized as follows. In section 2 we collect the necessary definitions for examining the conjecture. In section 3, we present the counterexample and show that it violates limited discontinuity as it must in view of Asheim et al. (2021, Proposition 2). Section 4 contains brief concluding remarks.

2 Framework and definitions

Let $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{N}$ be the set of real numbers, rational numbers, and natural numbers, respectively. We denote $\mathbb{N} \cup \{0\}$ by $\mathbb{N}_0$ and $\mathbb{N}_0 \cup \{\infty\}$ by A.

Let $Y = [0, 1]$ be the set of all possible utilities that any generation can achieve. Then, $X \equiv Y^{\mathbb{N}}$ is the set of all feasible utility streams. We denote an element of X by $x = (x_1, x_2, \cdots)$. For all $n \in \mathbb{N}$, $x_n \in Y$ represents the amount of utility that period-n generation owns. For $x, y \in X$, we write $x \geq y$ if $x_t \geq y_t$ for every $t \in \mathbb{N}$, and $x > y$ if $x \geq y$ and $x \neq y$. If $b \in Y$, we write $b_{\text{con}} = (b, b, b, \cdots)$.

A binary relation on X is denoted by $\succsim$, with its symmetric and asymmetric parts, denoted by $\sim$ and $\succ$, respectively, defined in the usual manner. A *social welfare order* (SWO) is,

by definition, a complete and transitive binary relation on X. Given a SWO $\succsim$, we say that $\succsim$ can be *represented* by a real-valued function, called a *social welfare function* (SWF), if there is a mapping $W : X \to \mathbb{R}$ such that for all $x, y \in X$, $x \succsim y$ if and only if $W(x) \geq W(y)$. We will be dealing with the following axioms that a SWF should satisfy

Definition 1 (Monotonicity)**.** The SWF W is monotone if, for all $x, y \in X$,

$$x > y \quad \Longrightarrow \quad W(x) \geq W(y).$$

A permutation π of $\mathbb{N}$ is a bijection on $\mathbb{N}$. Permutation π of $\mathbb{N}$ is finite if there exists $\tau \in \mathbb{N}$ such that $\pi(t) = t$ for all $t \geq \tau$. For $x \in X$, we let

$$x_\pi = (x_{\pi(1)}, x_{\pi(2)}, \ldots).$$

Definition 2 (Finite anonymity)**.** The SWF W satisfies finite anonymity if, for every $x \in X$ and every finite permutation π of $\mathbb{N}$,

$$W(x_\pi) = W(x).$$

Definition 3 (Strong anonymity)**.** The SWF W satisfies strong anonymity if, for every $x \in X$ and every permutation π of $\mathbb{N}$,

$$W(x_\pi) = W(x).$$

Definition 4 (Restricted dominance)**.** The SWF W satisfies restricted dominance if, for all $a, b \in Y$ with $a < b$,

$$W(a, b_{\text{con}}) < W(b_{\text{con}}).$$

For any $x \in X$, write

$$\ell(x) = \liminf_{t \to \infty} x_t.$$

Definition 5 (Liminf-Restricted Dominance)**.** The SWF W satisfies Liminf-Restricted Dominance if, for all $x, y \in X$ such that $x_i > y_i$ for some $i \in \mathbb{N}$ and $x_j = y_j$ for all $j \in \mathbb{N} \setminus \{i\}$,

$$\ell(x) > y_i \quad \Longrightarrow \quad W(x) > W(y).$$

Definition 6 (Stationarity)**.** The SWF W satisfies stationarity if, for all $x, y \in X$ with $x_1 = y_1$,

$$W(x) \geq W(y) \quad \Longleftrightarrow \quad W(x') \geq W(y'),$$

where $x' = (x_2, x_3, \cdots)$ and $y' = (y_2, y_3, \cdots)$.

For comparison with the impossibility result of Asheim et al. (2021, Proposition 2), we define the diagonal mapping

$$f_W : [0, 1] \to \mathbb{R}, \qquad f_W(c) = W(c_{\text{con}}).$$

Definition 7 (Limited discontinuity)**.** The SWF W satisfies limited discontinuity if the set of discontinuity points of f_W is not dense in $[0, 1]$.

3 A counterexample

In this section, we construct an example of a monotone SWF satisfying restricted dominance, stationarity, and finite anonymity that proves the conjecture to be false. It is useful to recall the definition of lexicographic order.

Definition 8. (Lexicographic order $\succeq_{\text{lex}}$ on $A^{\mathbb{N}}$) For all $x, y \in A^{\mathbb{N}}$, $x \succeq_{\text{lex}} y$ if and only if either $x_t = y_t$ for all $t \in \mathbb{N}$, or $x_T > y_T$ for some $T \in \mathbb{N}$ and $x_t = y_t$ for all $t \in \mathbb{N}$ such that $t < T$.

We first describe a lexicographic order on $A^{\mathbb{N}}$ and show that it admits a real-valued representation.

Lemma 1. There exists a real-valued function $\Phi : A^{\mathbb{N}} \to \mathbb{R}$ on $A^{\mathbb{N}}$ such that for all $u, v \in A^{\mathbb{N}}$,

$$u \preceq_{\text{lex}} v \quad \Longleftrightarrow \quad \Phi(u) \leq \Phi(v).$$

Proof. We construct the function explicitly. We define intervals J_a, one for each $a \in A$ as follows. (i) For $n \in \mathbb{N}_0$, we set

$$J_n = \left(\frac{1}{2} - \frac{1}{2^{2n+2}}, \frac{1}{2} - \frac{1}{2^{2n+3}}\right).^{1}$$

(ii) For $a = \infty$, we set

$$J_\infty = \left(\frac{3}{4}, \frac{7}{8}\right).$$

Observe that (i) the length of interval J_n is $\frac{1}{2^{2n+3}}$; (ii) for $i \neq j$, the closures $\bar{J}_i$ and $\bar{J}_j$ are disjoint, i.e., $\bar{J}_i \cap \bar{J}_j = \emptyset$, and (iii) ordered i.e., for $i < j$, the interval J_j lies entirely to the right of J_i.
Let $A^{<\mathbb{N}}$ denote the set of finite sequences with each element belonging to A. We take $I_\emptyset = (0, 1)$, and define recursively a sequence of sets by successive concatenation, starting from $I_\emptyset$ and then moving successively to $I_{\emptyset^\frown n_1}$, followed by $I_{\emptyset^\frown n_1^\frown n_2}$, etc. in the following manner. Suppose

$$I_\sigma = I_{\emptyset^\frown \sigma_1^\frown \cdots^\frown \sigma_n} = (\ell_\sigma, r_\sigma)$$

has been defined for every finite sequence $\sigma = (\sigma_1, \cdots, \sigma_n) \in A^n$ of length n. For $a \in A$, we write $J_a = (\alpha_a, \beta_a)$ and define

$$I_{\sigma^\frown a} = \big(\ell_\sigma + (r_\sigma - \ell_\sigma)\alpha_a, \ell_\sigma + (r_\sigma - \ell_\sigma)\beta_a\big).$$

The intervals $I_{\sigma^\frown a}$ are contained in I_σ, have ordered disjoint closures, and have length at most $\frac{1}{8}$ times the length of I_σ.[2] For $u = (u_1, u_2, \cdots) \in A^{\mathbb{N}}$, let

$$u|m = (u_1, \ldots, u_m)$$

be its initial segment of length m. The closed intervals

$$\overline{I_{u|1}} \supseteq \overline{I_{u|2}} \supseteq \overline{I_{u|3}} \supseteq \cdots$$

[1] Thus $J_0 = (\frac{1}{4}, \frac{3}{8})$, $J_1 = (\frac{7}{16}, \frac{15}{32})$, $\cdots$.

[2] Consider for example, $I_{\emptyset^\frown 0} = \big(0 + (1-0)\cdot\frac{1}{4}, 0 + (1-0)\cdot\frac{3}{8}\big) = \big(\frac{1}{4}, \frac{3}{8}\big)$, and $I_{\emptyset^\frown 1} = \big(0 + (1-0)\cdot\frac{7}{16}, 0 + (1-0)\cdot\frac{15}{32}\big) = \big(\frac{7}{16}, \frac{15}{32}\big)$. It is easy to verify that $I_{\emptyset^\frown 0^\frown 1} = \big(\frac{39}{128}, \frac{79}{256}\big)$.

are nested, and their lengths tend to zero. Hence their intersection is a singleton. Define $\Phi(u)$ to be the unique number in that intersection.

Now let $u <_{\text{lex}} v$. Let k be the first coordinate at which u and v differ. Then $u_i = v_i$ for $i < k$ and $u_k < v_k$. Let

$$\sigma = (u_1, \ldots, u_{k-1}) = (v_1, \ldots, v_{k-1}).$$

By construction, $\overline{I_{\sigma^\frown u_k}}$ lies strictly to the left of $\overline{I_{\sigma^\frown v_k}}$. Since

$$\Phi(u) \in \overline{I_{\sigma^\frown u_k}}, \qquad \Phi(v) \in \overline{I_{\sigma^\frown v_k}},$$

we get $\Phi(u) < \Phi(v)$. Thus strict lexicographic order is preserved. Further, if $\Phi(u) \leq \Phi(v)$ and $\neg u \leq_{\text{lex}} v$, then $v <_{\text{lex}} u$ since the lexicographic relation is complete, hence $\Phi(v) < \Phi(u)$, a contradiction. Therefore $u \leq_{\text{lex}} v$ if and only if $\Phi(u) \leq \Phi(v)$. □

We take $(q_k)_{k\in\mathbb{N}}$ to be the one-to-one enumeration of $\mathbb{Q}\cap(0,1)$ as detailed in the appendix. We call this enumeration $\mathbb{Q}(01)$, i.e., $\mathbb{Q}(01) = (q_1, q_2, \cdots)$. For $x \in X = Y^{\mathbb{N}}$ and $k \in \mathbb{N}$, define

$$N_k(x) = |\{t \in \mathbb{N} : x_t < q_k\}| \in A,$$

where $|\cdot|$ denote the cardinality of a given finite (or infinite) set. We refer to $N_k(x)$ as cut-count. We denote the profile

$$P(x) = (N_1(x), N_2(x), \ldots) \in A^{\mathbb{N}}.$$

Profiles are compared lexicographically, with smaller cut-counts being socially preferred. We define the SWF by using the function Φ described in Lemma 1 as follows.

$$W(x) = -\Phi(P(x)) = -\Phi\big(N_1(x), N_2(x), \cdots\big).^3$$

The minus sign reflects the convention that smaller cut-count profiles represent higher welfare.

Proposition 1. Let $Y = [0,1]$ and $X = Y^{\mathbb{N}}$. The function

$$W(x) = -\Phi\big(N_1(x), N_2(x), \cdots\big)$$

is a real-valued monotone SWF satisfying strong anonymity, Liminf-Restricted Dominance, restricted dominance, and stationarity.

Proof. By Lemma 1, Φ is a real-valued function on $A^{\mathbb{N}}$. Since $P(x) \in A^{\mathbb{N}}$ for every infinite utility stream x, the value $W(x) = -\Phi(P(x))$ is well-defined.
Monotonicity. Suppose $x, y \in X$ and $x_t \geq y_t$ for all $t \in \mathbb{N}$. For every q_k,

$$\{t : x_t < q_k\} \subseteq \{t : y_t < q_k\}.$$

Hence

$$N_k(x) \leq N_k(y) \qquad \text{for every } k.$$

[3]For instance, let $x = (0_{\text{con}})$, and $y = (1_{\text{con}})$. It is not difficult to verify that $W(x) = -\frac{6}{7}$ and $W(y) = -\frac{2}{7}$.

Therefore, $P(x) \leq_{\text{lex}} P(y)$. Hence, by Lemma 1 and by the above definition of W, $W(x) \geq W(y)$. Thus the SWF $W(\cdot)$ is monotone.

Strong anonymity. Let π be an arbitrary permutation of $\mathbb{N}$. For every k,

$$|\{t : x_{\pi(t)} < q_k\}| = |\{t : x_t < q_k\}|.$$

Thus $N_k(x_\pi) = N_k(x)$ for every k. Hence $P(x_\pi) = P(x)$ and $W(x_\pi) = W(x)$. Therefore strong anonymity holds, and finite anonymity follows immediately.

Liminf-Restricted Dominance. Let $x, y \in X$ be such that $x_i > y_i$ for some $i \in \mathbb{N}$ and $x_j = y_j$ for all $j \neq i$. Assume that $\ell(x) > y_i$. Since $x \geq y$, for every rational threshold q_k,

$$N_k(x) \leq N_k(y).$$

Choose a rational $q_m \in \mathbb{Q}(01)$ such that

$$y_i < q_m < \min\{x_i, \ell(x)\}.$$

This is possible by density of the rationals. Since $q_m < \ell(x)$, the set $\{t : x_t < q_m\}$ is finite. Moreover, because $x_i > q_m > y_i$ and $x_j = y_j$ for all $j \neq i$,

$$N_m(y) = N_m(x) + 1.$$

Thus the profiles $P(x)$ and $P(y)$ differ at some coordinate, and at every coordinate at which they differ the count for x is weakly smaller, with strict inequality at coordinate m. Therefore the first coordinate at which $P(x)$ and $P(y)$ differ favors x:

$$P(x) <_{\text{lex}} P(y).$$

Since $W = -\Phi(P(\cdot))$, it follows that

$$W(x) > W(y).$$

Hence W satisfies Liminf-Restricted Dominance. Since Liminf-Restricted Dominance implies restricted dominance (by letting $x = b_{\text{con}}$, $y = (a, b_{\text{con}})$, with $a < b$, $i = 1$, $x_i = b$ and $y_i = a$), it follows that W also satisfies restricted dominance.

The following claim would be useful for showing that the SWF satisfies stationarity.

Claim 1. *Let $u, v \in A^{\mathbb{N}}$ and let $e \in \{0,1\}^{\mathbb{N}}$. Define coordinate-wise addition, with $\infty + 0 = \infty + 1 = \infty$. Then*

$$u \leq_{\text{lex}} v \quad \Longleftrightarrow \quad u + e \leq_{\text{lex}} v + e.$$

Proof. $\Longrightarrow$ Suppose that $u \leq_{\text{lex}} v$. We distinguish between two cases: if $u = v$, the claim is immediate. For the second case, let $u_k < v_k$ and $u_j = v_j$ for all $j < k$. Thus, $u + e \leq_{\text{lex}} v + e$.

$\Longleftarrow$ Suppose that $u + e \leq_{\text{lex}} v + e$. Case 1: $u + e = v + e$ which implies $u = v$ and $u \leq_{\text{lex}} v$. Case 2: $u_k + e_k < v_k + e_k$ and $u_j + e_j = v_j + e_j$ for all $j < k$. Then, $u_k < v_k$ and $u_j = v_j$ for all $j < k$. Therefore, $u \leq_{\text{lex}} v$. □

Stationarity. Let $x, y \in X$ satisfy $x_1 = y_1 = a$. For each k,

$$N_k(x) = N_k(x') + \mathbf{1}_{\{a<q_k\}}, \text{ and } N_k(y) = N_k(y') + \mathbf{1}_{\{a<q_k\}},^4$$

when the cut-counts are all finite. Equality holds trivially in case the cut-counts are infinite as we define $\infty + 1 = \infty$ and $\infty + 0 = \infty$. Observe that the vector

$$e(a) = \left(\mathbf{1}_{\{a<q_1\}}, \mathbf{1}_{\{a<q_2\}}, \cdots\right)$$

is added coordinate-wise to both profiles $P(x')$ and $P(y')$ to obtain the profiles $P(x)$ and $P(y)$ respectively. Therefore, lexicographic order on $A^{\mathbb{N}}$ is preserved. Therefore

$$P(x) \preceq_{\text{lex}} P(y) \quad \Longleftrightarrow \quad P(x') \preceq_{\text{lex}} P(y').$$

Since $W = -\Phi(P(\cdot))$, this is equivalent to

$$W(x) \geq W(y) \quad \Longleftrightarrow \quad W(x') \geq W(y').$$

Thus stationarity holds. □

Remark 1. Proposition 1 proves that the conjecture in Asheim et al. (2021, Remark 1) is false. Further, in view of Asheim et al. (2021, Proposition 2), the SWF must violate the limited discontinuity axiom. We show that is indeed the case. Let

$$f_W(c) = W(c_{\text{con}}).$$

We show that every $q_k \in \mathbb{Q}(01)$ is a discontinuity point of f_W. Fix q_m. Choose $\varepsilon > 0$ such that $\varepsilon < q_m$ and the interval $(q_m - \varepsilon, q_m + \varepsilon)$ contains no member of the finite set $\{q_1, \ldots, q_{m-1}\}$. For every $b \in (q_m - \varepsilon, q_m)$, the first $m-1$ coordinates of $P(b_{\text{con}})$ and $P((q_m)_{\text{con}})$ agree, while at coordinate m,

$$N_m(b_{\text{con}}) = \infty, \qquad N_m((q_m)_{\text{con}}) = 0.$$

Let $\sigma = (N_1((q_m)_{\text{con}}), \cdots, N_{m-1}((q_m)_{\text{con}}))$. Then $\Phi(P((q_m)_{\text{con}}))$ lies in the closed interval $\overline{I_{\sigma^\frown 0}}$, while every such $\Phi(P(b_{\text{con}}))$ lies in $\overline{I_{\sigma^\frown \infty}}$. By the construction in Lemma 1, these two closed intervals are separated by a positive gap. Therefore $\Phi(P((q_m)_{\text{con}}))$ is separated by a positive distance from all $\Phi(P(b_{\text{con}}))$ with $b \in (q_m - \varepsilon, q_m)$. Since $W = -\Phi(P(\cdot))$, f_W is not left-continuous at q_m. Hence q_m is a discontinuity point of f_W. Because $\mathbb{Q} \cap (0,1)$ is dense in $[0,1]$, the discontinuity set of f_W is dense in $[0,1]$. So, W does not satisfy limited discontinuity.

4 Conclusion

Proposition 1 answers an open question posed in Asheim et al. (2021, Remark 1). Moreover, the SWF constructed in the proposition satisfies Strong Anonymity and Liminf-Restricted Dominance. The SWF employs a lexicographic threshold-count criterion, and it violates the continuity conditions used in Asheim et al. (2021), Asheim et al. (2022), and Asheim et al. (2025). Thus, the example does not conflict with those results; it shows that, without such continuity (a regularity condition), strong anonymity, stationarity, real-valued representability, and Liminf-Restricted Dominance can coexist.

[4] Recall the definition

$$\mathbf{1}_{\{a<q_k\}} = \begin{cases} 1 & \text{if } a < q_k, \\ 0 & \text{if } a \geq q_k. \end{cases}$$

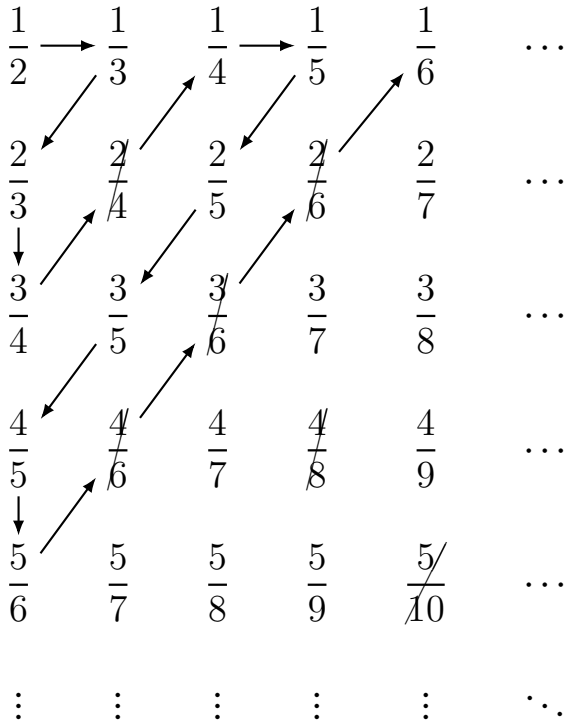
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Appendix: Enumeration of rational numbers in $(0, 1)$ for use in Proposition 1



Fixed enumeration of rational numbers in $\mathbb{Q}(01)$.

The crossed-out fractions are omitted because they repeat rational numbers that have already appeared in the enumeration. The first ten rational numbers are:

$$q_1 = \frac{1}{2},\ q_2 = \frac{1}{3},\ q_3 = \frac{2}{3},\ q_4 = \frac{3}{4},\ q_5 = \frac{1}{4},\ q_6 = \frac{1}{5},\ q_7 = \frac{2}{5},\ q_8 = \frac{3}{5},\ q_9 = \frac{4}{5},\ q_{10} = \frac{5}{6},\ \cdots$$